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Characterizations of ordered semigroups by the properties of their bipolar fuzzy ideals

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Abstract: In this paper we introduce bipolar fuzzy subsemigroup, bipolar fuzzy left (right) ideals and bipolar fuzzy bi-ideal in ordered semigroups. We characterize different classes of ordered semigroups by the properties of their bipolar fuzzy ideals and bipolar fuzzy bi-ideals.

Keywords: Ordered semigroups, Regular, intra-regular, and right weakly regular ordered semigroups, bipolar fuzzy ideal and bipolar fuzzy bi-ideal.

1 Introduction

The theory of fuzzy sets, proposed by Zadeh [1] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill defined to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, control engineering, expert systems, management science, operations research, pattern recognition, robotics, and others. There are several kinds of fuzzy set extensions, for example intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0,1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1,0)$ indicate that elements somewhat satisfy the implicit counter-property (see [2,3]). Lee [2] introduced the notion of bipolar-valued fuzzy sets. She introduced the concept of bipolar fuzzy subalgebras/ideals of a BCK/BCI-algebra, and investigated several properties. Jun et al. [4] applied the notion of bipolar-valued fuzzy set to finite state machines, and they introduced the notion of bipolar fuzzy finite state machines. Jun and Park [5] introduced the notion of bipolar fuzzy filter and bipolar fuzzy closed quasi filter in BCH-algebras, and investigated several properties. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar to each other. However, they are different from each other [6,2].

Kehayopulu and Tsingelis in [7] initiated the study of fuzzy ordered semigroups. They introduced the concept of fuzzy bi-ideal in ordered semigroups and characterized different classes (left and right simple, completely regular and strongly regular) of ordered semigroups in terms of fuzzy bi-ideals in [8]. In [9], they introduced the concept of fuzzy quasi-ideal in ordered semigroups and studied regular ordered semigroups in terms of fuzzy quasi-ideals.

In this paper we characterize different classes of ordered semigroups by the properties of their bipolar fuzzy ideals and bipolar fuzzy bi-ideals.

An ordered groupoid is a system $(S,\cdot,\leq)$ satisfying the following:

(i) $(S,\cdot)$ is a groupoid.
(ii) $(S,\leq)$ is a poset.
(iii) $(\forall a,b,x\in S)(a\leq b \implies ax\leq bx$ and $xa\leq xb)$.

If $(S,\cdot)$ is a semigroup and it satisfied (ii) and (iii), then $(S,\cdot,\leq)$ is called an ordered semigroup.

Let S be an ordered semigroup. For $A\subseteq S$, we denote $(A]=\{t\in S \mid t\leq h$ for some $h\in A\}$.

For $A,B\subseteq S$, we denote $AB=\{ab \mid a\in A$ and $b\in B\}$. For subsets A, B of S, we have $A\subseteq (A]$, $((A]]=(A]$. If

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$A \subseteq B$, then $(A] \subseteq (B]$, $(A](B] \subseteq (AB]$ [9]. A non-empty subset A of an ordered semigroup S is called a subsemigroup of S if $ab \in A$ for all $a,b \in A$. Let $(S,\cdot,\leq)$ be an ordered semigroup, $\emptyset \neq A \subseteq S$ is called a right (resp. left) ideal of S if (1) $AS \subseteq A$ (resp. $SA \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and a right ideal of S then A is called a two-sided ideal or simply an ideal of S. A subsemigroup B of an ordered semigroup S is called a bi-ideal of S if (1) $BSB \subseteq B$ and (2) If $a \in B$ and $S \ni b \leq a$, then $b \in B$ [8].

We denote by $R(a)$ (resp. $L(a)$ and $B(a)$) the right (resp. left and bi-) ideal of S generated by a $(a \in S)$. We have $R(a) = (a \cup aS]$, $L(a) = (a \cup Sa]$ and $B(a) = (a \cup a^2 \cup aSa]$. The ideal I of S generated by a is denoted by $I(a)$ and $I(a) = (a \cup aS \cup Sa \cup SaS]$ [10]. An ordered semigroup $(S,\cdot\leq)$ is called regular if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$, and S is called an intra-regular if for every $a \in S$ there exist $x,y \in S$ such that $a \leq xa^2y$ [11,12].

A bipolar fuzzy set μ in an ordered semigroup S is an object having the form

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$$

where $\mu^P : S \longrightarrow [0,1]$ and $\mu^N : S \longrightarrow [-1,0]$ are mappings. The positive membership degree $\mu^P(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$ and the negative membership degree $\mu^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$. If $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$. If $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$, but somewhat satisfies the counter-property of $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$. It is possible for an element x to be $\mu^P(x) \neq 0$ and $\mu^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain. For the sake of simplicity, we shall write $\mu = \left(\mu^P,\ \mu^N\right)$ for the bipolar fuzzy set $\mu = \left\{\left(x,\ \mu^P(x),\ \mu^N(x)\right) : x \in X\right\}$.

For $x \in S$, define

$$A_x = \{(y,z) \in S \times S \mid x \leq yz\}.$$

For two bipolar fuzzy subsets $\mu = \left(\mu^P,\ \mu^N\right)$ and $\lambda = (\lambda^P,\ \lambda^N)$ of S, the product of two bipolar fuzzy subsets is denoted by $\mu \circ \lambda$ and is defined as:

$$\left(\mu^P \circ \lambda^P\right)(x) = \begin{cases} \bigvee_{(s,t)\in A_x} \left\{\mu^P(s) \wedge \lambda^P(t)\right\} & \text{if } A_x \neq \emptyset \\ 0 & \text{if } A_x = \emptyset \end{cases}$$

and

$$\left(\mu^N \circ \lambda^N\right)(x) = \begin{cases} \bigwedge_{(s,t)\in A_x} \left\{\mu^N(s) \vee \lambda^N(t)\right\} & \text{if } A_x \neq \emptyset \\ 0 & \text{if } A_x = \emptyset \end{cases}$$

We denote by $BF(S)$ the set of all bipolar fuzzy subsets of S. One can easily see that the multiplication "$\circ$" on $BF(S)$ is well-defined and associative. We define order relation on $BF(S)$ as follows:

$\mu \preceq \lambda$ means that $\mu^P(x) \leq \lambda^P(x)$ and $\mu^N(x) \geq \lambda^N(x)$ for all $x \in S$. Clearly $(BF(S),\circ,\preceq)$ is an ordered semigroup.

Now we define some set theoretical operations on bipolar fuzzy sets. Let $\mu = \left(\mu^P,\ \mu^N\right)$ and $\lambda = (\lambda^P,\ \lambda^N)$ be two bipolar fuzzy subsets of S.

