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Linear Fractional Time Minimizing Transportation Problem with Impurities

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Abstract: : In this study, the solution procedure of Linear Fractional Time Minimizing Transportation Problem with Impurities (LFTMTPI) in the commodity to generate total transportation solution schedules, is going to be presented. This LFTMTPI is related to a lexicographic linear fractional time minimizing transportation problem with impurities. The partial flows constituting a feasible transportation schedule may be partitioned according to the actual and standard transportation time involved. An LFTMTPI algorithm is also presented to solve such real life fractional decision priority problems. This algorithm takes into account the special structure of the problem due to impurities in the commodity and depends heavily on the optimality conditions. The optimality conditions reflects nothing else than dual feasibility.

Keywords: Fractional Programming, Transportation Problem, Lexicographic, Impurity Constraint.

I. Introduction

Time minimizing transportation problem is required to find a feasible transportation schedule which minimizes the maximum of transportation time associated between a supply point and a demand point such that the distribution between the two points is positive. The time transportation problem is relevant in a variety of real life transportation situations e.g. Military transportation during war and emergencies, transportation of perishable goods, transportation in emergency situations. Khanna, Bakhshi and Arora [6] studied a time transportation problem, wherein there was a restriction on the total flow. Nikolic [1] demonstrated the total transportation time problem regarding the time of the active transportation routes. According to the author if the multiple optimal solutions exist, it was possible to include other criteria as second level of criteria and find the corresponding solutions. Sonia and Puri [7] considered a two level hierarchical balanced time minimizing transportation problem.

Transportation problems with fractional objective function are widely used as performance measures in many real life situations e.g., in the analysis of financial aspects of transportation enterprises and undertaking and in transportation management situations, where an individual, or a group of community is faced with the problem of maintaining good ratios between some very important crucial parameters concerned with the transportation of commodities from certain sources to various destinations. Sharma and Swarup [2] presented a transportation technique for time minimization in fractional functional programming problem with an objective function. Swarup [3] studied a transportation technique for linear fractional functional programming problem. Kanchan, Holland and Sahney [4] investigated transportation techniques in linear plus linear fractional programming having special structured objective function. In all Transportation Models it is assumed that the commodity is identical irrespective of its source and that the consumers have no preference relating to its supply point. However in many real life transportation situations in industries of coal, iron, cement etc., the commodity does vary in some characteristics according to its source and the final commodity mixture reaching the various destinations, may then be required to meet known specifications.

In this paper, to generate total transportation solution schedules, an algorithm is presented to solve linear fractional time minimizing transportation problem with impurities where the commodity can have different types of impurities, by relating it to a lexicographic linear fractional time minimizing transportation problem with impurities. This algorithm takes into account the special structure of the problem and depends heavily on the optimality conditions. The developed algorithm is also supported by a real life example of crude-ore transportation problem of Steel Authority of India Limited.

1. Mathematical Formulation

The LFTMTPI can be formulated as:

$$\min\ t = \max_{(i,j)} \left\{ \frac{t_{ij}^a}{t_{ij}^s} \,\middle|\, x_{ij} > 0 \right\} \tag{1}$$

subject to

$$\sum_{j=1}^{N} x_{ij} = a_i \qquad (i = 1,2,\ldots\ldots,M) \tag{2}$$

$$\sum_{i=1}^{M} x_{ij} = b_j \qquad (j = 1,2,\ldots\ldots,N) \tag{3}$$

$$\sum_{i=1}^{M} f_{ijk}\, x_{ij} \le q_{jk} \qquad (j = 1,2,\ldots\ldots,N; k = 1,2,\ldots\ldots,P) \tag{4}$$

$$x_{ij} \ge 0 \qquad (i = 1,2,\ldots\ldots,M;\ j = 1,2,\ldots\ldots,N) \tag{5}$$

where a_i is the quantity of the commodity available at the i^{th} source and b_j is the quantity of commodity required at the j^{th} destination. One unit of the commodity contains f_{ijk} units of P impurities $(k = 1,2,\ldots,P)$ when it is sent from the i^{th} source to the j^{th} destination. Customer j cannot receive more than q_{jk} units of impurity k and x_{ij} is the amount of the commodity transported from the i^{th} source to the j^{th} destination, $T^a = [t_{ij}^a]$ and $T^s = [t_{ij}^s]$ are two $(M \times N)$ time matrices where t_{ij}^a is the actual transportation time for transporting $x_{ij} > 0$ units from the i^{th} source to the j^{th} destination and t_{ij}^s is the standard transportation time for transporting $x_{ij} > 0$ units from the i^{th} source to the j^{th} destination. t_{ij}^a / t_{ij}^s is proportional contribution to the value of the fractional time objective function for shipping one unit of commodity from the i^{th} source to the j^{th} destination, and is independent of the amount of commodity for $x_{ij} > 0$ and t is fractional transportation time.

