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On the Calculation of Price Sensitivities with a Jump-Diffusion Structure

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Abstract: An integral part of successful risk management in modern financial markets is the accurate calculation of the price sensitivities of the underlying asset. There are a number of recent research papers that have focused on this important issue. A strand of literature has applied the finite difference method which is biased. Another strand of literature has made use of the Malliavin calculus within a jump diffusion framework. However, the existing papers have provided the price sensitivities by conditioning on some of the stochastic part of the complicated random process. The current paper provides price sensitivities in jump diffusion model without conditioning on any stochastic part in the model. These estimates are shown to be unbiased. Thus, the solution that is provided in this paper is expected to induce decision making under uncertainty more precise.

Keywords: Asset Pricing; Malliavin Calculus; Price sensitivity; Risk management; European options; Jump-diffusion models; Jump-times Poisson noise.

1 Introduction

The calculation of price sensitivities is a necessary input for successful financial risk management. It is widely agreed in the literature that the modeling of financial derivatives is more precise if the process of generating the future price of the underlying asset is modeled as a jump-diffusion process [1]. Several recent papers deal with this issue based on a jump-diffusion formula that generates the underlying asset price via a Brownian motion and a Poisson process jointly. However, the calculations of the price sensitivities are provided by conditioning on either the Brownian motion or the Poisson process. Thus, there is scope for further research on this important topic by providing unconditional calculations. The main contribution of the current paper is therefore to provide an approach for the calculation of the price sensitivities when the underlying asset price is generated by both a Brownian motion and a Poisson process simultaneously, i.e. a jump-diffusion model. These calculations are provided via the Malliavin calculus without conditioning on any random part in the jump-diffusion process. The crucial advantage of the Malliavin calculus is that it provides unbiased estimators compared to the commonly used finite difference approach that usually results in biased estimators. Consequently, the solution that is provided in this paper is expected to improve on the correctness of the calculations of the underlying price sensitivities for successful decision making under uncertainty. The suggested method can be used for the computation of the price sensitivities when the stochastic process describing the stock's price includes jumps. There are five price sensitivities of a trading position that are usually denoted as "Greeks" in the financial literature. The importance of a precise calculation of these Greeks is paramount pertinent to immunization of underlying risk. The change of the trading position with regard to the price of the underlying asset is called Delta. The change of the delta for a portfolio of options with regard to the price of the underlying asset is known as Gamma. Another source of risk in this context is signified as Vega, which represents the sensitivity of the trading position with regard to a change in the volatility of the underlying asset.

[1] One of the most applied models for option pricing is the Black and Scholes formula [5]. However, the Black and Scholes model suffers from the continuity of the Brownian motion and thus from the exclusion of jumps.

The change of the portfolio with regard to time under the ceteris paribus condition is known as Theta. Finally, the sensitivity of the trading position with regard to the interest rate is known as Rho in the literature. Each Greek measures a source of risk for the underlying trading position. Traders need to calculate their Greeks at the end of every trading day in order to take necessary action if the internal risk exceeds the prespecified levels in the underlying financial institution that the trader is linked to, in order to avoid dismissal. We utilize the Malliavin calculus to provide an accurate and operational solution for each price sensitivity. This approach is particularly useful since the price of the option is characterized by a stochastic structure that cannot be given in closed form. Therefore, the study of price sensitivities via this approach is very important in this context. Via the Malliavin calculus we can transform the differentiation into integration and thereby make the calculation of the unbiased price sensitivities operational. Most previous work on the price sensitivities makes use of the finite difference method, which can be a biased approach. However, the Malliavin method is unbiased and it is also less time consuming in terms of convergence. There has been some work done on this issue using the Malliavin method. The main contribution of this paper is to extend the Malliavin approach to calculating the price sensitivities when the price of the underlying asset follows a jump-diffusion process. To ensure the market is arbitrage free, one should find a probability equivalent to the historical one under which the discounted prices are martingale, based on the first part of fundamental theorem of the asset pricing.

The application of the Malliavin calculus to the computations of price sensitivities was introduced by [13] for markets with Brownian information. Their approach rests on the Malliavin derivative on the Wiener space and consists of two parts-namely the application of the chain rule and utilizing the fact that this derivative has an adjoint (Skorohod integral) which coincides with the Itô integral for adapted processes. Recently in [10], this method has been used for markets suffering from a financial crash. However, several papers employing this method have been developed for markets with jumps. For pure jump markets, [11] use the Poisson noise via the jump times, while [1] differentiate with respect to both the jump times and the amplitude of the jumps. For jump-diffusion models, [8] apply the Malliavin calculus with respect to the Brownian motion while conditioning on the Poisson component. [2] allow the Poisson noise to act on the amplitude of the jumps but not with regard to the timing of the jumps. However, in the previous works on price sensitivities there have not been any calculations that take into proper account both stochastic components simultaneously without conditioning on any part. More recently, for Lévy models, [18] acts also on the jumps amplitude to establish a formula for the calculation of the price sensitivities. In other recent papers, the robustness of the sensitivity is investigated. For instance, [3] study the robustness of the sensitivity in Lévy models using a new conditional density method. In [4] the authors extend the method developed in [8] to study the robustness of option prices in markets modeled by jump-diffusions. They approximate the small jumps by a continuous martingale with approximately scaled variance. [14] uses the same method to compute the Delta in a multidimensional jump-diffusion framework. In this paper, we account for the timing of the jumps in the computations. Naturally, the timing of the jump is at least as important if not more in financial terms as the size of the jump. For example, the timing of a financial crisis can be more important than the potential size of the crisis.