The infimum of two bipolar fuzzy sets $\mu = \left(\mu^P,\ \mu^N\right)$ and $\lambda = (\lambda^P, \lambda^N)$ will mean the following $\left(\mu^P \wedge \lambda^P\right)(x) = \mu^P(x) \wedge \lambda^P(x)$, $\left(\mu^N \vee \lambda^N\right)(x) = \mu^N(x) \vee \lambda^N(x)$. The supremum of two bipolar fuzzy sets μ and λ will mean the following $\left(\mu^P \vee \lambda^P\right)(x) = \mu^P(x) \vee \lambda^P(x)$, $\left(\mu^N \wedge \lambda^N\right)(x) = \mu^N(x) \wedge \lambda^N(x)$. More generally, If $\left\{\mu_i = \left(\mu_i^P,\ \mu_i^N\right) : i \in I\right\}$ is a family of bipolar fuzzy sets of X, then, $\bigwedge_{i\in I}\mu_i$ and $\bigvee_{i\in I}\mu_i$ are define by $\left(\bigwedge_{i\in I}\mu_i^P\right)(x) = \bigwedge_{i\in I}\left(\mu_i^P(x)\right)$, $\left(\bigvee_{i\in I}\mu_i^N\right)(x) = \bigvee_{i\in I}\left(\mu_i^N(x)\right)$ and $\left(\bigvee_{i\in I}\mu_i^P\right)(x) = \bigvee_{i\in I}\left(\mu_i^P(x)\right)$, $\left(\bigwedge_{i\in I}\mu_i^N\right)(x) = \bigwedge_{i\in I}\left(\mu_i^N(x)\right)$, respectively.

A bipolar fuzzy set $\mu = \left(\mu^P,\ \mu^N\right)$ of an ordered semigroup S is called a bipolar fuzzy right (resp. left) ideal of S if (1) $\mu^P(xy) \geq \mu^P(x)$ (resp. $\mu^P(xy) \geq \mu^P(y)$) and $\mu^N(xy) \leq \mu^N(x)$ (resp. $\mu^N(xy) \leq \mu^N(y)$) for every $x,y \in S$. (2) If $x \leq y$, then $\mu^P(x) \geq \mu^P(y)$ and $\mu^N(x) \leq \mu^N(y)$. A bipolar fuzzy set $\mu = \left(\mu^P,\ \mu^N\right)$ of an ordered semigroup S is called a bipolar fuzzy ideal of S if it is both a bipolar fuzzy right and a bipolar fuzzy left ideal of S. One can easily see that a bipolar fuzzy set $\mu = \left(\mu^P,\ \mu^N\right)$ is a bipolar fuzzy ideal of S if and only if the following assertions hold:

(1) $\mu^P(xy) \geq \max\left\{\mu^P(x),\ \mu^P(y)\right\}$ and $\mu^N(xy) \leq \min\left\{\mu^N(x),\ \mu^N(y)\right\}$for every $x,y \in S$.

(2) If $x \leq y$, then $\mu^P(x) \geq \mu^P(y)$ and $\mu^N(x) \leq \mu^N(y)$.

Let $(S,\cdot,\leq)$ be an ordered semigroup. A bipolar fuzzy set $\mu = \left(\mu^P,\ \mu^N\right)$ of S is called a bipolar fuzzy subsemigroup of S if:

$$\mu^P(xy) \geq \min\left\{\mu^P(x), \mu^P(y)\right\} \text{ and}$$
$$\mu^N(xy) \leq \max\left\{\mu^N(x), \mu^N(y)\right\}$$

for every $x,y \in S$.

A bipolar fuzzy subsemigroup μ of S is called a bipolar fuzzy bi-ideal of S if it satisfies the following

(1)$\mu^P(xyz) \geq \min\left\{\mu^P(x), \mu^P(z)\right\}$ and $\mu^N(xyz) \leq \max\left\{\mu^N(x), \mu^N(z)\right\}$ for every $x,y,z \in S$.

(2) For $x,y, \in S$, if $x \leq y$, then $\mu^P(x) \geq \mu^P(y)$ and $\mu^N(x) \leq \mu^N(y)$.

Obviously every one sided bipolar fuzzy ideal of an ordered semigroup is a bipolar fuzzy bi-ideal but the converse is not true.

For an ordered semigroup S, the bipolar fuzzy subsets $0=\left(0^P,\ 0^N\right)$ and $1=\left(1^P,\ 1^N\right)$ of S having the form "$0 = \left\{\left(x,\ 0^P(x),\ 0^N(x)\right) \mid x\in S\right\}$" and "$1=\left\{\left(x,\ 1^P(x),\ 1^N(x)\right) \mid x\in S\right\}$", where $0^P(x), 0^N(x)$, $1^P(x)$ and $1^N(x)$ are defined as follows:

$0^P(x)=0=0^N(x),\ 1^P(x)=1$ and $1^N(x)=-1\ \forall x\in S.$

Clearly, the bipolar fuzzy subset 0 (resp. 1) of S is the least (resp. the greatest) element of the ordered set $(BF(S), \preceq)$ (that is, $0\preceq\mu$ and $\mu\preceq 1$ for every $\mu\in BF(S)$). The bipolar fuzzy set 0 is the zero element of $(BF(S),\ \circ,\ \preceq)$ (that is, $\mu^P\circ 0^P=0^P\circ\mu^P=0=\mu^N\circ 0^N=0^N\circ\mu^N$ and $0^P\leq\mu^P$, $1^N\leq\mu^N$and $1^P\geq\mu^P$, $0^N\geq\mu^N$ for every $\mu\in BF(S)$).

If A is a non-empty subset of an ordered groupoid $(S,\cdot,\leq)$, then the bipolar fuzzy characteristic function of A denoted and defined by

$\chi_A=\left\{\left(x,\ \chi_A^P(x),\ \chi_A^N(x)\right) : x\in S\right\}$,

where χ_A^P and χ_A^N are defined by

$$\chi_A^P(x)=\begin{cases}1 & \text{if } x\in A\\ 0 & \text{otherwise}\end{cases}$$

and

$$\chi_A^N(x)=\begin{cases}-1 & \text{if } x\in A\\ 0 & \text{otherwise}\end{cases}$$

2 Intra-regular Ordered Semigroups

In this section we characterize intra-regular ordered semigroups in terms of bipolar fuzzy ideals.

2.1 Lemma [9]

Let $(S,\cdot,\leq)$ be an ordered semigroup. The following are equivalent:

(1) S is intra-regular.

(2) $R\cap L\subseteq(LR]$ for every right ideal R and every left ideal L of S.

(3) $R(a)\cap L(a)\subseteq(L(a)R(a)]$ for every $a\in S$.

2.2 Lemma

Let A be a non-empty subset of an ordered semigroup $(S,\cdot,\leq)$. Then A is a right (resp. left, two-sided, bi-) ideal of S if and only if the characteristic function $\chi_A=\left\langle\chi_A^P,\chi_A^N\right\rangle$ is a bipolar fuzzy right (resp. left, two-sided, bi-) ideal of S.

