Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090516

On the Stabilization of a Generalized Lagrange-Poisson Problem

J. A. Vera[∗]

Centro Universitario de la Defensa. Academia General del Aire. Universidad Politécnica de Cartagena, 30720-Santiago de la Ribera, Región de Murcia, Spain

Received: 17 Jan. 2015, Revised: 18 Apr. 2015, Accepted: 19 Apr. 2015
Published online: 1 Sep. 2015

Abstract: In this work we consider the Hamiltonian dynamics of a generalized Lagrange–Poisson problem whose fixed point performs high-frequency vertical periodic oscillations of small amplitude. Using the generalized Kapitza averaging method we obtain a sufficient condition for stabilizing the unstable upright position.

Keywords: Periodic solution, symmetrical gyrostat, nonlinear oscillations

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction

By a *generalized Lagrange–Poisson problem* (see for instance [24] for more details) we are specifically referring to a mechanical system formed by a symmetric gyrostat that has a fixed point P and the centre of mass of the gyrostat lies on its dynamic-symmetry axis. The forces acting on the gyrostat deriving from a Newtonian symmetric potential that is to say a potential $U(k_3)$, being U a smooth function like to the case of Lagrange-Poisson for a rigid body with a fixed point. The variable k_3 is the third component of the Poisson vector of the system.

It is known that a gyrostat is a mechanical system S made of a rigid body S_1 to which other bodies S_2 are connected; these other bodies may be deformable or rigid, but must not be rigidly connected to S_1, so that the movements of S_2 with respect to S_1 do not modify the distribution of mass within the compound system S.

For instance, we can envision a rigid main body S_1, designated as the *platform*, supporting additional bodies S_2, which possess axial symmetry and are designated as *rotors*. These rotors may rotate with respect to the platform in such a way that the mass distribution within the system as a whole is not altered; this will produce an internal angular momentum, designated as *gyrostatic momentum*, which will be normally regarded as a constant. Note that when this constant vector is zero, the motion of the system is reduced to the motion of a rigid body.

Vito Volterra was the first to introduce the concept of a gyrostat in [25], in order to study the motion of the Earth's polar axis and explaining variations in the Earth's latitude by means of internal movements that do not alter the planets's distribution of mass.

The general study of the dynamics of gyrostats has been presented extensively in the classic literature about this topic. Hamiltonian formulations of such dynamics are the main tools used in the formulation of these problems (see for instance [7], or [21]). Various aspects related to these problems are discused, for example, the existence of periodic solutions, bifurcations, or chaos, in various gyrostat motion problems ([20],[8],[6],[23]), integrability and first integrals for the problem (see [12],[3], [4]) or equilibria and stabilities in rigid bodies and gyrostats, either with fixed point or in orbit (see [16],[17],[5],[1],[19],[2],[15],[22],[9]).

In this work we consider the Hamiltonian dynamics of a generalized Lagrange–Poisson problem whose fixed point P performs high-frequency vertical periodic oscillations of small amplitude. Using the generalized Kapitza averaging method (see [10], [11]) we obtain a sufficient condition for stabilizing the unstable upright position.

∗ Corresponding author e-mail: juanantonio.vera@cud.upct.es

2 Hamiltonian formulation of the problem

In the generalized Lagrange–Poisson problem we will assume that the point P executes vertical motion in accordance with the periodic law $\xi(t)$ about a certain fixed point O. Suppose $OXYZ$ is a inertial system of coordinates and $Pxyz$ is a system of coordinates, rigidly attached to the gyrostat, whose axes coincide with the principal axes of inertia of the body for the point P. We will specify the orientation of the system of coordinates $Pxyz$ with respect to $OXYZ$ using the Euler angles (ψ,φ,θ).

The kinetic energy of the body is given by the expression

$$\mathscr{T} =\frac{1}{2}m\left\|\mathbf{v}_P\right\|^2+m\,\mathbf{v}_P\cdot\left(\omega\times\overrightarrow{PG}\right)+\frac{1}{2}\omega^t\mathbb{I}\omega+\mathbf{l}\cdot\omega \quad (1)$$

where $\mathbb{I} = diag(I_1,I_1,I_3)$ is the diagonal tensor of inertia and m is the mass. On the other hand, in projections onto the $Pxyz$ axes, we have $\omega=(p,q,r)$ the vector of the absolute velocity of rotation and $\mathbf{l}=(0,0,l)$ is the gyrostatic momentum of the gyrostat. The other vectors have the following coordinates

$$\overrightarrow{PG}=(0,0,z_G)^t,\ \omega\times\overrightarrow{PG}=(qz_G,-pz_G,0)^t$$

with

$$\mathbf{v}_P=\frac{d\xi}{dt}(\sin\theta\sin\varphi,\sin\theta\cos\varphi,\cos\theta)^t.$$