The After this introduction the remaining part of the paper is organized as follows: Section two presents the the model. In Section three we apply the Malliavin calculus to derive the formula for computing the Greeks. The last Section concludes the paper. The appendix is devoted to Brownian and Poisson Malliavin derivatives.

2 The model

In this paper we apply the Malliavin calculus to compute Greeks for options with payoff $f(S_T)$ for discontinuous models. We consider a European option with maturity T. We assume that the underlying asset price is driven by a jump diffusion model. The dynamic of $(S_t)_{t\in[0,T]}$ under a risk neutral measure Q is

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t [dW_t + dM_t],\ t \in [0,T],\quad S_0 = x > 0,$$

where $r = (r_t)_{t\in[0,T]}$ and $\sigma = (\sigma_t)_{t\in[0,T]}$ are deterministic functions denoting respectively the interest rate and the volatility and the two processes $W = (W_t)_{t\in[0,T]}$ and $M = (M_t := N_t - t)_{t\in[0,T]}$ are respectively a Q-Brownian motion and a Q-compensated Poisson process. This means

$$S_T = x\exp\left(\int_0^T \sigma_t dW_t + \int_0^T (r_t - \sigma_t - \frac{1}{2}\sigma_t^2)dt\right) \times \prod_{k=1}^{k=N_T}(1+\sigma_{T_k}),$$

where $(T_k)_{k\geq 1}$ denotes the jump times of $(N_t)_{t\in[0,T]}$. Consider a standard Poisson process $N = (N_t)_{t\in\mathbb{R}_+}$ with jump times $(T_i)_{i\in\mathbb{N}}$ and let H denote the Cameron-Martin space

$$H = \left\{ u = \int_0^{\cdot} \dot{u}_t dt : \dot{u} \in L^2(\mathbb{R}_+) \right\}$$

for $u \in H$ and a smooth functional $F_n = f(T_1,\ldots,T_n)$, $f \in \mathcal{C}_b^1(\mathbb{R}^n), n \geq 1$ of the Poisson process, we let

$$D_u^N F_n := -\sum_{k=1}^{k=n} u_{T_k} \partial_k f(T_1,\ldots,T_n).$$

Unfortunately, N_t does not belong to $\mathrm{Dom}(D^N)$, i.e. the domain of D^N. Therefore, any underlying asset price $(S_t)_{t\in\mathbb{R}_+}$ formulated as

$$dS_t = \mu_t S_t dt + \sigma_t S_{t^-}(dN_t - dt),\ \in \mathbb{R}_+,\quad S_0 = x > 0,$$

does not belong to $\mathrm{Dom}(D^N)$. Nevertheless, for $T \in \mathbb{R}_+$, $\int_0^T S_t dt \in \mathrm{Dom}(D^N)$ since it can be written as

$$\int_0^T S_t dt = \sum_{k\geq 0} \int_{T_k\wedge T}^{T_{k+1}\wedge T} x e^{\int_0^t (\mu_t - \sigma_t)dt} \prod_{i=0}^{i=k} (1+\sigma_{T_i})dt.$$

In [11] only options with payoff of the form $f(\int_0^T S_t dt)$ were considered and those with payoff $f(S_T)$ were excluded. However, this paper can deal with both forms of the payoffs.
To show the point consider a smooth functional $F = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n$, $m \geq 1$, where $F_n := f(T_1,\ldots,T_n)$ and $f \in \mathcal{C}_b^1(\mathbb{R}^n)$. Let

$$\tilde{D}_u^N F := \sum_{n=1}^{n=m} 1_{\{N_T=n\}} D_u^N F_n.$$

$\tilde{D}^N$ is a derivative and it has an adjoint (Skorohod integral) which coincides with the Itô integral for adapted processes (see Prop.4 and Prop.5 in the appendix). Moreover N_t belong to $\mathrm{Dom}(\tilde{D}^N)$. In the following section we show how the Greeks can be calculated.

3 Computations of Greeks

In this section we compute the Greeks using Malliavin calculus for European options with maturity T and payoff $f(S_T)$. Let $C = E[f(S_T^\zeta)]$ be the price of the option, where ζ is a parameter taking the values: $S_0 = x$, the volatility σ, or the interest rate r. The computations of Greeks by the Malliavin approach rest on the integration by parts formula given in the following proposition[2].

[2] See [13] for the Brownian case and [11] for the Poisson case.