*Proof.*Let A be a right ideal of S. For any $x,y\in S$, if $x\in A$ then we have $xy\in A$ so

$$\chi_A^P(x)=\chi_A^P(xy)=1 \text{ and } \chi_A^N(x)=\chi_A^N(xy)=-1.$$

If $x\notin A$ then we have

$$\chi_A^P(xy)\geq 0=\chi_A^P(x) \text{ and } \chi_A^N(xy)\leq 0=\chi_A^N(x).$$

Combining the above two cases we have $\chi_A^P(xy)\geq\chi_A^P(x)$ and $\chi_A^N(xy)\leq\chi_A^N(x)$. Let $x\leq y$, if $y\in A$, then $x\in A$ and so $\chi_A^P(x)=\chi_A^P(y)=1$ and $\chi_A^N(y)=\chi_A^N(x)=-1$. If $y\notin A$, then $\chi_A^P(x)\geq 0=\chi_A^P(y)$ and $\chi_A^N(x)\leq 0=\chi_A^N(y)$. By the above two cases we have $\chi_A^P(x)\geq\chi_A^P(y)$ and $\chi_A^N(x)\leq\chi_A^N(y)$. Hence $\chi_A=\left\langle\chi_A^P,\chi_A^N\right\rangle$ is a bipolar fuzzy right ideal of S.

Conversely, assume that $\chi_A=\left\langle\chi_A^P,\chi_A^N\right\rangle$ is a bipolar fuzzy right ideal of S. Then for any $x\in A$ and $y\in S$, we have $\chi_A^P(xy)\geq\chi_A^P(x)=1$ and $\chi_A^N(xy)\leq\chi_A^N(x)=-1$. This implies that $\chi_A^P(xy)=1$ and $\chi_A^N(xy)=-1$. Thus $xy\in A\implies AS\subseteq A$. Now if $y\in A$ and $S\ni x\leq y$, then $\chi_A^P(x)\geq\chi_A^P(y)=1$ and $\chi_A^N(y)=-1\geq\chi_A^N(x)$. This implies that $\chi_A^P(x)=1$ and $\chi_A^N(x)=-1$, that is $x\in A$. So A is a right ideal of S.

Similarly we can prove the other cases.

2.3 Proposition

Let S be an ordered semigroup, $A,B\subseteq S$ and $\{A_i \mid i\in I\}$ be a family of subsets of S. Then

(1) $A\subseteq B$ if and only if $\chi_A^P\leq\chi_B^P$ and $\chi_A^N\geq\chi_B^N$.

(2) $A=B$ if and only if $\chi_A^P=\chi_B^P$ and $\chi_A^N=\chi_B^N$.

(3) $\underset{i\in I}{\wedge}\chi_{A_i}^P=\chi_{\underset{i\in I}{\cap}A_i}^P$ and $\underset{i\in I}{\vee}\chi_{A_i}^N=\chi_{\underset{i\in I}{\cap}A_i}^N$.

(4) $\underset{i\in I}{\vee}\chi_{A_i}^P=\chi_{\underset{i\in I}{\cup}A_i}^P$ and $\underset{i\in I}{\wedge}\chi_{A_i}^N=\chi_{\underset{i\in I}{\cup}A_i}^N$.

*Proof.*Straightforward.

2.4 Proposition

Let $(S,\cdot,\leq)$ be an ordered groupoid and $A,B\subseteq S$. Then $\chi_A^P\circ\chi_B^P=\chi_{(AB]}^P$ and $\chi_A^N\circ\chi_B^N=\chi_{(AB]}^N$.

*Proof.*Let $x\in S$. If $x\in(AB]$, then $\chi_{(AB]}^P(x)=1$ and $\chi_{(AB]}^N(x)=-1$. Since $x\leq ab$ for some $a\in A$ and $b\in B$, we have $(a,b)\in A_x$, and $A_x\neq\emptyset$. Thus we have

$$\begin{aligned}\left(\chi_A^P\circ\chi_B^P\right)(x) &= \underset{(y,z)\in A_x}{\vee}\min\left\{\chi_A^P(y),\chi_B^P(z)\right\}\\ &\geq \min\left\{\chi_A^P(a),\chi_B^P(b)\right\}=1\end{aligned}$$

and
$$\begin{aligned}\left(\chi_A^N\circ\chi_B^N\right)(x) &= \underset{(y,z)\in A_x}{\wedge}\max\left\{\chi_A^N(y),\chi_B^N(z)\right\}\\ &\leq \max\left\{\chi_A^N(a),\chi_B^N(b)\right\}=-1\end{aligned}$$

Therefore $\left(\chi_A^P\circ\chi_B^P\right)(x)=1=\chi_{(AB]}^P(x)$ and $\left(\chi_A^N\circ\chi_B^N\right)(x)=-1=\chi_{(AB]}^N(x)$.

If $x \notin (AB]$, then $\chi^P_{(AB]}(x) = 0$ and $\chi^N_{(AB]}(x) = 0$. Let $(y,z) \in A_x$ be such that $x \leq yz$. Then $y \notin A$ or $z \notin B$. Thus

$$\begin{aligned}\left(\chi^P_A \circ \chi^P_B\right)(x) &= \bigvee_{(y,z)\in A_x} \min\left\{\chi^P_A(y), \chi^P_B(z)\right\} \\ &= 0 \\ \text{and } \left(\chi^N_A \circ \chi^N_B\right)(x) &= \bigwedge_{(y,z)\in A_x} \max\left\{\chi^N_A(y), \chi^N_B(z)\right\} \\ &= 0.\end{aligned}$$

Therefore in both the cases, we have $\chi^P_A \circ \chi^P_B = \chi^P_{(AB]}$ and $\chi^N_A \circ \chi^N_B = \chi^N_{(AB]}$.

2.5 Proposition

Let S be an ordered semigroup, $\mu = \left(\mu^P, \mu^N\right)$ a bipolar fuzzy right ideal and $\lambda = \left(\lambda^P, \lambda^N\right)$ a bipolar fuzzy left ideal of S. Then $\mu^P \circ \lambda^P \leq \mu^P \wedge \lambda^P$ and $\mu^N \circ \lambda^N \geq \mu^N \vee \lambda^N$.