The potential energy of the system is

$$\mathscr{U}(t,\theta)=mg\xi(t)+U(\cos\theta).$$

The Lagrangian of the system $\mathscr{L}=\mathscr{T}-\mathscr{U}$ are given by

$$\begin{aligned}\mathscr{L} =&\frac{1}{2}m\left(\frac{d\xi}{dt}\right)^2-mz_G\sin\theta\left(\frac{d\xi}{dt}\right)\left(\frac{d\theta}{dt}\right)+\\ &\frac{1}{2}\left(I_1\left(\sin^2\theta\left(\frac{d\psi}{dt}\right)^2+\left(\frac{d\theta}{dt}\right)^2\right)+\right.\\ &I_3\left(\cos\theta\left(\frac{d\psi}{dt}\right)+\left(\frac{d\varphi}{dt}\right)\right)^2+\\ &\left.l\left(\cos\theta\left(\frac{d\psi}{dt}\right)+\left(\frac{d\varphi}{dt}\right)\right)\right)-\mathscr{U}(t,\theta).\end{aligned} \quad (2)$$

The coordinates ψ and φ are cyclical, and the momenta corresponding to them are given by

$$\begin{aligned}p_\psi=&I_1\left(\frac{d\psi}{dt}\right)\sin^2\theta+I_3\cos\theta\left(\cos\theta\left(\frac{d\psi}{dt}\right)+\frac{d\phi}{dt}\right)+\\ &l\cos\theta\\ p_\varphi=&I_3\left(\cos\theta\left(\frac{d\psi}{dt}\right)+\frac{d\phi}{dt}\right)+l\end{aligned} \quad (3)$$

We will introduce the notation $p_\psi=a,\ p_\varphi=b$ for the constant quantities p_ψ and p_φ (where a and b are constants). We then have from (3)

$$\frac{d\psi}{dt}=\frac{a-b\cos\theta}{\sin^2\theta},\ \frac{d\varphi}{dt}=\frac{bI_1-l}{I_3}-\frac{(a-b\cos\theta)\cos\theta}{\sin^2\theta}. \quad (4)$$

The momentum p_θ, corresponding to the positional coordinate θ, depends on the motion of the point P and given by the equation

$$p_\theta=I_1\left(\frac{d\theta}{dt}\right)-mz_G\sin\theta\frac{d\xi}{dt}. \quad (5)$$

From (4), (3) and (5) and using the Legendre transformation we obtain the following expression for the Hamilton function of the system. (unimportant terms which are functions of time or are constant are omitted)

$$\mathscr{H}=\frac{\left(p_\theta+mz_G\left(\frac{d\xi}{dt}\right)\sin\theta\right)^2}{2I_1}+\frac{(a-b\cos\theta)^2}{2I_1\sin^2\theta}+U(\cos\theta). \quad (6)$$

The Hamiltonian (6) corresponds to a system with one and half degree of freedom with generalized coordinate θ.

3 The Generalized Kapitza method of averaging

Let's discuss the one-dimensional motion of a classical particle in the time-independent potential $U(x)$ and under a $T-$periodical force

$$f(x,t)=\sum_{n=1}^{\infty}\left(A_n(x)\cos(n\omega t)+B_n(x)\sin(n\omega t)\right) \quad (7)$$

which varies in time with a high frequency ω (A_n,B_n are functions of the coordinates only). If we put T_U for a characteristic time of the motion which the particle would execute in the field U alone, then by a "high" frequency $\omega\equiv 2\pi/T$ we mean such that $\omega>>2\pi/T_U$. The coefficients A_n and B_n are given by

$$A_n(x)=\frac{2}{T}\int_0^T f(x,t)\cos(n\omega t)dt,$$
$$B_n(x)=\frac{2}{T}\int_0^T f(x,t)\sin(n\omega t)dt$$

The equation of the particle motion are

$$\frac{d^2x}{dt^2}=-\frac{dU}{dx}+f(x,t) \quad (8)$$

We present the movement as a slow path and at the same time execute fast but small oscillations of frequency ω about the path

$$x(t)=X(t)+\xi(t)$$

where $\xi(t)$ corresponds to these small oscillations. The mean value of the function $\xi(t)$ over its period T is zero, and the function $X(t)$ changes only slightly in that time. Denoting this average by a bar, we therefore have $\bar{x} = X(t)$. Under these considerations holds the following result.

Theorem 1(Kapitza). *If X_0 is a equilibrium of*

$$\frac{d^2X}{dt^2} = -\frac{dU_{eff}}{dX}$$

with

$$U_{eff}(X) = U(X) + \frac{1}{4\omega^2}\sum_{n=1}^{\infty}\frac{(A_n^2(X)+B_n^2(X))}{n^2}$$

then

$$x(t) = X_0 - \sum_{n=1}^{\infty}\frac{1}{\omega^2 n^2}(A_n^2(X_0)\cos(n\omega t) + B_n^2(X_0)\sin(n\omega t))$$

$$+h.o.t$$

is a quasi-periodic solution of (8). If X_0 is a minimum of U_{eff} the quasi-periodic solution is Lyapunov stable and if X_0 is a maximum of U_{eff} the quasi-periodic solution is unstable.

