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Soliton Solutions of the Perturbed Resonant Nonlinear Schrödinger's Equation with Full Nonlinearity by Semi-Inverse Variational Principle

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Abstract: This paper carries out the integration of the resonant nonlinear Schrödinger's equation in presence of perturbation terms that are considered with full nonlinearity. The three types of nonlinear media are studied. They are the cubic nonlinearity, power law and log law nonlinearity. The semi-inverse variational principle is applied to extract the analytical soliton solution.

Keywords: solitons; integrability; Kerr law

1 Introduction

The nonlinear Schrödinger's equation (NLSE) plays a vital role in various areas of STEM disciplines [1–20]. It appears in the study of nonlinear optics, plasma physics, mathematical biosciences, quantum mechanics, fluid dynamics and several other disciplines. The main feature of the NLSE is that it supports soliton solution which makes it very widely applicable. Solitons are stable nonlinear waves or pulses and is the outcome of a delicate balance between dispersion and nonlinearity. Therefore, these solitons are the essential fabrics that dictate our daily lives. For example, these waves are stable pulses that transport information through optical fibers over trans-continental and trans-oceanic distances in a matter of a few femto-seconds. Other examples of solitons in our daily lives are in Bose-Einstein condensates, α-helix proteins in clinical sciences, nuclear physics and several others.

Therefore it is imperative to focus deeply into the integrability aspects of the NLSE that will reveal soliton solutions. This paper will study the extended as well as the generalized version of the NLSE with a few Hamiltonian perturbation terms that are going to be taken into consideration with full nonlinearity. There are three types of nonlinearity that will be addressed. They are the cubic NLSE, power law nonlinearity and the log law nonlinearity. While in the first two types of nonlinearity, it is the soliton solution that will be obtained, the third law nonlinearity will give Gausson solutions. The semi-inverse variational principle (SVP) will be implemented to extract the soliton and Gausson solutions to the perturbed resonant NLSE. This principle is basically an inverse problem approach.

2 GOVERNING EQUATION

The perturbed resonant NLSE with full nonlinearity that is going to studied in this paper by the SVP is given by

$$
\begin{aligned}
& iq_t + aq_{xx} + b\left(\frac{|q|_{xx}}{|q|}\right) q + cF\left(|q|^2\right) q \\
= \; & i\alpha q_x + i\lambda\left(|q|^{2m} q\right)_x + i\nu\left(|q|^{2m}\right)_x q \\
& + i\theta\,|q|^{2m} q_x + \sigma\frac{q^*_{xx}}{|q|^2} q^2 \qquad (1)
\end{aligned}
$$

Here in (1), on the left hand side, the first term represents the linear evolution of the soliton pulse. The coefficient of a is the group velocity dispersion while the coefficient of c is the nonlinear term. The coefficient of b is the quantum or Bohm potential that appears in the context of chiral solitons in quantum Hall effect [5,7,10,13]. It is also seen in the context of Madelung fluid in quantum mechanics [17]. For the perturbation terms on the right

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hand side, the coefficient of α is the inter-modal dispersion that shows up in nonlinear optics. Then, the coefficient of λ is the self-steepening term that is also studied in nonlinear optics in order to avoid the formation of shock waves during soliton transmission through optical fibers. The coefficients of ν and θ are due to nonlinear dispersions. Finally, the σ-term is from plasma physics for solitons in relativistic plasmas [1,9,14]. The index m represents the full nonlinearity parameter. The independent variables are x and t that represent spatial and temporal variables respectively. The dependent variable is $q(x,t)$ that is the complex valued wave profile for the perturbed resonant NLSE. The functional F represents the general form of nonlinear media and $F\left(|q|^2\right)q$ is k times continuously differentiable, so that

$$F\left(|q|^2\right)q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n)\times(-m,m);R^2\right). \tag{2}$$

In order to look into a brief history of this problem, it must be noted that several special cases of this problem have been studied in the past. In particular, the special case with $b = \sigma = \theta = 0$, for cubic and power law nonlinearity, was covered in 2009 [12,16]. Additionally, the case of log-law nonlinearity with the same special values of the parameters was addressed in 2011 [4]. The case with right hand side equation set to zero but with chiral nonlinearity was addressed on several occasions, namely during 2011 and 2012 [5,7,10]. The exact bright and dark soliton solutions by the ansatz method was also obtained in 2012 [17]. Thus, this paper is thus going to address equation (1) on the most generalized setting so far, and the tool of integration is going to be the SVP.