*Proof.*Let $x \in S$. If $A_x = \emptyset$, then $\left(\mu^P \circ \lambda^P\right)(x) = 0 = \left(\mu^N \circ \lambda^N\right)(x)$. Thus $\left(\mu^P \wedge \lambda^P\right)(x) \geq 0 = \left(\mu^P \circ \lambda^P\right)(x)$ and $\left(\mu^N \vee \lambda^N\right)(x) \leq 0 = \left(\mu^N \circ \lambda^N\right)(x)$. If $A_x \neq \emptyset$, then we have

$$(\mu^P \circ \lambda^P)(x) = \bigvee_{(b,c)\in A_x} \min\left\{\mu^P(b),\ \lambda^P(c)\right\}$$

$$\text{and } (\mu^N \circ \lambda^N)(x) = \bigwedge_{(b,c)\in A_x} \max\left\{\mu^N(b),\ \lambda^N(c)\right\}.$$

Let $(b,c) \in A_x$. Then $x \leq bc$. Thus $\mu^P(x) \geq \mu^P(bc) \geq \mu^P(b)$ and $\mu^N(x) \leq \mu^N(bc) \leq \mu^N(b)$. Similarly $\lambda^P(x) \geq \lambda^P(bc) \geq \lambda^P(c)$ and $\lambda^N(x) \leq \lambda^N(bc) \leq \lambda^N(c)$. Thus

$$\begin{aligned}(\mu^P \circ \lambda^P)(x) &= \bigvee_{(b,c)\in A_x} \min\left\{\mu^P(b),\ \lambda^P(c)\right\} \\ &\leq \bigvee_{(b,c)\in A_x} \min\left\{\mu^P(x),\ \lambda^P(x)\right\} \\ &= \left(\mu^P \wedge \lambda^P\right)(x)\end{aligned}$$

and

$$\begin{aligned}(\mu^N \circ \lambda^N)(x) &= \bigwedge_{(b,c)\in A_x} \max\left\{\mu^N(b),\ \lambda^N(c)\right\} \\ &\geq \bigwedge_{(b,c)\in A_x} \max\left\{\mu^N(x),\ \lambda^N(x)\right\} \\ &= \left(\mu^N \vee \lambda^N\right)(x).\end{aligned}$$

This implies that $\mu^P \circ \lambda^P \leq \mu^P \wedge \lambda^P$ and $\mu^N \circ \lambda^N \geq \mu^N \vee \lambda^N$.

2.6 Theorem

An ordered semigroup S is intra-regular if and only if for every bipolar fuzzy right ideal $\mu = \left(\mu^P, \mu^N\right)$ and every bipolar fuzzy left ideal $\lambda = \left(\lambda^P, \lambda^N\right)$ of S, we have $\mu^P \wedge \lambda^P \leq \lambda^P \circ \mu^P$ and $\mu^N \vee \lambda^N \geq \lambda^N \circ \mu^N$.

*Proof.*Let S be an intra-regular ordered semigroup, $\mu = \left(\mu^P, \mu^N\right)$ a bipolar fuzzy right ideal, and $\lambda = \left(\lambda^P, \lambda^N\right)$ a bipolar fuzzy left ideal of S. Since S is intra-regular, so for each $x \in S$ there exist $a, b \in S$ such that $x \leq ax^2b = (ax)(xb)$. Thus $(ax, xb) \in A_x$. Since $A_x \neq \emptyset$, we have

$$\begin{aligned}\left(\lambda^P \circ \mu^P\right)(x) &= \bigvee_{(b,c)\in A_x} \min\left\{\lambda^P(b), \mu^P(c)\right\} \\ &\geq \min\left\{\lambda^P(ax), \mu^P(xb)\right\} \\ &\geq \min\left\{\lambda^P(x), \mu^P(x)\right\} \\ &= \left(\mu^P \wedge \lambda^P\right)(x)\end{aligned}$$

and

$$\begin{aligned}\left(\lambda^N \circ \mu^N\right)(x) &= \bigwedge_{(b,c)\in A_x} \max\left\{\lambda^N(b), \mu^N(c)\right\} \\ &\leq \max\left\{\lambda^N(ax), \mu^N(xb)\right\} \\ &\leq \max\left\{\lambda^N(x), \mu^N(x)\right\} \\ &= \left(\mu^N \vee \lambda^N\right)(x).\end{aligned}$$

Thus we have $\mu^P \wedge \lambda^P \leq \lambda^P \circ \mu^P$ and $\mu^N \vee \lambda^N \geq \lambda^N \circ \mu^N$.

Conversely, assume that $\mu^P \wedge \lambda^P \leq \lambda^P \circ \mu^P$ and $\mu^N \vee \lambda^N \geq \lambda^N \circ \mu^N$ for every bipolar fuzzy right ideal μ and every bipolar fuzzy left ideal λ of S. Let R be a right ideal and L be a left ideal of S. Then by Lemma 2.2, χ_R is a bipolar fuzzy right ideal and χ_L is a bipolar fuzzy left ideal of S. Thus by hypothesis and Propositions 2.3 and 2.4, we have

$$\begin{aligned}\chi^P_{R\cap L} &= \chi^P_R \wedge \chi^P_L \leq \chi^P_L \circ \chi^P_R = \chi^P_{(LR]} \\ \text{and } \chi^N_{R\cap L} &= \chi^N_R \vee \chi^N_L \geq \chi^N_L \circ \chi^N_R = \chi^N_{(LR]}.\end{aligned}$$

This implies that $R \cap L \subseteq LR$. Hence by Lemma 2.1, S is intra-regular.

3 Regular and Intra-regular Ordered Semigroups

In this section we characterize regular and intra-regular ordered semigroups in terms of bipolar fuzzy ideals.

3.1 Lemma [9]

Let S be an ordered semigroup. Then S is regular if and only if for every right ideal R and every left ideal L of S we have $R \cap L = (RL]$.

3.2 Lemma [13]

Let S be an ordered semigroup. Then the following are equivalent:

(1) S is both regular and intra-regular.

(2) $R\cap L\subseteq(RL]\cap(LR]$ for every right ideal R and every left ideal L of S.

(3) $R(a)\cap L(a)\subseteq(R(a)L(a)]\cap(L(a)R(a)]$ for every $a\in S$.

3.3 Theorem

An ordered semigroup S is regular if and only if for every bipolar fuzzy right ideal $\mu=(\mu^P,\mu^N)$ and every bipolar fuzzy left ideal $\lambda=(\lambda^P,\lambda^N)$ of S, we have $\mu^P\wedge\lambda^P=\mu^P\circ\lambda^P$ and $\mu^N\vee\lambda^N=\mu^N\circ\lambda^N$.

*Proof.*Let S be a regular ordred semigroup and $\mu=(\mu^P,\ \mu^N)$ be a bipolar fuzzy right ideal, $\lambda=(\lambda^P,\ \lambda^N)$ be a bipolar fuzzy left ideal of S. For $x\in S$ there exists $a\in S$ such that $x\le(xa)x$. Then $(xa,x)\in A_x$. Thus we have

$$\begin{aligned}(\mu^P\circ\lambda^P)(x)&=\bigvee_{(b,c)\in A_x}\min\left\{\mu^P(b),\ \lambda^P(c)\right\}\\&\ge\min\left\{\mu^P(xa),\ \lambda^P(x)\right\}\\&\ge\min\left\{\mu^P(x),\ \lambda^P(x)\right\}\\&=\left(\mu^P\wedge\lambda^P\right)(x)\\\text{and }(\mu^N\circ\lambda^N)(x)&=\bigwedge_{(b,c)\in A_x}\max\left\{\mu^N(b),\ \lambda^N(c)\right\}\\&\le\max\left\{\mu^N(xa),\ \lambda^N(x)\right\}\\&\le\max\left\{\mu^N(x),\ \lambda^N(x)\right\}\\&=\left(\mu^N\vee\lambda^N\right)(x).\end{aligned}$$

Thus $\mu^P\wedge\lambda^P\le\mu^P\circ\lambda^P$ and $\mu^N\vee\lambda^N\ge\mu^N\circ\lambda^N$. By Proposition 2.5, $\mu^P\circ\lambda^P\le\mu^P\wedge\lambda^P$ and $\mu^N\circ\lambda^N\ge\mu^N\vee\lambda^N$ are always true. Hence $\mu^P\circ\lambda^P=\mu^P\wedge\lambda^P$ and $\mu^N\circ\lambda^N=\mu^N\vee\lambda^N$.