*Proof.*The Taylor's expansion in powers of ξ up to the first order term provides us

$$\frac{dU}{dx} = \frac{dU}{dX} + \xi\frac{d^2U}{dX^2}$$

Substituting the above expression in (8) we have

$$\frac{d^2X}{dt^2} + \frac{d\xi}{dt} = -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + f(X,t) + \xi\frac{df}{dX} \tag{9}$$

This equation involves both fast and slow terms, which must evidently be separately equal. For the fast term we can put simply

$$\frac{d^2\xi}{dt^2} = f(X,t) \tag{10}$$

and the slow term with small oscillations is

$$\frac{d^2X}{dt^2} = -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + \xi\frac{df}{dX}.$$

Integrating (10) with the function f given by (7) and regarding X as a constant, we get

$$\xi = -\frac{1}{\omega^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(A_n^2\cos(n\omega t) + B_n^2\sin(n\omega t)).$$

Next we average equation (9) with respect to the time interval $[0,T]$. Since $\overline{\xi} = 0$ and and $\overline{f} = 0$ we obtain

$$\frac{d^2X}{dt^2} = -\frac{dU}{dX} + \overline{\xi\frac{df}{dX}}$$

and

$$\frac{df}{dX} = \sum_{n=1}^{\infty}\left(\frac{dA_n}{dX}\cos(n\omega t) + \frac{dB_n}{dX}\sin(n\omega t)\right).$$

Then we apply the time averaging

$$\overline{\xi\frac{df}{dX}} = -\frac{1}{\omega^2}\sum_{k,j=1}^{\infty}\left(\frac{A_k\frac{dA_j}{dX}}{k^2}\overline{\cos(k\omega t)\cos(j\omega t)} + \frac{B_k\frac{dA_j}{dX}}{k^2}\overline{\sin(k\omega t)\cos(j\omega t)} + \frac{A_k\frac{dB_j}{dX}}{k^2}\overline{\cos(k\omega t)\sin(j\omega t)} + \frac{B_k\frac{dB_j}{dX}}{k^2}\overline{\sin(k\omega t)\sin(j\omega t)}\right).$$

Since

$$\overline{\sin(k\omega t)\cos(j\omega t)} = \overline{\cos(k\omega t)\sin(j\omega t)} = 0$$

if $k \neq j$ and

$$\overline{\cos(k\omega t)\cos(j\omega t)} = \overline{\sin(k\omega t)\sin(j\omega t)} = \frac{1}{2}$$

if $k = j$ we obtain

$$\overline{\xi\frac{df}{dX}} = -\frac{1}{4\omega^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{dA_n^2}{dX} + \frac{dB_n^2}{dX}\right).$$

Thus

$$m\frac{d^2X}{dt^2} = -\frac{dU}{dX} - \frac{1}{4\omega^2}\frac{d}{dX}\left(\sum_{n=1}^{\infty}\frac{A_n^2+B_n^2}{n^2}\right).$$

The previous equation can be written as

$$m\frac{d^2X}{dt^2} = -\frac{dU_{eff}}{dX},$$

where the effective potential energy is defined as

$$U_{eff}(X) = U(X) + \frac{1}{4\omega^2}\sum_{n=1}^{\infty}\frac{A_n^2(X)+B_n^2(X)}{n^2}.$$

4 Main Results

Using the Generalized Kapitza averaging we obtain the following result.

Theorem 2.*We consider a generalized Lagrange–Poisson problem that the point of suspension P executes vertical motion about a certain fixed point O in accordance with the $T-$periodic law*

$$\xi(t) = -\frac{1}{\omega^2}\sum_{n=1}^{\infty}\left(\frac{A_n}{n^2}\cos(n\omega t) + \frac{B_n}{n^2}\sin(n\omega t)\right)$$

, then the effective potential energy of the system obtained by mean of the Generalized Kapitza averaging is

$$U_{eff}(\alpha) = \mathscr{U}(\alpha) + K\sin^2\alpha$$

with

$$\mathscr{U}(\alpha) = \frac{(a-b\cos\alpha)^2}{2I_1\sin^2\alpha} + U(\cos\alpha)$$

and

$$K = \left(\frac{mz_G}{2I_1\omega}\right)^2 \sum_{n=1}^{\infty} \frac{A_n^2+B_n^2}{n^2}.$$