2.1 SEMI-INVERSE VARIATIONAL PRINCIPLE

In order to apply the SVP to (1), the starting hypothesis is the traveling wave solution that is given by

$$q(x,t) = g(s)e^{i\phi}, \tag{3}$$

where $g(s)$ represents the shape of the wave profile and

$$s = x - vt, \tag{4}$$

with v being the velocity of the wave. The phase component $\phi(x,t)$ is defined as

$$\phi = -\kappa x + \omega t + \sigma_0. \tag{5}$$

where κ represents the soliton frequency and ω is the soliton wave number while σ_0 represent the phase constant. Therefore, substituting this hypothesis into (1) and decomposing into real and imaginary parts yield the following two equations

$$\begin{aligned}&(a+b-\sigma)g'' - \left(\omega - \alpha\kappa + a\kappa^2\right)g\\ -\ &(\lambda+\theta)\kappa g^{2m+1} + cF\left(g^2\right)g = 0\end{aligned} \tag{6}$$

and

$$(\nu + 2a\kappa + \alpha) + \{(2m+1)\lambda + 2m\nu + \theta\}g^{2m} = 0 \tag{7}$$

respectively. Setting the coefficients of the linearly independent functions in (7) to zero yields the soliton velocity as

$$\nu = -\alpha - 2a\kappa \tag{8}$$

and the constraint conditions between the parameters is given by

$$(2m+1)\lambda + 2m\nu + \theta = 0 \tag{9}$$

It needs to be noted that the velocity of the soliton as well as the constraint condition holds irrespective of the type of nonlinearity in question. Now, multiplying both sides of the real part equation (6) by g' and integrating leads to

$$\begin{aligned}&(a+b-\sigma)\left(g'\right)^2 - \left(\omega - \alpha\kappa + a\kappa^2\right)g^2\\ -\ &(\lambda+\theta)\kappa\frac{g^{2m+2}}{m+1} + 2c\int F\left(g^2\right)g\,dg = K\end{aligned} \tag{10}$$

where K represents the integration constant. Then, the stationary integral J is defined to be

$$J = \int_{-\infty}^{\infty} K ds \tag{11}$$

which therefore is

$$\begin{aligned}J = \int_{-\infty}^{\infty}&\left[(a+b-\sigma)\left(g'\right)^2 - \left(\omega - \alpha\kappa + a\kappa^2\right)g^2\right.\\ -\ &\left.(\lambda+\theta)\kappa\frac{g^{2m+2}}{m+1} + 2c\int F\left(g^2\right)g\,dg\right]ds\end{aligned} \tag{12}$$

Finally, the 1-soliton solution hypothesis is taken to be

$$g(s) = Af\left[\operatorname{sech}(Bs)\right], \tag{13}$$

or

$$g(s) = Ae^{-B^2s^2} \tag{14}$$

where A is the amplitude and B is the inverse width of the soliton or Gausson. The functional f in (13) depends on cubic or power law nonlinearity. Then SVP states that the amplitude can be retrieved from the coupled system of equations given by

$$\frac{\partial J}{\partial A} = 0 \tag{15}$$

and

$$\frac{\partial J}{\partial B} = 0. \tag{16}$$

3 APPLICATIONS

The SVP that was developed in the previous section will now be applied to the three types of nonlinearity that will be detailed in the following three subsections. The explicit value of the soliton amplitude A and the inverse width B are going to be determined for each of these nonlinear cases.

3.1 CUBIC NONLINEARITY

In this case, $F(u) = u$ and this is commonly referred to as the cubic Schrödinger's equation and in the context of nonlinear optics this is referred to Kerr law nonlinearity. This type of cubic NLSE is very commonly studied in plasma physics, solitons due to α-helix proteins in mathematical biosciences, deep water waves in fluid dynamics as well as nonlinear optics. With cubic nonlinearity, the governing equation (1) reduces to

$$\begin{aligned} & iq_t + aq_{xx} + b\left(\frac{|q|_{xx}}{|q|}\right)q + c|q|^2 q \\ &= i\alpha q_x + i\lambda\left(|q|^{2m}q\right)_x + i\nu\left(|q|^{2m}\right)_x q \\ &+ i\theta|q|^{2m}q_x + \sigma\frac{q^*_{xx}}{|q|^2}q^2 \end{aligned} \tag{17}$$

so that the corresponding stationary integral is

$$\begin{aligned} J &= \int_{-\infty}^{\infty}\left[(a+b-\sigma)\left(g'\right)^2 - \left(\omega - \alpha\kappa + a\kappa^2\right)g^2\right]ds \\ &- \int_{-\infty}^{\infty}\left[(\lambda+\theta)\kappa\frac{g^{2m+2}}{m+1} + \frac{cg^4}{2}\right]ds \end{aligned} \tag{18}$$