Conversely, assume that $\mu^P\wedge\lambda^P=\mu^P\circ\lambda^P$ and $\mu^N\circ\lambda^N=\mu^N\vee\lambda^N$ for every bipolar fuzzy right ideal μ and every bipolar fuzzy left ideal λ of S. Let R be a right ideal and L be a left ideal of S. Then by Lemma 2.2, χ_R is a bipolar fuzzy right ideal and χ_L is a bipolar fuzzy left ideal of S. Thus by hypothesis and Propositions 2.3 and 2.4, we have

$$\begin{aligned}&\chi^P_{R\cap L}=\chi^P_R\wedge\chi^P_L=\chi^P_R\circ\chi^P_L=\chi^P_{(RL]}\\\text{and }&\chi^N_{R\cap L}=\chi^N_R\vee\chi^N_L=\chi^N_R\circ\chi^N_L=\chi^N_{(RL]}.\end{aligned}$$

This implies that $R\cap L=RL$. Hence by Lemma 3.1, S is regular.

3.4 Theorem

An ordered semigroup S is both regular and intra-regular if and only if for every bipolar fuzzy right ideal $\mu=(\mu^P,\mu^N)$ and every bipolar fuzzy left ideal $\lambda=(\lambda^P,\lambda^N)$ of S, we have $\mu^P\wedge\lambda^P\le(\mu^P\circ\lambda^P)\wedge(\lambda^P\circ\mu^P)$ and $\mu^N\vee\lambda^N\ge(\mu^N\circ\lambda^N)\vee(\lambda^N\circ\mu^N)$.

*Proof.*Follows from Theorems 2.6 and 3.3.

3.5 Lemma [13]

Let $(S,\cdot,\le)$ be an ordered semigroup. Then the following are equivalent:

(1) S is regular.

(2) $B\cap I\cap L\subseteq(BIL]$ for every bi-ideal B, every ideal I and every left ideal L of S.

(3) $B(a)\cap I(a)\cap L(a)\subseteq(B(a)I(a)L(a)]$ for every $a\in S$.

3.6 Theorem

An ordered semigroup S is regular if and only if for every bipolar fuzzy bi-ideal $\mu=(\mu^P,\mu^N)$, every bipolar fuzzy ideal $\lambda=(\lambda^P,\lambda^N)$ and every bipolar fuzzy left ideal $\varsigma=(\varsigma^P,\varsigma^N)$ of S, we have

$$\mu^P\wedge\lambda^P\wedge\varsigma^P\le\mu^P\circ\lambda^P\circ\varsigma^P\text{ and }\mu^N\vee\lambda^N\vee\varsigma^N\ge\mu^N\circ\lambda^N\circ\varsigma^N.$$

*Proof.*Let S be a regular ordered semigroup and $\mu=(\mu^P,\mu^N)$ a bipolar fuzzy bi-ideal, $\lambda=(\lambda^P,\lambda^N)$ a bipolar fuzzy ideal and $\varsigma=(\varsigma^P,\varsigma^N)$ a bipolar fuzzy left ideal of S. For $a\in S$, there exists $x\in S$ such that

$$a\le a(xa)\le(axa)(xa)\le(axa)(xaxa).$$

Then $(axa,xaxa)\in A_a$. Since $A_a\neq\emptyset$, we have

$$\begin{aligned}\left(\mu^P\circ\lambda^P\circ\varsigma^P\right)(a)&=\bigvee_{(y,z)\in A_a}\min\left\{\mu^P(y),\ \left(\lambda^P\circ\varsigma^P\right)(z)\right\}\\&\ge\min\left\{\mu^P(axa),\ \left(\lambda^P\circ\varsigma^P\right)(xaxa)\right\}\\&=\min\left\{\begin{array}{c}\mu^P(axa),\\\bigvee_{(s,t)\in Axaxa}\min\left\{\lambda^P(s),\ \left(\varsigma^P\right)(t)\right\}\end{array}\right\}\\&\ge\min\left\{\mu^P(axa),\ \min\left\{\lambda^P(xa),\ \left(\varsigma^P\right)(xa)\right\}\right\}\\&\ge\min\left\{\mu^P(a),\ \min\left\{\lambda^P(a),\ \left(\varsigma^P\right)(a)\right\}\right\}\\(\text{because, }\mu&=\left(\mu^P,\mu^N\right)\text{ is a bipolar fuzzy bi-ideal,}\\\lambda&=\left(\lambda^P,\lambda^N\right)\text{ is a bipolar fuzzy ideal and}\\\varsigma&=\left(\varsigma^P,\varsigma^N\right)\text{ is a bipolar fuzzy left ideal of }S)\\&=\left(\mu^P\wedge\lambda^P\wedge\varsigma^P\right)(a)\end{aligned}$$

and

$$\begin{aligned}
\left(\mu^N\circ\lambda^N\circ\varsigma^N\right)(a) &= \bigwedge_{(y,z)\in A_a}\max\left\{\mu^N(y),\ \left(\lambda^N\circ\varsigma^N\right)(z)\right\}\\
&\leq \max\left\{\mu^N(axa),\ \left(\lambda^N\circ\varsigma^N\right)(xaxa)\right\}\\
&= \max\left\{\begin{array}{c}\mu^N(axa),\\ \bigwedge_{(s,t)\in A_{xaxa}}\max\left\{\begin{array}{c}\lambda^N(s),\\ \left(\varsigma^N\right)(t)\end{array}\right\}\end{array}\right\}\\
&\leq \max\left\{\begin{array}{c}\mu^N(axa),\\ \max\left\{\lambda^N(xa),\ \left(\varsigma^N\right)(xa)\right\}\end{array}\right\}\\
&\leq \max\left\{\mu^N(a),\ \max\left\{\begin{array}{c}\lambda^N(a),\\ \left(\varsigma^N\right)(a)\end{array}\right\}\right\}
\end{aligned}$$

