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Recurrence Relations for Single and Product Moments of Generalized Order Statistics for Rayleigh Distribution

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In this small note we have developed the recurrence relations for single and product moments of Generalized Order Statistics for the Rayleigh distribution. These recurrence relations can be used to develop the relationship for moments of ordinary order statistics, record statistics and other ordered random variable techniques.

Keywords: Rayleigh distribution, generalized order statistics, recurrence relations.

1 Introduction

The distribution theory for ordered random variables has been widely used in practice in many situations. Several special distribution theories are available to tackle with the problems where the random sample from some probability distribution can be arranged under certain conditions. Some of these special distribution theories include the ordinary order statistics, record statistics, k-th record statistics and others. A comprehensive review of these techniques can be found in David and Nagaraja (2003), Ahsanullah (1995) and Nevzorov (2001).

The Generalized Order Statistics (GOS) has been introduced by Kamps (1995) as a unified theory for ordered random variables. The probability distribution of r-th generalized ordered statistics developed by Kamps (1995) is given as

$$f_{r,n,m,k}(x) = \begin{cases} \frac{C_{r-1}}{(r-1)!}[1-F(x)^{\gamma_r - 1} f(x) g_m^{r-1}\{F(x)\}, & -\infty < x < \infty; \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

where

$$C_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad \gamma_j = k + (n-j)(m+1),$$

and

$$g_m(x) = \begin{cases} -\ln(1-x), & m = -1; \\ \{1-(1-x)\}^{m+1}/(m+1), & m \neq -1. \end{cases} \quad (1.2)$$

The distribution (1.1) can be used to obtain the distribution of GOS for any specific distribution $F(x)$. Kamps (1995) has shown that (1.1) can serve as basis for many other probability distributions for ordered random variables under different choices of the constants involved.

The development of GOS extended the horizons for distributional theories of ordered random variables. Many people jumped into this area of probability distributions with its emergence. Roudsari and Cramer (1999) developed the extreme rates of convergence for the distribution of GOS. The general theory for distributions of sequential and generalized order statistics has been developed by Cramer and Kamps (2003) without imposing and condition on the parameters. Some characterizations of the probability distributions via the regression of GOS have been obtained by Bieniek and Szynal (2003). Some recurrence relations for moments of GOS for different distributions have been considered by Ahsanullah (2000), Ahmad and Fawzy (2003) and Al-Hussain *et al.* (2005) among others.

Recurrence relations are useful to characterize the distribution and to reduce the number of operation necessary to obtain a general form for the function under consideration. The organization of the paper is as follows: Recurrence relation for the single moments and for product moments are presented in Sections 2 and 3 respectively. Some concluding remarks are given in Section 4.

2 Recurrence Relations for Single Moments

In this section we have developed the recurrence relations for single moments of GOS for Rayleigh distribution. A random variable X is said to have a Rayleigh distribution if its probability density function is given as

$$f(x) = 2\theta x e^{-\theta x^2}, \ \ 0 \le x < \infty, \ \ \theta > 0. \quad (2.1)$$

The distribution function corresponding to (2.1) is

$$F(x) = 1 - e^{-\theta x^2}. \quad (2.2)$$

The $(p-1)$th moment of GOS for (2.1) is given as

$$\begin{aligned} \mu_{r,n,m,k}^{p-1} = E(X_{r,n,m,k}^{p-1}) &= \int_0^\infty x^{p-1} \frac{C_{r-1}}{(r-1)!} \{1-F(x)\}^{\gamma_r - 1} f(x) g_m^{r-1}\{F(x)\} dx \\ &= \frac{2\theta C_{r-1}}{(r-1)!} \int_0^\infty x^p e^{-\theta x^2 \gamma_r} \left[\frac{1}{m+1} \Big(1 - e^{-\theta x^2 (m+1)} \Big) \right]^{r-1} dx. \end{aligned}$$