For the cubic NLSE, the 1-soliton solution hypothesis is [17]

$$g(s) = A\,\mathrm{sech}(Bs) \tag{19}$$

Substituting this hypothesis into (18) and carrying out the integration reduces the stationary integral to

$$\begin{aligned} J &= \frac{2}{3}(a+b-\sigma)A^2B - 2\left(\omega-\alpha\kappa+a\kappa^2\right)\frac{A^2}{B} \\ &- \frac{2m(\lambda+\theta)\kappa}{(m+1)(2m+1)}\frac{A^{2m+2}}{B}\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} + \frac{2cA^4}{3B} \end{aligned} \tag{20}$$

Then equations (15) and (16) in this case, after simplification, are respectively given by

$$\begin{aligned} & (a+b-\sigma)B^2 - \left(\omega-\alpha\kappa+a\kappa^2\right) \\ &- \frac{3m(\lambda+\theta)\kappa A^{2m}}{(2m+1)}\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} + 2cA^2 = 0 \end{aligned} \tag{21}$$

$$\begin{aligned} & (a+b-\sigma)B^2 + 3\left(\omega-\alpha\kappa+a\kappa^2\right) \\ &+ \frac{3m(\lambda+\theta)\kappa A^{2m}}{(m+1)(2m+1)}\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} - cA^2 = 0 \end{aligned} \tag{22}$$

Solving the coupled system of equations (21) and (22) leads to the polynomial equation for the amplitude A as

$$\begin{aligned} & \frac{3m(m+2)(\lambda+\theta)\kappa A^{2m}}{(m+1)(2m+1)}\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} - 3cA^2 \\ &+ 4\left(\omega-\alpha\kappa+a\kappa^2\right) = 0 \end{aligned} \tag{23}$$

From (23), it can be easily seen that the soliton amplitude A can be explicitely obtained provided $m = 1$, 2 or 3. Once the amplitude A is available, the width B can be recovered from the realation

$$B = \frac{1}{2\sqrt{a+b-\sigma}}\left[\frac{3m(3m+2)(\lambda+\theta)\kappa A^{2m}}{(m+1)(2m+1)}\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} - 5cA^2\right]^{\frac{1}{2}} \tag{24}$$

that can be obtained from (21) and (22). Equation (24) immediately poses a restriction

$$a+b > \sigma \tag{25}$$

Therefore the 1-soliton solution to (17) is given by

$$q(x,t) = A\,\mathrm{sech}[B(x-vt)]e^{i(-\kappa x+\omega t+\sigma_0)} \tag{26}$$

where the amplitude A and the width B are respectively given by (23) and (24) and the velocity of the soliton is given by (8). This solution is valid as long as the constraint conditions given by (9) and (25) hold.

3.2 POWER LAW NONLINEARITY

In this subsection, the perturbed resonant NLSE will be studied with power law nonlinearity. In this case, $F(u) = u^n$. Power law nonlinearity is also studied in the context of nonlinear optics where special case of optical fibers are designed with soliton transmission in mind [2]. In this case, equation (1) modifies to

$$\begin{aligned} & iq_t + aq_{xx} + b\left(\frac{|q|_{xx}}{|q|}\right)q + c|q|^{2n}q \\ &= i\alpha q_x + i\lambda\left(|q|^{2m}q\right)_x + i\nu\left(|q|^{2m}\right)_x q \\ &+ i\theta|q|^{2m}q_x + \sigma\frac{q^*_{xx}}{|q|^2}q^2 \end{aligned} \tag{27}$$

where the restriction is

$$0 < n < 2 \tag{28}$$

to avoid soliton collapse and in particular

$$n \neq 2 \tag{29}$$

in order to eliminate self-focussing singularity in nonlinear optics [11]. This leads to the stationary integral being

$$\begin{aligned} J &= \int_{-\infty}^{\infty}\left[(a+b-\sigma)\left(g'\right)^2 - \left(\omega-\alpha\kappa+a\kappa^2\right)g^2\right]ds \\ &- \int_{-\infty}^{\infty}\left[(\lambda+\theta)\kappa\frac{g^{2m+2}}{m+1} + \frac{cg^{2n+2}}{n+1}\right]ds \end{aligned} \tag{30}$$