(because, $\mu = \left(\mu^P,\mu^N\right)$ is a bipolar fuzzy bi-ideal, $\lambda = \left(\lambda^P,\lambda^N\right)$ is a bipolar fuzzy ideal and $\varsigma = \left(\varsigma^P,\varsigma^N\right)$ is a bipolar fuzzy left ideal of S)

$$= \left(\mu^N\vee\lambda^N\vee\varsigma^N\right)(a).$$

Thus we have

$$\mu^P\wedge\lambda^P\wedge\varsigma^P \leq \mu^P\circ\lambda^P\circ\varsigma^P \text{ and}$$
$$\mu^N\vee\lambda^N\vee\varsigma^N \geq \mu^N\circ\lambda^N\circ\varsigma^N.$$

Conversely, assume that $\mu^P\wedge\lambda^P\wedge\varsigma^P \leq \mu^P\circ\lambda^P\circ\varsigma^P$ and $\mu^N\vee\lambda^N\vee\varsigma^N \geq \mu^N\circ\lambda^N\circ\varsigma^N$, for every bipolar fuzzy bi-ideal $\mu = \left(\mu^P,\mu^N\right)$, every bipolar fuzzy ideal $\lambda = \left(\lambda^P,\lambda^N\right)$ and every bipolar fuzzy left ideal $\varsigma = \left(\varsigma^P,\varsigma^N\right)$ of S.

Let B be a bi-ideal, I be an ideal and L be a left ideal of S. Then by Lemma 2.2, $\chi_B = \left(\chi_B^P,\chi_B^N\right)$ is a bipolar fuzzy bi-ideal, $\chi_I = \left(\chi_I^P,\chi_I^N\right)$ a bipolar fuzzy ideal and $\chi_L = \left(\chi_L^P,\chi_L^N\right)$ a bipolar fuzzy left ideal of S. Thus by hypothesis and Proposition 2.3 and 2.4, we have

$$\chi^P_{B\cap I\cap L} = \chi_B^P\wedge\chi_I^P\wedge\chi_L^P \leq \chi_B^P\circ\chi_I^P\circ\chi_L^P = \chi^P_{(BIL]}$$
$$\text{and } \chi^N_{B\cap I\cap L} = \chi_B^N\vee\chi_I^N\vee\chi_L^N \geq \chi_B^N\circ\chi_I^N\circ\chi_L^N = \chi^N_{(BIL]}.$$

This implies $B\cap I\cap L \subseteq (BIL]$. Hence by Lemma 3.5, S is regular.

3.7 Lemma [13]

Let S be an ordered semigroup. Then the following are equivalent:

(1) S is regular.

(2) $B\cap L \subseteq (BL]$, for every bi-ideal B and every left ideal L of S.

(3) $B(a)\cap L(a) \subseteq (B(a)L(a)]$ for every $a\in S$.

(4) $B\cap R \subseteq (RB]$ for every bi-ideal B and every right ideal R of S.

(5) $B(a)\cap R(a) \subseteq (R(a)B(a)]$ for every $a\in S$.

3.8 Theorem

An ordered semigroup S is regular if and only if for every bipolar fuzzy bi-ideal $\mu = \left(\mu^P,\mu^N\right)$ and every bipolar fuzzy left ideal $\lambda = \left(\lambda^P,\lambda^N\right)$ of S, we have $\mu^P\wedge\lambda^P \leq \mu^P\circ\lambda^P$ and $\mu^N\vee\lambda^N \geq \mu^N\circ\lambda^N$.

Proof. Let S be a regular ordered semigroup, $\mu = \left(\mu^P,\mu^N\right)$ a bipolar fuzzy bi-ideal and $\lambda = \left(\lambda^P,\lambda^N\right)$ a bipolar fuzzy left ideal of S. For $a\in S$, there exists $x\in S$ such that $a\leq a(xa)\leq(axa)(xa)$. Then $(axa,xa)\in A_a$. Since $A_a\neq\emptyset$, we have

$$\begin{aligned}
\left(\mu^P\circ\lambda^P\right)(a) &= \bigvee_{(y,z)\in A_a}\min\left\{\mu^P(y),\ \lambda^P(z)\right\}\\
&\geq \min\left\{\mu^P(axa),\ \lambda^P(xa)\right\}\\
&\geq \min\left\{\mu^P(a),\ \lambda^P(a)\right\}
\end{aligned}$$

(because, $\mu = \left(\mu^P,\mu^N\right)$ is bipolar fuzzy bi-ideal and $\lambda = \left(\lambda^P,\lambda^N\right)$ a bipolar fuzzy left ideal of S)

$$= \left(\mu^P\wedge\lambda^P\right)(a)$$

and

$$\begin{aligned}
\left(\mu^N\circ\lambda^N\right)(a) &= \bigwedge_{(y,z)\in A_a}\max\left\{\mu^N(y),\ \lambda^N(z)\right\}\\
&\leq \max\left\{\mu^N(axa),\ \lambda^N(xa)\right\}\\
&\leq \max\left\{\mu^N(a),\ \lambda^N(a)\right\}
\end{aligned}$$

(because, $\mu = \left(\mu^P,\mu^N\right)$ is bipolar fuzzy bi-ideal and $\lambda = \left(\lambda^P,\lambda^N\right)$ a bipolar fuzzy left ideal of S)

$$= \left(\mu^N\vee\lambda^N\right)(a).$$

Conversely, assume that $\mu^P\wedge\lambda^P\leq\mu^P\circ\lambda^P$ and $\mu^N\vee\lambda^N\geq\mu^N\circ\lambda^N$, for every bipolar fuzzy bi-ideal $\mu = \left(\mu^P,\mu^N\right)$ and bipolar fuzzy left ideal $\lambda = \left(\lambda^P,\lambda^N\right)$ of S. As every bipolar fuzzy right ideal of S is a bipolar fuzzy bi-ideal of S, so for every bipolar fuzzy right ideal $\mu = \left(\mu^P,\mu^N\right)$ and for every bipolar fuzzy left ideal $\lambda = \left(\lambda^P,\lambda^N\right)$ of S, we have $\mu^P\wedge\lambda^P\preceq\mu^P\circ\lambda^P$ and $\mu^N\vee\lambda^N\succeq\mu^N\circ\lambda^N$. Hence by Theorem 3.3, S is regular.

4 Weakly Regular Ordered Semigroups

In this section we characterize right weakly regular ordered semigroups in terms of bipolar fuzzy ideals and bipolar fuuzy bi-ideals.

4.1 Definition

An ordered semigroup S is called right weakly regular ordered semigroup if for every $a\in S$ there exist $x,y\in S$ such that $a\leq axay$.

4.2 Lemma [13]

The following are equivalent for an ordered semigroup S.

(1) S is right weakly regular.

(2) $R \cap I \subseteq (RI]$ for every right ideal R and two-sided ideal I of S.

(3) $R(a) \cap I(a) \subseteq (R(a)I(a)]$ for every $a \in S$.