Making the transformation $w = \theta x^2$, we have

$$\mu^{p-1}_{r,n,m,k} = \frac{C_{r-1}}{\theta^{(p-1)/2}(r-1)!}\int_0^\infty w^{(p-1)/2}e^{-w\gamma_r}\left[\frac{1}{m+1}\Big(1-e^{-w(m+1)}\Big)\right]^{r-1}dw. \tag{2.3}$$

Integrating (2.3) by parts and simplifying, we have

$$\mu^{p-1}_{r,n,m,k} = \frac{2\gamma_r}{(p+1)}\frac{C_{r-1}}{(r-1)!}\frac{1}{\theta^{(p-1)/2}}\int_0^\infty w^{(p+1)/2}e^{-w\gamma_r}\left[\frac{1}{m+1}\Big(1-e^{-w(m+1)}\Big)\right]^{r-1}dw$$
$$-\frac{(m+1)}{(p+1)/2}\frac{C_{r-1}}{(r-2)!}\frac{1}{\theta^{(p+1)/2}}\int_0^\infty w^{(p+1)/2}e^{-w\gamma_{r-1}}\left[\frac{1}{m+1}\Big(1-e^{-w(m+1)}\Big)\right]^{r-2}dw.$$

or

$$\mu^{p-1}_{r,n,m,k} = \frac{2\gamma_r}{(p+1)}\mu^{p+1}_{r,n,m,k} - \frac{2(m+1)}{(p+1)}\mu^{p+1}_{r-1,n,m,k}.$$

Rearranging above equation, we have

$$(p+1)\mu^{p-1}_{r,n,m,k} = 2\Big[\gamma_r\mu^{p+1}_{r,n,m,k} - (m+1)\mu^{p+1}_{r-1,n,m,k}\Big]$$

or

$$\mu^{p+1}_{r,n,m,k} = \frac{1}{2\gamma_r}\Big((p+1)\mu^{p-1}_{r,n,m,k} + (m+1)\mu^{p+1}_{r-1,n,m,k}\Big). \tag{2.4}$$

Expression (2.4) can be used to obtain the moments of any order for generalized order statistics for Rayleigh distribution. Further, relation (2.4) can also be used to obtain the recurrence relation for single moments of simple order statistics, record values and sequential order statistics for Rayleigh distribution, as all three are special cases of generalized order statistics.

3 Recurrence Relation for Product Moments

In this section we have obtained the recurrence relation for product moments of generalized order statistics for Rayleigh distribution. The product moments of GOS are defined as

$$\mu^{p,q}_{r,s,n,m,k} = \int_0^\infty\int_x^\infty x^p y^q f_{r,s,n,m,k}(x,y)dydx.$$

$$\mu^{p,q}_{r,s,n,m,k} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^p y^q\big[1-F(x)\big]^m\big[g_mF(x)\big]^{r-1}$$
$$\cdot\big[g_mF(y)-g_mF(x)\big]^{s-r-1}\big[1-F(y)\big]^{\gamma_s-1}f(x)f(y)dydx. \tag{3.1}$$

Using the density function of Rayleigh distribution in (3.1), we have

$$\mu^{p,q}_{r,s,n,m,k} = \frac{4\theta^2 kC_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^{p+1}y^{q+1}e^{-\theta x^2(m+1)}\left[\frac{1}{m+1}\left(1-e^{-\theta x^2(m+1)}\right)\right]^{r-1}$$

$$\cdot\left[\frac{1}{m+1}\left(e^{-\theta x^2(m+1)}-e^{-\theta y^2(m+1)}\right)\right]^{s-r-1}e^{-\theta y^2\gamma_s}dydx,$$

$$\mu^{p,q}_{r,s,n,m,k} = \frac{4\theta^2 kC_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^{p+1}e^{-\theta x^2(m+1)}\left[\frac{1}{m+1}\left(1-e^{-\theta x^2(m+1)}\right)\right]^{r-1}$$

$$\cdot I^q_{r,s,n,m,k}(y)dx. \tag{3.2}$$

where

$$I^q_{r,s,n,m,k}(y)dx = \int_x^\infty y^{q+1}\left[\frac{1}{m+1}\left(e^{-\theta x^2(m+1)}-e^{-\theta y^2(m+1)}\right)\right]^{s-r-1}e^{-\theta y^2\gamma_s}dy.$$

The integral $I^q_{r,s,n,m,k}(y)$ can be evaluated in two different situations, which situations are given below:

Case I. $s = r+1$.