Proposition 1 *Let I be an open interval of* $\mathbb{R}$, $(F^{\zeta})_{\zeta\in I}$ *and* $(G^{\zeta})_{\zeta\in I}$ *be two families of random functionals in* $\mathrm{Dom}(\tilde{D}^N)\bigcap \mathrm{Dom}(D^W)$, *continuously differentiable with respect to the parameter* $\zeta\in I$. *Let* $(u_t)_{t\in[0,T]}$ *be a process satisfying*

$$(\tilde{D}_u^N + D_u^W)F^{\zeta} \neq 0, \quad a.s. \text{ on } \{\partial_{\zeta}F^{\zeta}\neq 0\}, \quad \zeta\in I,$$

such that $uG^{\zeta}\partial_{\zeta}F^{\zeta}/(\tilde{D}_u^N + D_u^W)F^{\zeta}$ is continuous in ζ in $\mathrm{Dom}(\delta^N)\bigcap \mathrm{Dom}(\delta^W)$ and $\int_0^T \dot{u}_t dt = 0$. We have

$$\frac{\partial}{\partial\zeta}E\left[G^{\zeta} f\left(F^{\zeta}\right)\right] = E\left[f\left(F^{\zeta}\right)\left(\frac{G^{\zeta}\partial_{\zeta}F^{\zeta}}{(\tilde{D}_u^N + D_u^W)F^{\zeta}}\delta^N(u) - \tilde{D}_u^N\left(\frac{G^{\zeta}\partial_{\zeta}F^{\zeta}}{(\tilde{D}_u^N + D_u^W)F^{\zeta}}\right)\right)\right]$$
$$+E\left[f\left(F^{\zeta}\right)\left(\frac{G^{\zeta}\partial_{\zeta}F^{\zeta}}{(\tilde{D}_u^N + D_u^W)F^{\zeta}}\delta^W(u) - D_u^W\left(\frac{G^{\zeta}\partial_{\zeta}F^{\zeta}}{(\tilde{D}_u^N + D_u^W)F^{\zeta}}\right)\right)\right] + E\left[f\left(F^{\zeta}\right)\partial_{\zeta}G^{\zeta}\right],$$

for any function f such that $f\left(F^{\zeta}\right)\in L^2(\Omega)$, $\zeta\in I$.

Proof. For function $f\in \mathcal{C}_b^{\infty}(\mathbb{R})$, we have

$$\frac{\partial}{\partial\zeta}E\left[G^{\zeta} f\left(F^{\zeta}\right)\right] = E\left[G^{\zeta}\partial_{\zeta} f\left(F^{\zeta}\right)\right] + E\left[f\left(F^{\zeta}\right)\partial_{\zeta}G^{\zeta}\right]$$
$$= E\left[G^{\zeta}\partial_{\zeta}F^{\zeta}\frac{(\tilde{D}_u^N + D_u^W)f\left(F^{\zeta}\right)}{(\tilde{D}_u^N + D_u^W)F^{\zeta}}\right] + E\left[f\left(F^{\zeta}\right)\partial_{\zeta}G^{\zeta}\right]$$

Then we conclude using Propositions 4 and 6 from the appendix A. The extension to $f\left(F^{\zeta}\right)\in L^2(\Omega)$ with $\zeta\in I$, can be obtained from the same argument as in p. 400 of [13] for the Brownian case and in [11] p. 167 for the Poisson case, using the bound

$$\left|\frac{\partial}{\partial\zeta}E\left[G^{\zeta} f_n(F^{\zeta})\right] - E\left[f\left(F^{\zeta}\right)\left(V^{\zeta}(\delta^N(u)+\delta^W(u)) - (\tilde{D}_u^N + D_u^W)V^{\zeta} + \partial_{\zeta}G^{\zeta}\right)\right]\right|$$
$$\leq \left\| f(F^{\zeta}) - f_n(F^{\zeta})\right\|_{L^2(\Omega)}\left\| V^{\zeta}(\delta^N(u)+\delta^W(u)) - (\tilde{D}_u^N + D_u^W)V^{\zeta} + \partial_{\zeta}G^{\zeta}\right\|_{L^2(\Omega)},$$

and an approximating sequence $(f_n)_{n\in\mathbb{N}}$ of smooth functions, where $V^{\zeta} := G^{\zeta}\partial_{\zeta}F^{\zeta}/(\tilde{D}_u^N + D_u^W)F^{\zeta}$.

It is worth mentioning here that the Malliavin method is unbiased. Indeed, the Malliavin method provides a new representation of the sensitivities if $\Upsilon := \frac{\partial}{\partial\zeta}E\left[G^{\zeta} f\left(F^{\zeta}\right)\right]$ is one of the price sensitivities, and Υ_M is its value using Malliavin method then we have by Proposition. 1 $\Upsilon = \Upsilon_M$ and thus $E[\Upsilon_M] = E[\Upsilon]$. In the other hand, the Finite Differences method gives an approximation of the sensitivities and we have $E[\Upsilon_F] \neq E[\Upsilon]$.