4.3 Theorem

An ordered semigroup S is right weakly regular if and only if for every bipolar fuzzy right ideal $\mu = (\mu^P, \mu^N)$ and every bipolar fuzzy ideal $\lambda = (\lambda^P, \lambda^N)$ of S, we have $\mu^P \wedge \lambda^P \leq \mu^P \circ \lambda^P$ and $\mu^N \vee \lambda^N \geq \mu^N \circ \lambda^N$.

*Proof.*Let S be a right weakly regular ordered semigroup, $\mu = (\mu^P, \mu^N)$ a bipolar fuzzy right ideal and $\lambda = (\lambda^P, \lambda^N)$ a bipolar fuzzy ideal of S. Since S is right weakly regular, for $a \in S$ there exist $x, y \in S$ such that $a \leq (ax)(ay)$. Thus $(ax, ay) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}
(\mu^P \circ \lambda^P)(a) &= \bigvee_{(y,z)\in A_a} \min\{\mu^P(y), \lambda^P(z)\} \\
&\geq \min\{\mu^P(ax), \lambda^P(ay)\} \\
&\text{(because } \mu = (\mu^P, \mu^N) \text{ is a bipolar fuzzy right ideal and} \\
&\quad \lambda = (\lambda^P, \lambda^N) \text{ a bipolar fuzzy ideal of } S) \\
&\geq \min\{\mu^P(a), \lambda^P(a)\} = (\mu^P \wedge \lambda^P)(a)
\end{aligned}$$

and

$$\begin{aligned}
(\mu^N \circ \lambda^N)(a) &= \bigwedge_{(y,z)\in A_a} \max\{\mu^N(y), \lambda^N(z)\} \\
&\leq \max\{\mu^N(ax), \lambda^N(ay)\} \\
&\text{(because } \mu = (\mu^P, \mu^N) \text{ is a bipolar fuzzy right ideal and} \\
&\quad \lambda = (\lambda^P, \lambda^N) \text{ a bipolar fuzzy ideal of } S) \\
&\leq \max\{\mu^N(a), \lambda^N(a)\} = (\mu^N \vee \lambda^N)(a).
\end{aligned}$$

Conversely, assume that $\mu^P \wedge \lambda^P \leq \mu^P \circ \lambda^P$ and $\mu^N \vee \lambda^N \geq \mu^N \circ \lambda^N$ for every bipolar fuzzy right ideal $\mu = (\mu^P, \mu^N)$ and every bipolar fuzzy ideal $\lambda = (\lambda^P, \lambda^N)$ of S. Let R be a right ideal and I be an ideal of S. Then by Lemma 2.2, $\chi_R = (\chi_R^P, \chi_R^N)$ is a bipolar fuzzy right ideal and $\chi_I = (\chi_I^P, \chi_I^N)$ is a bipolar fuzzy ideal of S. Then by hypothesis and Propositions 2.3 and 2.4, we have

$$\chi_{R\cap I}^P = \chi_R^P \wedge \chi_I^P \leq \chi_R^P \circ \chi_I^P = \chi_{(RI]}^P$$

and $\chi_{R\cap I}^N = \chi_R^N \vee \chi_I^N \geq \chi_R^N \circ \chi_I^N = \chi_{(RI]}^N$

This implies that $R \cap I \subseteq (RI]$. Hence by Lemma 4.2, S is right weakly regular.

4.4 Lemma [13]

Let S be an ordered semigroup. Then the following are equivalent:

(1) S is right weakly regular.

(2) $B \cap I \cap R \subseteq (BIR]$ for every bi-ideal B, every right ideal R and every ideal I of S.

(3) $B(a) \cap I(a) \cap R(a) \subseteq (B(a)I(a)R(a)]$ for every $a \in S$.

4.5 Theorem

An ordered semigroup S is right weakly regular if and only if for every bipolar fuzzy bi- ideal $\mu = (\mu^P, \mu^N)$, every bipolar fuzzy ideal $\lambda = (\lambda^P, \lambda^N)$ and every bipolar fuzzy right ideal $\varsigma = (\varsigma^P, \varsigma^N)$ of S, we have

$\mu^P \wedge \lambda^P \wedge \varsigma^P \leq \mu^P \circ \lambda^P \circ \varsigma^P$ and
$\mu^N \vee \lambda^N \vee \varsigma^N \geq \mu^N \circ \lambda^N \circ \varsigma^N$.

*Proof.*Let S be a right weakly regular ordered semigroup and $\mu = (\mu^P, \mu^N)$ a bipolar fuzzy bi-ideal, $\lambda = (\lambda^P, \lambda^N)$ a bipolar fuzzy ideal and $\varsigma = (\varsigma^P, \varsigma^N)$ a bipolar fuzzy right ideal of S. For $a \in S$, there exist $x, y \in S$ such that

$$a \leq a(xay) \leq (axa)(xayy).$$

Then $(axa, xay^2) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}
(\mu^P \circ \lambda^P \circ \varsigma^P)(a) &= \bigvee_{(y,z)\in A_a} \min\{\mu^P(y),\ (\lambda^P \circ \varsigma^P)(z)\} \\
&\geq \min\{\mu^P(axa),\ (\lambda^P \circ \varsigma^P)(xay^2)\} \\
&= \min\left\{ \begin{array}{c} \mu^P(axa), \\ \bigvee_{(s,t)\in A_{xay^2}} \min\left\{ \begin{array}{c} \lambda^P(s), \\ (\varsigma^P)(t) \end{array} \right\} \end{array} \right\} \\
&\geq \min\left\{\mu^P(axa),\ \min\left\{ \begin{array}{c} \lambda^P(xax), \\ (\varsigma^P)(ay^3) \end{array} \right\}\right\} \\
&\text{(because, } xay^2 \leq x(axay)y^2 = xaxay^3) \\
&\geq \min\{\mu^P(a),\ \min\{\lambda^P(a),\ (\varsigma^P)(a)\}\} \\
&\text{(because, } \mu = (\mu^P, \mu^N) \text{ is a bipolar fuzzy bi-ideal,} \\
&\quad \lambda = (\lambda^P, \lambda^N) \text{ is a bipolar fuzzy ideal and} \\
&\quad \varsigma = (\varsigma^P, \varsigma^N) \text{ is a bipolar fuzzy right ideal of } S) \\
&= (\mu^P \wedge \lambda^P \wedge \varsigma^P)(a)
\end{aligned}$$

and

$$\begin{aligned}
\left(\mu^N\circ\lambda^N\circ\varsigma^N\right)(a) &= \bigwedge_{(y,z)\in A_a}\max\left\{\mu^N(y),\ \left(\lambda^N\circ\varsigma^N\right)(z)\right\}\\
&\leq \max\left\{\mu^N(axa),\ \left(\lambda^N\circ\varsigma^N\right)(xay^2)\right\}\\
&= \max\left\{\begin{array}{c}\mu^N(axa),\\ \bigwedge_{(s,t)\in A_{xay^2}}\max\left\{\begin{array}{c}\lambda^N(s),\\ \left(\varsigma^N\right)(t)\end{array}\right\}\end{array}\right\}\\
&\leq \max\left\{\begin{array}{c}\mu^N(axa),\\ \max\left\{\begin{array}{c}\lambda^N(xax),\\ \left(\varsigma^N\right)(ay^3)\end{array}\right\}\end{array}\right\}
\end{aligned}$$