*Proof.*By means of the generating function

$$G(\alpha, p_\theta) = -\alpha p_\theta + mz_G\left(\frac{d\xi}{dt}\right)\cos\alpha$$

we obtain the following canonical transformation $(\theta, p_\theta) \to (\alpha, p)$ with

$$\begin{aligned} \theta &= -\frac{\partial G}{\partial p_\theta} = \alpha \\ p &= -\frac{\partial G}{\partial \alpha} = p_\theta + mz_G\left(\frac{d\xi}{dt}\right)\sin\alpha \end{aligned}.$$

The Hamiltonian (6) in the variables (α, p) are given by

$$\mathscr{K} = \mathscr{H} + \frac{\partial G}{\partial t} = \frac{p^2}{2I_1} + \frac{(a-b\cos\alpha)^2}{2I_1\sin^2\alpha} + mz_G\left(\frac{d^2\xi}{dt^2}\right)\cos\alpha + U(\cos\alpha).$$

The Hamiltonian differential equations are

$$\begin{aligned} \frac{d\alpha}{dt} &= \frac{p}{I_1}, \\ \frac{dp}{dt} &= -\frac{d}{d\alpha}\left(\frac{(a-b\cos\alpha)^2}{2I_1\sin^2\alpha} + mz_G\left(\frac{d^2\xi}{dt^2}\right)\cos\alpha + U(\cos\alpha)\right). \end{aligned}$$

The second order equation in the variable α is

$$\frac{d^2\alpha}{dt^2} = -\frac{d\mathscr{U}}{d\alpha} + f(\alpha, t)$$

with

$$\mathscr{U}(\alpha) = \frac{1}{I_1}\left(\frac{(a-b\cos\alpha)^2}{2I_1\sin^2\alpha} + U(\cos\alpha)\right)$$

and

$$f(\alpha,t) = \frac{mz_G}{I_1}\left(\frac{d^2\xi}{dt^2}\right)\sin\alpha = \frac{mz_G}{I_1}\sum_{n=1}^{\infty}\left(A_n\sin\alpha\cos(n\omega t) + B_n\sin\alpha\sin(n\omega t)\right).$$

Using Theorem 1, the effective potential of the problem is

$$U_{eff}(\alpha) = \mathscr{U}(\alpha) + K\sin^2\alpha$$

with

$$K = \left(\frac{mz_G}{2I_1\omega}\right)^2 \sum_{n=1}^{\infty} \frac{A_n^2+B_n^2}{n^2}. \quad (11)$$

Corollary 1.*We consider a sleeping generalized Lagrange–Poisson problem on the unstable upright position. Then*

$$\xi(t) = -\frac{1}{\omega^2}\sum_{n=1}^{\infty}\left(\frac{A_n}{n^2}\cos(n\omega t) + \frac{B_n}{n^2}\sin(n\omega t)\right) \quad (12)$$

with

$$\left(\frac{mz_G}{2I_1\omega}\right)^2 \sum_{n=1}^{\infty} \frac{A_n^2+B_n^2}{n^2} > \frac{4I_1 \left.\frac{dU}{d\alpha}\right|_{\alpha=0} - a^2}{8I_1^2}$$

has the property of stabilizing the unstable upright position.

*Proof.*A sleeping generalized Lagrange–Poisson problem on the unstable upright position verifies the two relations $a = b$ and

$$a^2 < 4I_1 \left.\frac{dU}{d\alpha}\right|_{\alpha=0}$$

, see ([24]). On the other hand, if the suspension point performs high-frequency vertical periodic oscillations of equation (12), then the effective potential is

$$U_{eff}(\alpha) = \frac{1}{I_1}\left(\frac{a^2}{2I_1}\frac{1-\cos\alpha}{1+\cos\alpha} + U(\cos\alpha)\right) + K\sin^2\alpha$$

with K given by (11). The function $U_{eff}(\alpha)$ has a local extremum in $\alpha = 0$ for

$$\left.\frac{d^2U_{eff}}{d\alpha^2}\right|_{\alpha=0} = 2K + \frac{\frac{a^2}{4I_1} - \left.\frac{dU}{d\alpha}\right|_{\alpha=0}}{I_1} \neq 0.$$

If

$$K > \frac{4I_1 \left.\frac{dU}{d\alpha}\right|_{\alpha=0} - a^2}{8I_1^2} \quad (13)$$

the point $\alpha = 0$ is a minimum of U_{eff} and the sleeping upright position is Lyapunov stable.

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J. A. Vera is Associate Professor of Applied Mathematics at Centro Universitario de la Defensa in Murcia. He received the PhD degree in "Applied Mathematics" at Universidad Politcnica de Cartagena (Spain). He is referee of many international journals in the frame of pure and applied mathematics, physics and engeenering. His main research interests are: dynamical systems with emphasis in hamiltonian dynamics, celestial mechanics and dynamical astronomy.