3.1 Delta, Rho, Vega

Consider an option with payoff $f\left(F^{\zeta}\right)$. The Greeks $\text{Delta} := \frac{\partial C}{\partial x}$, $\text{Rho} = \frac{\partial C}{\partial r}$ and $\text{Vega} = \frac{\partial C}{\partial \sigma}$ can be computed based on Proposition. 6 (presented in the appendix). That is

$$\frac{\partial}{\partial\zeta}E\left[f\left(F^{\zeta}\right)\right] = E\left[f\left(F^{\zeta}\right)\left(L^{\zeta}(\delta^N(u)+\delta^W(u)) - (\tilde{D}_u^N + D_u^W)L^{\zeta}\right)\right], \qquad (1)$$

where we let $G^{\zeta}=1$ and $L^{\zeta}:=\dfrac{\partial_{\zeta}F^{\zeta}}{(\tilde{D}_u^N+D_u^W)F^{\zeta}}$. As an example we compute the delta[3] of an European option using (1) with $\zeta=x$, $f(F^{\zeta})=f(S_T)$, and $\partial_{\zeta}F^{\zeta}=\partial_x S_T=\dfrac{1}{x}S_T$. We have

$$\text{Delta}=\partial_x E\left[e^{-\int_t^T r_s ds}f(S_T)\right]=e^{-\int_t^T r_s ds}E\left[f(S_T)\left(L^x(\delta^N+\delta^W)(u)-(\tilde{D}_u^N+D_u^W)L^x\right)\right],$$

where

$$L^x=\frac{1}{x}\frac{S_T}{(\tilde{D}_u^N+D_u^W)S_T}=\frac{1}{x}\frac{1}{\int_0^T u_t\sigma_t dt-\int_0^T \frac{u_t\sigma_t'}{1+\sigma_t}dN_t}.$$

And

$$D_u^W L^x=0$$

$$\tilde{D}_u^N L^x=(L^x)^2\tilde{D}_u^N\left(\int_0^T\frac{u_t\sigma_t'}{1+\sigma_t}dN_t\right)=-(L^x)^2\left(\int_0^T u_t\partial_t\frac{u_t\sigma_t'}{1+\sigma_t}dN_t\right)$$

$$=-(L^x)^2\int_0^T\frac{u_t}{1+\sigma_t}\left((\sigma_t'u_t'+u_t\sigma_t'')-\frac{u_t(\sigma_t')^2}{1+\sigma_t}\right)dN_t,$$

here we supposed that $\sigma'\neq 0$.
We can use

$$\delta^N(u)=\int_0^T\dot{u}_t dN_t=\int_0^T\dot{u}_t(dN_t-dt)=\sum_{k\geq 0}\dot{u}_{T_k},$$

$$\delta^W(v)=\int_0^T v_t dW_t=\sum_{j\geq 1}v_{t_{j-1}}(W_{tj}-W_{t_{j-1}}),$$

for u predictable in H such that $\int_0^T\dot{u}_t dt=0$ and v adapted.

3.2 Gamma

For computing the $\text{Gamma}=\dfrac{\partial^2 C}{\partial x^2}$, we let $H^x:=L^x(\delta^N(u)+\delta^W(u))-(\tilde{D}_u^N+D_u^W)L^x$. By using (1) and Proposition. 1 the following formula is obtained:

$$\text{Gamma}=e^{-\int_t^T r_s ds}\frac{\partial^2}{\partial x^2}E\left[f(F^x)\right]=e^{-\int_t^T r_s ds}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}E\left[f(F^x)\right]\right)=e^{-\int_t^T r_s ds}\frac{\partial}{\partial x}E\left[f(F^x)H^x\right]$$

$$=e^{-\int_t^T r_s ds}\left\{E\left[f(F^x)\left(\frac{H^x\partial_x F^x}{(\tilde{D}_u^N+D_u^W)F^x}\delta^N(u)-\tilde{D}_u^N\left(\frac{H^x\partial_x F^x}{(\tilde{D}_u^N+D_u^W)F^x}\right)\right)\right]\right.$$

$$\left.+E\left[f(F^x)\left(\frac{H^x\partial_x F^x}{(\tilde{D}_u^N+D_u^W)F^x}\delta^W(u)-D_u^W\left(\frac{H^x\partial_x F^x}{(\tilde{D}_u^N+D_u^W)F^x}\right)\right)\right]+E\left[f(F^x)\partial_x H^x\right]\right\}.$$

This formula can be used to calculate the change of Delta with regard to the underlying initial price more accurately.

[3] We can use the same techniques for Rho and Vega.

5 Conclusions

Making use of options is a common practice in financial markets by investors and other financial agents for neutralizing or reducing the price risk of the underlying asset. Thus, option pricing is an integral part of modern financial risk management. The accurate calculation of price sensitivities is a vital input in financial risk management models. Previous literature suggests that modeling financial derivatives based on a jump-diffusion process for generating the future price of the underlying asset is more precise. Several recent papers attempt to tackle this problem based on a jump-diffusion model that consists of a Brownian motion component and a Poisson process component jointly. Nevertheless, the price sensitivities are calculated by conditioning on one of the stochastic parts, either the Brownian motion or the Poisson process. Using Malliavin calculus, this paper provides the calculation of the price sensitivities in a situation in which the underlying asset price is generated by both a Brownian motion and a Poisson process simultaneously, i.e. a jump-diffusion model, without any conditioning on any random part in the jump-diffusion process. It is also shown that the Malliavin calculus provides unbiased estimators unlike the commonly used finite difference approach that are usually utilized in the existing literature.