(because, $xay^2\leq x(axay)y^2=xaxay^3$)

$$\begin{aligned}
&\leq \max\left\{\begin{array}{c}\mu^N(a),\\ \max\left\{\lambda^N(a),\ \left(\varsigma^N\right)(a)\right\}\end{array}\right\}
\end{aligned}$$

(because, $\mu=\left(\mu^P,\mu^N\right)$ is a bipolar fuzzy bi-ideal, $\lambda=\left(\lambda^P,\lambda^N\right)$ is a bipolar fuzzy ideal and $\varsigma=\left(\varsigma^P,\varsigma^N\right)$ is a bipolar fuzzy right ideal of S)

$$= \left(\mu^N\vee\lambda^N\vee\varsigma^N\right)(a).$$

Therefore

$\mu^P\wedge\lambda^P\wedge\varsigma^P\leq\mu^P\circ\lambda^P\circ\varsigma^P$ and
$\mu^N\vee\lambda^N\vee\varsigma^N\geq\mu^N\circ\lambda^N\circ\varsigma^N$.

Conversely, assume that $\mu^P\wedge\lambda^P\wedge\varsigma^P\leq\mu^P\circ\lambda^P\circ\varsigma^P$ and $\mu^N\vee\lambda^N\vee\varsigma^N\geq\mu^N\circ\lambda^N\circ\varsigma^N$ for every bipolar fuzzy bi-ideal $\mu=\left(\mu^P,\mu^N\right)$, every bipolar fuzzy ideal $\lambda=\left(\lambda^P,\lambda^N\right)$ and every bipolar fuzzy right ideal $\varsigma=\left(\varsigma^P,\varsigma^N\right)$ of S. Let B be a bi-ideal, I be an ideal and R be a right ideal of S. Then by Lemma 2.2, $\chi_B=\left(\chi_B^P,\chi_B^N\right)$ is a bipolar fuzzy bi-ideal, $\chi_I=\left(\chi_I^P,\chi_I^N\right)$ a bipolar fuzzy ideal and $\chi_R=\left(\chi_R^P,\chi_R^N\right)$ a bipolar fuzzy right ideal of S. Thus by hypothesis and Propositions 2.3 and 2.4, we have

$$\chi^P_{B\cap I\cap R}=\chi^P_B\wedge\chi^P_I\wedge\chi^P_R\leq\chi^P_B\circ\chi^P_I\circ\chi^P_R=\chi^P_{(BIR]}$$

and $\chi^N_{B\cap I\cap R}=\chi^N_B\vee\chi^N_I\vee\chi^N_R\geq\chi^N_B\circ\chi^N_I\circ\chi^N_R=\chi^N_{(BIR]}$

Thus $B\cap I\cap R\subseteq(BIR]$. Hence by Lemma 4.4, S is right weakly regular ordered semigroup.

4.6 Lemma[13]

Let S be an ordered semigroup. Then the following are equivalent:

(1) S is right weakly regular.

(2) $B\cap I\subseteq(BI]$, for every bi-ideal B and every ideal I of S.

(3) $B(a)\cap I(a)\subseteq(B(a)I(a)]$ for every $a\in S$.

4.7 Theorem

An ordered semigroup S is right weakly regular if and only if for every bipolar fuzzy bi-ideal $\mu=\left(\mu^P,\mu^N\right)$ and every bipolar fuzzy ideal $\lambda=\left(\lambda^P,\lambda^N\right)$ of S, we have $\mu^P\wedge\lambda^P\leq\mu^P\circ\lambda^P$ and $\mu^N\vee\lambda^N\geq\mu^N\circ\lambda^N$.

*Proof.*Let S be a right weakly regular ordered semigroup, $\mu=\left(\mu^P,\mu^N\right)$ a bipolar fuzzy bi-ideal and $\lambda=\left(\lambda^P,\lambda^N\right)$ a bipolar fuzzy ideal of S. For $a\in S$, there exist $x,y\in S$ such that $a\leq axay\leq axaxayy=(axa)\left(xay^2\right)$. Thus $\left(axa,xay^2\right)\in A_a$. Since $A_a\neq\emptyset$, we have

$$\begin{aligned}
\left(\mu^P\circ\lambda^P\right)(a) &= \bigvee_{(y,z)\in A_a}\min\left\{\mu^P(y),\ \lambda^P(z)\right\}\\
&\geq \min\left\{\mu^P(axa),\ \lambda^P(xay^2)\right\}\\
&\geq \min\left\{\mu^P(a),\ \lambda^P(a)\right\}
\end{aligned}$$

(because, $\mu=\left(\mu^P,\mu^N\right)$ is a bipolar fuzzy bi-ideal and $\lambda=\left(\lambda^P,\lambda^N\right)$ a bipolar fuzzy ideal of S)

$$= \left(\mu^P\wedge\lambda^P\right)(a)$$

and

$$\begin{aligned}
\left(\mu^N\circ\lambda^N\right)(a) &= \bigwedge_{(y,z)\in A_a}\max\left\{\mu^N(y),\ \lambda^N(z)\right\}\\
&\leq \max\left\{\mu^N(axa),\ \lambda^N(xay^2)\right\}\\
&\leq \max\left\{\mu^N(a),\ \lambda^N(a)\right\}
\end{aligned}$$

(because, $\mu=\left(\mu^P,\mu^N\right)$ is a bipolar fuzzy bi-ideal and $\lambda=\left(\lambda^P,\lambda^N\right)$ a bipolar fuzzy ideal of S)

$$= \left(\mu^N\vee\lambda^N\right)(a).$$

Conversely, assume that $\mu^P\wedge\lambda^P\leq\mu^P\circ\lambda^P$ and $\mu^N\vee\lambda^N\geq\mu^N\circ\lambda^N$, for every bipolar fuzzy bi-ideal $\mu=\left(\mu^P,\mu^N\right)$ and every bipolar fuzzy ideal $\lambda=\left(\lambda^P,\lambda^N\right)$ of S. As every bipolar fuzzy right ideal of S is a bipolar fuzzy bi-ideal of S, so by hypothesis $\mu^P\wedge\lambda^P\leq\mu^P\circ\lambda^P$ and $\mu^N\vee\lambda^N\geq\mu^N\circ\lambda^N$ for every bipolar fuzzy right ideal $\mu=\left(\mu^P,\mu^N\right)$ and for every bipolar fuzzy ideal $\lambda=\left(\lambda^P,\lambda^N\right)$ of S. Hence by Theorem 4.5, S is right weakly regular.

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