In this case the integral $I^q_{r,s,n,m,k}(y)$ is

$$I^q_{r,s,n,m,k}(y)dx = \int_x^\infty y^{q+1}e^{-\theta y^2\gamma_{r+1}}dy.$$

Making the transformation $w = \theta y^2$, we have

$$I^q_{r,s,n,m,k}(y) = \frac{1}{2\theta}\frac{x^q}{\gamma_{r+1}}e^{-\theta x^2\gamma_{r+1}} + \frac{q}{2\gamma_{(r+1)}}\frac{1}{2\theta^{q/2+1}}\int_{\theta x^2}^\infty w^{q/2-1}e^{-w\gamma_{r+1}}dw.$$

or

$$I^q_{r,s,n,m,k}(y) = \frac{1}{2\theta}\frac{x^q}{\gamma_{r+1}}e^{-\theta x^2\gamma_{r+1}} + \frac{q}{2\gamma_{r+1}}I^{q-1}_{r,r+1,n,m,k}(y). \tag{3.3}$$

Putting the value of $I^q_{r,s,n,m,k}(y)$ from (3.3) in (3.2) we have

$$\mu^{p,q}_{r,r+1,n,m,k} = \frac{4\theta^2 kC_r}{(r-1)!}\int_0^\infty x^{p+1}e^{-\theta x^2(m+1)}\left[\frac{1}{m+1}\left(1-e^{-\theta x^2(m+1)}\right)\right]^{r-1}$$

$$\cdot\frac{1}{2\theta}\frac{x^q}{\gamma_{r+1}}e^{-\theta x^2\gamma_{r+1}} + \frac{q}{2\gamma_{r+1}}I^{q-1}_{r,r+1,n,m,k}(y)\Bigg]dx.$$

Simplifying above expression, we have

$$\mu^{p,q}_{r,r+1,n,m,k} = \frac{2\theta kC_r}{(r-1)!\gamma_{r+1}}\int_0^\infty x^{p+q+1}e^{-\theta x^2\gamma_r}\left[\frac{1}{m+1}\left(1-e^{-\theta x^2(m+1)}\right)\right]^{r-1}dx$$

$$+\frac{2\theta^2 kq}{\gamma_{r+1}}\int_0^\infty x^{p+1}e^{-\theta x^2(m+1)}\left[\frac{1}{m+1}\left(1-e^{-\theta x^2(m+1)}\right)\right]^{r-1}I^{q-1}_{r,r+1,n,m,k}(y)dx.$$

$$\mu^{p,q}_{r,r+1,n,m,k} = \frac{2\theta k C_{r-1}}{(r-1)!} \int_0^\infty x^{p+q+1} e^{-\theta x^2 \gamma_r} \left[\frac{1}{m+1}\Big(1 - e^{-\theta x^2 (m+1)}\Big)\right]^{r-1} dx$$
$$+ \frac{2\theta^2 kq}{\gamma_{r+1}} \int_0^\infty x^{p+1} e^{-\theta x^2 (m+1)} \left[\frac{1}{m+1}\Big(1 - e^{-\theta x^2 (m+1)}\Big)\right]^{r-1} I^{q-1}_{r,r+1,n,m,k}(y) dx.$$

Simplifying above equation we get the following recurrence relation for product moments of joint GOS when $s = r + 1$

$$\mu^{p,q}_{r,r+1,n,m,k} = \mu^{p+q}_{r,n,m,k} + \frac{q}{2\gamma_{r+1}} \mu^{p,q-1}_{r,r+1,n,m,k}. \tag{3.4}$$

Case II. $s > r + 1$.