Appendix

We give a brief presentation of the Malliavin derivative on the Wiener space and its adjoint. The new version of the Poisson gradient introduced in [21] is also presented. The Poissonian operator is a derivative and it admits an adjoint which coincides with the Poissonian Itô integral for adapted processes. For more details about the Malliavin calculus we refer[4] to [17] and [15] on the Wiener space and to [6], [7], [9], [12], [16], [19] and [21] on the Poisson space.

1. Malliavin derivative on the Wiener space

We denote by P the set of random variables $F:\Omega\to\mathrm{R}$, such that F has the representation

$$F(\omega)=f\left(\int_0^T f_1(t)dW_t,\dots,\int_0^T f_n(t)dW_t\right),$$

where $f(x_1,\dots,x_n)=\sum_\alpha a_\alpha x^\alpha$ is a polynomial in n variables $x_1,\dots,x_n$ and deterministic functions $f_i\in L^2([0,T])$. Let $\|.\|_{1,2}$ be the norm

$$\|F\|_{1,2}:=\|F\|_{L^2(\Omega)}+\|D^W_. F\|_{L^2([0,T]\times\Omega)},\quad F\in \mathrm{Dom}\,(D^W).$$

We have $\mathrm{P}\subset \mathrm{Dom}\,(D^W)$ and the following Proposition holds:

Proposition 2 *Given* $F=f\left(\int_0^T f_1(t)dW_t,\dots,\int_0^T f_n(t)dW_t\right)\in\mathrm{P}$. *We have*

$$D^W_t F=\sum_{k=1}^{k=n}\frac{\partial f}{\partial x_k}\left(\int_0^T f_1(t)dW_t,\dots,\int_0^T f_n(t)dW_t\right)f_k(t).$$

From now on, for any stochastic process u and for $F\in\mathrm{Dom}\,(D^W)$ such that $u_.D^W_.F\in L^2([0,T])$ we let

$$D^W_u F:=\langle D^W F,u\rangle_{L^2([0,T])}:=\int_0^T u_t D^W_t F dt.$$

Skorohod integral

Let δ^W be the Skorohod integral on the Wiener space. The next proposition is well known, it says that δ^W is the adjoint of D^W and is an extension of the Itô integral (see for example [17]).

Proposition 3 *a) Let* $u\in\mathrm{Dom}\,(\delta^W)$ *and* $F\in\mathrm{Dom}\,(D^W)$, *we have*

[4] The list is not exhaustive.

$$E[F\delta^W(u)] = E[D_u^W F], \quad \text{For every } F \in \mathrm{Dom}(D^W).$$

b) Consider a $L^2(\Omega\times[0,T])$-adapted stochastic process $u=(u_t)_{t\in[0,T]}$. We have

$$\delta^W(u) = \int_0^T u_t dW_t.$$

c) Let $F\in \mathrm{Dom}(D^W)$ and $u\in \mathrm{Dom}(\delta^W)$ such that $uF\in \mathrm{Dom}(\delta^W)$ thus

$$\delta^W(uF) = F\delta^W(u) - D_u^W F.$$

2. Poisson derivative

Let S denote the set of smooth functionals

$$F = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n, \quad where \ \ F_n = f_n(T_1,\cdots,T_n) \in \mathrm{Dom}(D^N), \quad m\in \mathbb{N}^* = \{1,2,\ldots\},$$

and for $1\le n\le m$, $f_n \in \mathcal{C}_b^1(\mathbb{R}^n)$.

Definition 1 *Given an element* u *of the Cameron-Martin space* H *and* $F\in S$ *as in the above, we define the gradient*[5]

$$\tilde{D}_u^N F := \sum_{n=1}^{n=m} 1_{\{N_T=n\}} D_u^N F_n = \sum_{n=1}^{n=m} 1_{\{N_T=n\}}\left(-\sum_{k=1}^{k=n} u_{T_k}\partial_k f_n(T_1,\cdots,T_n)\right). \qquad (2)$$

The next proposition shows that the gradient $\tilde{D}^N$ is a derivative.

Proposition 4 *Consider* $F=\sum_{n=1}^{n=m}1_{\{N_T=n\}}F_n$ *and* $G=\sum_{n=1}^{n=m}1_{\{N_T=n\}}G_n$ *two smooth functionals in* S, *where* $F_n = f_n(T_1,\cdots,T_n)\in \mathrm{Dom}(D^N)$ *and* $G_n = G_n(T_1,\cdots,T_n)\in \mathrm{Dom}(D^N)$. *We have*

$$\tilde{D}_u^N(FG) = F\tilde{D}_u^N G + G\tilde{D}_u^N F.$$

Proof. We have

$$FG = \Big(\sum_{n=1}^{n=m}1_{\{N_T=n\}}F_n\Big)\Big(\sum_{l=1}^{l=m}1_{\{N_T=l\}}G_l\Big) = \sum_{n=1}^{n=m}1_{\{N_T=n\}}F_nG_n.$$

Thanks to the chain rule of the gradient D^N, we have

$$\tilde{D}_u^N(FG) = \sum_{n=1}^{n=m}1_{\{N_T=n\}}F_n D_u^N G_n + \sum_{n=1}^{n=m}1_{\{N_T=n\}}G_n D_u^N F_n$$
$$= F\tilde{D}_u^N G + G\tilde{D}_u^N F.$$

This ends the proof.