For power law nonlinearity, the 1-soliton solution ansatz is [17]

$$g(s) = A\,\mathrm{sech}^{\frac{1}{n}}(Bs) \tag{31}$$

Substituting this hypothesis into (30) and carrying out the integration reduces the stationary integral to

$$\begin{aligned} J &= \left\{\frac{a+b-\sigma}{n(n+2)}A^2B - \left(\omega-\alpha\kappa+a\kappa^2\right)\frac{A^2}{B}\right. \\ &+ \left.\frac{2c}{(n+1)(n+2)}\frac{A^{2n+2}}{B}\right\}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} \\ &- \frac{(\lambda+\theta)\kappa}{m+1}\frac{A^{2m+2}}{B}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m+1}{n}+\frac{1}{2}\right)} \end{aligned} \tag{32}$$

Then equations (15) and (16) in this case, after simplification, are respectively given by

$$\frac{a+b-\sigma}{n(n+2)}B^2-\left(\omega-\alpha\kappa+a\kappa^2\right)$$
$$-\left(\lambda+\theta\right)\kappa A^{2m}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{m+1}{n}+\frac{1}{2}\right)}+\frac{2c}{n+2}A^{2n}=0 \quad (33)$$

and

$$\frac{a+b-\sigma}{n(n+2)}B^2+\left(\omega-\alpha\kappa+a\kappa^2\right)$$
$$+\frac{(\lambda+\theta)\kappa}{m+1}A^{2m}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{m+1}{n}+\frac{1}{2}\right)}$$
$$-\frac{2c}{(n+1)(n+2)}A^{2n}=0 \quad (34)$$

Equations (33) and (34) leads to the polynomial equation for the soliton amplitude A as

$$\frac{(m+2)(\lambda+\theta)\kappa}{2(m+1)}A^{2m}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{m+1}{n}+\frac{1}{2}\right)}$$
$$-\frac{c}{n+1}A^{2n}+\left(\omega-\alpha\kappa+a\kappa^2\right)=0 \quad (35)$$

Again from (32) and (33), the soliton width width B can be recovered from

$$B=\left[F\left\{\frac{m(\lambda+\theta)\kappa}{m+1}A^{2m}G-\frac{2nc}{(n+1)(n+2)}A^{2n}\right\}\right]^{\frac{1}{2}} \quad (36)$$

where $F=\frac{n(n+2)}{2(a+b-\sigma)}$, $G=\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{m+1}{n}+\frac{1}{2}\right)}$.

Therefore, the 1-soliton solution to (27) is given by

$$q(x,t)=A\,\mathrm{sech}^{\frac{1}{n}}[B(x-vt)]e^{i(-\kappa x+\omega t+\sigma_0)} \quad (37)$$

where the amplitude A and the width B are respectively given by (35) and (36) and the velocity of the soliton is still given by (8). This solution is valid as long as the constraint conditions given by (9) and (25) hold.

4 LOG LAW NONLINEARITY

In the case of log law nonlinearity, $F(u)=\ln u$, and thus the perturbed resonant NLSE is given by

$$iq_t+aq_{xx}+b\left(\frac{|q|_{xx}}{|q|}\right)q+2cq\ln q$$
$$=i\alpha q_x+i\lambda\left(|q|^{2m}q\right)_x+i\nu\left(|q|^{2m}\right)_x q$$
$$+i\theta|q|^{2m}q_x+\sigma\frac{q_{xx}^*}{|q|^2}q^2 \quad (38)$$

so that the stationary integral in this case transforms to

$$J=\int_{-\infty}^{\infty}\left[(a+b-\sigma)\left(g'\right)^2-\left(\omega-\alpha\kappa+a\kappa^2\right)g^2\right.$$
$$\left.-(\lambda+\theta)\kappa\frac{g^{2m+2}}{m+1}+cg^2\left(2\ln g-1\right)\right]ds \quad (39)$$