In this case the integral $I^q_{r,s,n,m,k}(y)$ is given as

$$I^q_{r,s,n,m,k}(y) = \int_x^\infty y^{q+1} \left[\frac{1}{m+1}\Big(e^{-\theta x^2 (m+1)} - e^{-\theta y^2 (m+1)}\Big)\right]^{s-r-1} e^{-\theta y^2 \gamma_s}\, dy.$$

Making the transformation $w = \theta y^2$, we have

$$I^q_{r,s,n,m,k}(y) = \frac{1}{2\theta^{q/2+1}} \int_{\theta x^2}^\infty w^{q/2} \left[\frac{1}{m+1}\Big(e^{-\theta x^2 (m+1)} - e^{-w(m+1)}\Big)\right]^{s-r-1} e^{-w\gamma_s} dw. \tag{3.5}$$

Integrating (3.5) by parts and simplifying, we get

$$I^q_{r,s,n,m,k}(y) = \frac{(m+1)(s-r-1)}{\gamma_s} \frac{1}{2\theta^{q/2+1}} \int_{\theta x^2}^\infty w^{q/2} \left[\frac{1}{m+1}\Big(e^{-\theta x^2 (m+1)}\right.$$
$$\left. - e^{-w(m+1)}\Big)\right]^{s-r-2} e^{-w\gamma_{s-1}} dw + \frac{q}{2\gamma_s} \frac{1}{2\theta^{q/2+1}} \int_{\theta x^2}^\infty w^{q/2-1}$$
$$\cdot \left[\frac{1}{m+1}\Big(e^{-\theta x^2 (m+1)} - e^{-w(m+1)}\Big)\right]^{s-r-1} e^{-w\gamma_s} dw.$$

Simplifying we have

$$I^q_{r,s,n,m,k}(y) = \frac{(m+1)(s-r-1)}{\gamma_s} I^q_{r,s-1,n,m,k}(y) + \frac{q}{2\gamma_s} I^{q-1}_{r,s,n,m,k}(y). \tag{3.6}$$

Using (3.6) in (3.2), we have

$$\mu^{p,q}_{r,s,n,m,k} = \frac{(m+1)}{\gamma_s} \frac{4\theta^2 C_{s-1}}{(r-1)!(s-r-1)!} \int_0^\infty x^{p+1} e^{-\theta x^2 (m+1)}$$
$$\cdot \left[\frac{1}{m+1}\Big(1 - e^{-\theta x^2 (m+1)}\Big)\right]^{r-1} I^q_{r,s-1,n,m,k}(y) dx + \frac{q}{2\gamma_s} \frac{4\theta^2 C_{s-1}}{(r-1)!(s-r-1)!}$$
$$\cdot \int_0^\infty x^{p+1} [e^{-\theta x^2 (m+1)}] \left[\frac{1}{(m+1)}\{1 - e^{-\theta x^2 (m+1)}\}\right]^{r-1} I^{q-1}_{r,s,n,m,k}(y) dx.$$

Simplifying the above equation, we get the following recurrence relation for product moments of GOS when $s > r + 1$

$$\mu^{p,q}_{r,s,n,m,k} = \frac{(m+1)}{\gamma_s}\mu^{p,q}_{r,s-1,n,m,k} + \frac{q}{2\gamma_s}\mu^{p,q-1}_{r,s,n,m,k}. \tag{3.7}$$

The recurrence relations (3.4) and (3.7) can be used to obtain the similar expressions for simple order statistics and record statistics of Rayleigh distribution.

4 Conclusion

The recurrence relations for single and product moments of generalized order statistics for the Rayleigh distribution are derived in this paper. Following the same procedure one may obtain the recurrence relation for single and product moment for We ibull and other useful continuous distributions.

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