Remark 1 *Let* $\mathrm{Dom}(\tilde{D}^N)$ *be the domain of* $\tilde{D}^N$.

1. $\mathrm{Dom}(D^N)\subset \mathrm{Dom}(\tilde{D}^N)$. In fact any $F\in \mathrm{Dom}(D^N)$ can be written as $F=\sum_{n=1}^{n=\infty}1_{\{N_T=n\}}F$. Here $m=\infty$ and we have $\tilde{D}_u^N F = \sum_{n=1}^{n=\infty}1_{\{N_T=n\}}D_u^N F = D_u^N F$.

[5] See [21], section 7.3.

2. $\mathrm{Dom}(\tilde{D}^N)$ contains N_T and $\tilde{D}_u^N N_T = 0$, since $N_T = \sum_{n\geq 0} 1_{\{N_T=n\}} n.$

5.2.1 Adjoint

The following proposition gives the adjoint gradient for D^N, it is well-known, cf. e.g. [7], [19], [20].

Proposition 5 *Consider* $F \in \mathrm{Dom}(D^N)$ *and* $u \in H$ *, we have*

a) The gradient D^N is closable and admits an adjoint δ^N such that

$$E[D_u^N F] = E[F\delta^N(u)].$$

b) For $u \in \mathrm{Dom}(\delta^N)$ such that $uF \in \mathrm{Dom}(\delta^N)$ we have

$$\delta^N(uF) = F\delta^N(u) - D_u^N F.$$

c) Moreover δ^N coincides with the compensated Poisson stochastic integral on the adapted processes in $L^2(\Omega; H)$:

$$\delta^N(u) = \int_0^T \dot{u}_t(dN_t - dt).$$

To be able to use the Malliavin method for the computations of Greeks we need to show first the existence of an adjoint for $\tilde{D}^N$ satisfying the properties of δ^N listed in Proposition 5. The relationship between $\tilde{D}^N$ and D^N will be very helpful. In fact, we have δ^N is the adjoint of $\tilde{D}^N$ as it is shown in the following proposition[6].

Proposition 6 *With previous notations:*

A) $\tilde{D}^N$ is closable and admits δ^N as adjoint. Moreover, if $F = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n$ in S with $F_n = f_n(T_1, \cdots, T_n) \in \mathrm{Dom}(D^N)$ and $u \in H$ such that $\int_0^T \dot{u}_t dt = 0$ then

$$E[\tilde{D}_u^N F] = E[F\delta^N(u)].$$

b) For $F, G \in \mathrm{Dom}(\tilde{D}^N)$ and $u \in \mathrm{Dom}(\delta^N)$ with $\int_0^T \dot{u}_t dt = 0$:

$$E\left[G\tilde{D}_u^N F\right] = E\left[F(G\delta^N(u) - \tilde{D}_u^N G)\right]$$

Proof of the Proposition. 6

First, we need the following lemma.

Lemma 1 *Consider* $u \in H$ *such that* $\int_0^T \dot{u}_t dt = 0$ *and a smooth functional* $f(T_1, \cdots, T_n) \in \mathrm{Dom}(D^N)$*, we have*

$$E\left[D_u^N f(T_1, \cdots, T_n) \mid N_T = n\right] = E\left[f(T_1, \cdots, T_n)\delta^N(u) \mid N_T = n\right].$$

Proof of the Lemma 1.

Proof. Let $u \in H$ such that $\int_0^T \dot{u}_t dt = 0$. We follow [20], Lemma 1. We consider the simplex $\Delta_n = \{(t_1, \ldots, t_n) \in [0,T]^n : 0 \leq t_1 < \cdots < t_n\}$. We have for $f \in L^2(\Delta_n, dt_1, \ldots, dt_n)$,

$$E[f(T_1, \cdots, T_n) \mid N_T = n] = \frac{n!}{T^n} \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \cdots, t_n) dt_1 \ldots dt_n.$$

And

$$E[D_u^N f(T_1, \cdots, T_n) \mid N_T = n] = -\sum_{k=1}^{k=n} I_k,$$

[6] The proof of this proposition can be found in [21]: Section 7.3. However another proof is provided here with the condition $\int_0^T \dot{u}_t dt = 0$.

where

$$I_k := \frac{n!}{T^n}\int_0^T\int_0^{t_n}\ldots\int_0^{t_2}u_{t_k}\partial_k f(t_1,\cdots,t_n)dt_1\ldots dt_n.$$

We have by integration by parts

$$\int_0^{t_2}u_{t_1}\partial_1 f(t_1,\cdots,t_n)dt_1 = -\int_0^{t_2}\dot{u}_{t_1} f(t_1,\cdots,t_n)dt_1 + u_{t_2} f(t_2,t_2,\cdots,t_n).$$