Now, choosing the Gausson ansatz given by (14), the stationary integral (39) reduces to

$$J=2(a+b-\sigma)A^2B\sqrt{\pi}-\left(\omega-\alpha\kappa+a\kappa^2\right)\frac{A^2}{B}\sqrt{\pi}$$
$$-\frac{(\lambda+\theta)\kappa A^{2m+2}}{4\sqrt{2}(m+1)^{\frac{5}{2}}}\sqrt{\pi}+\frac{\sqrt{2}cA^2}{4B}\left(4\ln A-3\right)\sqrt{\pi} \quad (40)$$

Then, equation (15) and (16), in this case gives

$$2(a+b-\sigma)B^2\sqrt{\pi}-\left(\omega-\alpha\kappa+a\kappa^2\right)$$
$$-\frac{(\lambda+\theta)\kappa A^{2m}}{4\sqrt{2}(m+1)^{\frac{3}{2}}}+\sqrt{2}c\ln A=\frac{\sqrt{2}c}{4} \quad (41)$$

and

$$2(a+b-\sigma)B^2\sqrt{\pi}+\left(\omega-\alpha\kappa+a\kappa^2\right)$$
$$-\frac{(\lambda+\theta)\kappa A^{2m}}{4\sqrt{2}(m+1)^{\frac{5}{2}}}-\sqrt{2}c\ln A=-\frac{3\sqrt{2}c}{4} \quad (42)$$

respectively. Solving this coupled system yields the amplitude of the Gausson as

$$A=\exp\left\{-\frac{1}{2m}W\left[-\frac{2mc_1}{c_2}\exp\left(\frac{2mc_3}{c_2}\right)\right]+\frac{c_3}{c_2}\right\} \quad (43)$$

where

$$c_1=\frac{(\lambda+\theta)\kappa}{4\sqrt{2}(m+1)^{\frac{5}{2}}} \quad (44)$$

$$c_2=2\sqrt{2}c \quad (45)$$

$$c_3=2\left(\omega-\alpha\kappa+a\kappa^2\right)+\sqrt{2}c \quad (46)$$

and $W(x)$ is the Lambert's W-function that is defined to be the inverse of

$$f(x)=xe^x \quad (47)$$

Equation (43) introduces the constraint condition, from the definition of Lambert's function, as

$$\exp\left\{\frac{c_2+2mc_3}{c_2}\right\}\leq\frac{c_2}{2mc_1} \quad (48)$$

Once the amplitude of the Gausson is available, the width can be obtained from

$$B=\frac{1}{2\sqrt{a+b-\sigma}}\left[\frac{(m+2)(\lambda+\theta)\kappa A^{2m}}{4\sqrt{2}(m+1)^{\frac{5}{2}}}-\frac{2\sqrt{2}c}{4}\right]^{\frac{1}{2}} \quad (49)$$

which can be recovered from (41) and (42). Hence, finally, the Gausson solution of (38) is given by

$$q(x,t)=Ae^{-B^2(x-vt)^2}e^{i(-\kappa x+\omega t+\sigma_0)} \quad (50)$$

where the Gausson amplitude and the inverse width can be obtained from (43) and (49) respectively. Besides (25), the additional constraint condition in this case is given by (48) that must also hold in order for the Gaussons to exist. The velocity is again seen in (8) along with the restriction (9).

5 CONCLUSIONS

This paper addressed the perturbed resonant NLSE where the perturbation terms are considered with full nonlinearity. The SVP is applied to carry out the integration of the perturbed resonant NLSE. Thus the soliton solutions were obtained. There are three types of nonlinearity that are discussed in this paper. They are the cubic nonlinearity, power-law nonlinearity and finally the log law nonlinearity. This SVP is essentially an inverse problem mechanism that integrates the perturbed NLSE. The solutions are obtained and the soliton parameters are in terms of gamma functions and Lambert's W-function. Several constraint conditions automatically fell out from the mathematical structure of the solution parameters.

This paper encompassed several studies that were conducted in the past. The special cases of the results of the paper were already obtained before as indicated. Thus, the application of the NLSE in nonlinear optics, plasma physics, chiral solitons in nuclear physics and mathematical biosciences are all collectively studied in this paper. Therefore, these generalized results are going to serve as the starting point for further investigation of the NLSE in this direction. Thus, the future of this area of research stands on a strong footing. One immediate expansion of this research is to look into several other forms of nonlinear media, such as the parabolic law nonlinearity, polynomial law nonlinearity, dual and triple power law nonlinearity as well as the saturable law nonlinearity, just to name a few. This is just the tip of the iceberg.

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