Thus,

$$I_1 = A_1 + B_2,$$

where

$$A_1 := -\frac{n!}{T^n}\int_0^T\int_0^{t_n}\cdots\int_0^{t_2}\dot{u}_{t_1} f(t_1,\cdots,t_n)dt_1\cdots dt_n$$

$$B_2 := \frac{n!}{T^n}\int_0^T\int_0^{t_n}\cdots\int_0^{t_3}u_{t_2} f(t_2,t_2,\cdots,t_n)dt_2\cdots dt_n.$$

We have

$$\begin{aligned}
&\int_0^{t_3}u_{t_2}\int_0^{t_2}\partial_2 f(t_1,\cdots,t_n)dt_1dt_2\\
&= \int_0^{t_3}u_{t_2}\partial_2\int_0^{t_2} f(t_1,\cdots,t_n)dt_1dt_2 - \int_0^{t_3}u_{t_2} f(t_2,t_2,\cdots,t_n)dt_2\\
&= -\int_0^{t_3}\dot{u}_{t_2}\int_0^{t_2} f(t_1,t_2,\cdots,t_n)dt_1dt_2 + u_{t_3}\int_0^{t_3} f(t_1,t_3,t_3,t_4,\cdots,t_n)dt_1\\
&- \int_0^{t_3}u_{t_2} f(t_2,t_2,\cdots,t_n)dt_2.
\end{aligned}$$

Thus,

$$I_2 = A_2 - B_2 + B_3,$$

where

$$A_2 := -\frac{n!}{T^n}\int_0^T\int_0^{t_n}\cdots\int_0^{t_3}\dot{u}_{t_2}\int_0^{t_2} f(t_1,t_2,\cdots,t_n)dt_1\cdots dt_n$$

$$B_3 := \frac{n!}{T^n}\int_0^T\int_0^{t_n}\cdots\int_0^{t_4}u_{t_3}\int_0^{t_3} f(t_1,t_3,t_3,t_4,\cdots,t_n)dt_1dt_3\cdots dt_n.$$

By using the same argument of the above, for any $k\in\{3,\ldots,n-1\}$, we have

$$\begin{aligned}
&\int_0^{t_{k+1}}u_{t_k}\left(\int_0^{t_k}\cdots\int_0^{t_2}\partial_k f(t_1,\cdots,t_n)dt_1\cdots dt_{k-1}\right)dt_k\\
&= \int_0^{t_{k+1}}u_{t_k}\partial_k\left(\int_0^{t_k}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\cdots dt_{k-1}\right)dt_k\\
&- \int_0^{t_{k+1}}u_{t_k}\int_0^{t_k}\int_0^{t_{k-2}}\cdots\int_0^{t_2} f(t_1,\cdots,t_{k-2},t_k,t_k,\cdots,t_n)dt_1\cdots dt_{k-2}dt_k\\
&= -\int_0^{t_{k+1}}\dot{u}_{t_k}\int_0^{t_k}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\cdots dt_k\\
&+ \int_0^{t_{k+2}}u_{t_{k+1}}\int_0^{t_{k+1}}\int_0^{t_{k-1}}\cdot\int_0^{t_2} f(t_1,\cdot,t_{k-1},t_{k+1},t_{k+1},\cdot,t_n)dt_1\cdot dt_{k-1}dt_{k+1}\cdot dt_n\\
&- \int_0^{t_{k+1}}u_{t_k}\int_0^{t_k}\int_0^{t_{k-2}}\cdots\int_0^{t_2} f(t_1,\cdots,t_{k-2},t_k,t_k,\cdots,t_n)dt_1\cdots dt_{k-2}dt_k.
\end{aligned}$$

Thus for $k\in\{3,\cdots,n-1\}$, we have

$$I_k = A_k - B_k + B_{k+1},$$

where

$$A_k := -\frac{n!}{T^n}\int_0^T\int_0^{t_n}\cdots\int_0^{t_{k+1}}\dot{u}_{t_k}\int_0^{t_k}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\cdots dt_n,$$

$$B_k := \frac{n!}{T^n}\int_0^T\int_0^{t_n}\cdot\int_0^{t_{k+1}}u_{t_k}\int_0^{t_k}\int_0^{t_{k-2}}\cdot\int_0^{t_2} f(t_1,\cdot,t_{k-2},t_k,t_k,\cdot,t_n)dt_1\cdot d\hat{t}_{k-1}\cdot dt_n,$$

$d\hat{t}_k$ denotes the absence of dt_k.

Let

$$A_n := -\frac{n!}{T^n}\int_0^T\dot{u}_{t_n}\int_0^{t_n}\cdots\int_0^{t_k}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\cdots dt_n,$$

$$B_n := \frac{n!}{T^n}\int_0^T u_{t_n}\int_0^{t_n}\int_0^{t_{n-2}}\cdots\int_0^{t_2} f(t_1,\cdots,t_{n-2},t_n,t_n)dt_1\cdots d\hat{t}_{n-1}dt_n,$$

we have

$$I_n = \frac{n!}{T^n}\int_0^T u_{t_n}\int_0^{t_n}\cdots\int_0^{t_2}\partial_n f(t_1,\cdots,t_n)dt_1\cdots dt_n$$

$$= \frac{n!}{T^n}\int_0^T u_{t_n}\partial_n\int_0^{t_n}\int_0^{t_{n-1}}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\cdots dt_n - B_n$$

$$= A_n + \frac{n!}{T^n}u_T\int_0^T\int_0^{t_{n-1}}\cdots\int_0^{t_2} f(t_1,\cdots,t_{n-1},T)dt_1\cdots dt_{n-1} - B_n$$

$$= A_n - B_n,$$

since $\int_0^T \dot{u}_t dt = 0.$

Thus

$$\sum_{k=1}^{k=n} I_k = (A_1+B_2)+(A_2-B_2+B_3)+\sum_{k=3}^{k=n-1}(A_k-B_k+B_{k+1})+A_n-B_n$$

$$= \sum_{k=1}^{k=n} A_k.$$

Then,

$$E[D_u^N f(T_1,\cdots,T_n)\,|\,N_T=n] = -\sum_{k=1}^{k=n} I_k = -\sum_{k=1}^{k=n} A_k$$

$$= \sum_{k=1}^{k=n}\frac{n!}{T^n}\int_0^T\int_0^{t_n}\ldots\int_0^{t_2}\dot{u}_{t_k} f(t_1,\cdots,t_n)dt_1\ldots dt_n$$

$$= E\left[f(T_1,\cdots,T_n)\left(\sum_{k=1}^{k=n}\dot{u}_{T_k}\right)\,|\,N_T=n\right].$$

Now to show that

$$E\left[f(T_1,\cdots,T_n)\left(\sum_{k=1}^{k=n}\dot{u}_{T_k}\right)\,|\,N_T=n\right] = E\left[f(T_1,\cdots,T_n)\delta^N(u)\,|\,N_T=n\right],$$

it is sufficient to prove

$$E\left[f(T_1,\cdots,T_n)\left(\sum_{k>n}\dot{u}_{T_k} - \int_{T_n}^{\infty}\dot{u}_t dt\right)\,|\,N_T=n\right] = 0,$$

since $\int_0^T \dot{u}_t dt = 0$. Recall that for $k > n$ we have

$$E\left[f(T_1,\cdots,T_n,\cdots,T_k)\,|\,N_T=n\right] =$$

$$\frac{n!}{T^n}e^{-T}\int_0^\infty e^{-t_k}\int_0^{t_k}\dots\int_0^{t_{n+1}}\int_0^{t_n}\dots\int_0^{t_2} f(t_1,\cdots,t_n,\cdots,t_k)dt_1\dots dt_k.$$

Therefore for $k > n$

$$E[F\dot{u}_{T_k} \mid N_T = n] = \frac{n!}{T^n}e^{-T}\int_0^\infty \dot{u}_{t_k} e^{-t_k}\int_0^{t_k}\dots\int_0^{t_n}\dots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\dots dt_k$$

$$= \frac{n!}{T^n}e^{-T}\int_0^\infty u_{t_k} e^{-t_k}\int_0^{t_k}\dots\int_0^{t_n}\dots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\dots dt_k$$

$$-\frac{n!}{T^n}e^{-T}\int_0^\infty u_{t_{k-1}} e^{-t_{k-1}}\int_0^{t_{k-1}}\dots\int_0^{t_n}\dots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\dots dt_{k-1}$$

$$= E[F(u_{T_k} - u_{T_{k-1}}) \mid N_T = n]$$

$$= E\left[F\int_{T_{k-1}}^{T_k} \dot{u}_t dt \mid N_T = n\right].$$

Then $\quad E[D_u^N f(T_1,\cdots,T_n) \mid N_T = n] = E\left[f(T_1,\cdots,T_n)\left(\sum_{k=1}^{\infty}\dot{u}_{T_k} - \int_{T_n}^{\infty}\dot{u}_t dt\right) \mid N_T = n\right],$

this ends the proof.

Now we can give the proof of Prop. 6.

Proof. a) We have using Lemma. 1 for any $u \in H$ such that $\int_0^T \dot{u}_t dt = 0,$

$$E[1_{\{N_T=n\}} D_u^N F_n] = \sum_{i=1}^{\infty} E[1_{\{N_T=n\}} D_u^N F_n \mid N_T = i]P(N_T = i)$$

$$= 1_{\{N_T=n\}} E[D_u^N F_n \mid N_T = n]P(N_T = n)$$

$$= E\left[1_{\{N_T=n\}} F_n \delta^N(u) \mid N_T = n\right]P(N_T = n)$$

$$= E\left[1_{\{N_T=n\}} F_n \delta^N(u)\right].$$

Thus,

$$E[\tilde{D}_u^N F] = E[\sum_{n=1}^{n=m} 1_{\{N_T=n\}} D_u^N F_n]$$

$$= \sum_{n=1}^{n=m} E\left[1_{\{N_T=n\}} F_n \delta^N(u)\right] = E[F\delta^N(u)].$$

b) Using the chain rule of $\tilde{D}^N$ and a) we obtain

$$E\left[G\tilde{D}_u^N F\right] = E\left[\tilde{D}_u^N(FG) - F\tilde{D}_u^N G\right] = E\left[F(G\delta^N(u) - \tilde{D}_u^N G)\right].$$

The proof is completed.

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