Setting $M' = \{1,2,\ldots,M\}$, $N' = \{1,2,\ldots,N\}$, $P' = \{1,2,\ldots,P\}$, $J' = \{(i,j) \mid i \in M', j \in N'\}$, the LFTMTPI can be rewritten as:

$$\min\ t = \left[\max_{(i,j)} \left\{ \frac{t_{ij}^a}{t_{ij}^s} \,\middle|\, x_{ij} > 0 \right\} \,\middle|\, \begin{array}{l} \sum_{j \in N'} x_{ij} = a_i, \quad for\ all\ i \in M' \\ \sum_{i \in M'} x_{ij} = b_j, \quad for\ all\ j \in N' \\ \sum_{i \in M'} f_{ijk}\, x_{ij} \le q_{jk}, \quad for\ all\ j \in N', k \in P' \\ x_{ij} \ge 0, \quad for\ all\ (i,j) \in J' \end{array} \right]$$

It is assumed that $a_i > 0$, $i \in M'$; $b_j > 0$, $j \in N'$, and the consistency condition for the existence of the solution to the problem is $\sum_{i \in M'} a_i = \sum_{j \in N'} b_j$.

2. Lexicographic Linear Fractional Time Minimizing Transportation Problem with Impurities

If the transportation system decision maker decides to minimize the fractional time objective function of the LFTMTPI, this concept may be represented by a vector-valued fractional objective function which is to be minimized lexicographically. The LFTMTPI can be easily formulated as a Lexicographic Linear Fractional Time Minimizing Transportation Problem with Impurities (LLFTMTPI):

$$\text{lexmin} \left[\Im = \frac{\sum_{(i,j)\in J'} \alpha_{ij} x_{ij}}{\sum_{(i,j)\in J'} \beta_{ij} x_{ij}} \left| \begin{array}{l} \sum_{j \in N'} x_{ij} = a_i, \quad for\ all\ i \in M' \\ \sum_{i \in M'} x_{ij} = b_j, \quad for\ all\ j \in N' \\ \sum_{i \in M'} f_{ijk}\ x_{ij} \le q_{jk}, \quad for\ all\ (i,j) \in J' \\ x_{ij} \ge 0, \quad for\ all\ (i,j) \in J' \end{array} \right. \right] \tag{6}$$

with $\alpha_{ij} := [e_c]$, $(i,j) \in \xi_c^a$, $c = (1,2,\ldots,g)$

and $\beta_{ij} := [e_d]$, $(i,j) \in \xi_d^s$, $d = (g+1,\ldots,h)$

here $\alpha_{ij}, \beta_{ij} \in \mathrm{IR}^{\mathrm{h}}, \mathrm{IR}^{\mathrm{h}}$ be the set of real numbers.

Following the usual method of solution, the first stage is to introduce slack variables $x_{M+k,j}$ into the impurities:

$$\sum_i f_{ijk}\ x_{ij} + x_{M+k,j} = q_{jk} \tag{7}$$

$$x_{M+k,j} \ge 0$$

(8)

There are a total of $MN + NP$ variables including slacks and $NP + M + N$ equations. Because of the conditions imposed on the a_i and the b_j, one of the equations (2) and (3) is dependent and so a basic feasible solution contains $NP + M + N - 1$ basic variables.

3. Vector-valued Dual Variables and Optimality Conditions

Consider the vector-valued dual variables u_i^1, u_i^2, $(i \in M')$; v_j^1, v_j^2, $(j \in N')$ and w_{jk}^1, w_{jk}^2, $(j \in N', k \in P')$ defined such that:

$$\alpha_{ij} - \left(u_i^1 + v_j^1 + \sum_{k \in P'} w_{jk}^1\ f_{ijk} \right) = 0$$

$$\beta_{ij} - \left(u_i^2 + v_j^2 + \sum_{k \in P'} w_{jk}^2\ f_{ijk} \right) = 0$$

(for those i, j for which x_{ij} is in the basis)

and

$$w_{jk}^1 = 0$$

$$w_{jk}^2 = 0$$

(for those j, k for which $x_{M+k,j}$ is in the basis)

Also, let

$$\alpha'_{ij} = \alpha_{ij} - \left(u_i^1 + v_j^1 + \sum_{k \in P'} w_{jk}^1 \ f_{ijk} \right) \tag{9}$$

$$\beta'_{ij} = \beta_{ij} - \left(u_i^2 + v_j^2 + \sum_{k \in P'} w_{jk}^2 \ f_{ijk} \right) \tag{10}$$

These vector-valued dual variables $u_i^1, u_i^2,\ v_j^1, v_j^2,\ \ w_{jk}^1, w_{jk}^2$ can be obtained and then for non-basic variables, $\alpha'_{ij}, \beta'_{ij}$ can be determined by the relations (9) and (10).

Now, in order to derive the optimality conditions, the fractional objective function $\Im$ of LLFTMTPI in equation (6) is expressed in terms of the non-basic variables.
Let

$$\Im = \frac{\sum_{(i,j) \in J'} \alpha_{ij} x_{ij}}{\sum_{(i,j) \in J'} \beta_{ij} x_{ij}} = \frac{A}{B} \tag{11}$$

then

$$A = \sum_{i \in M'} \sum_{j \in N'} \alpha_{ij} x_{ij} + \sum_{i \in M'} u_i^1 \left(a_i - \sum_{j \in N'} x_{ij} \right) + \sum_{j \in N'} v_j^1 \left(b_j - \sum_{i \in M'} x_{ij} \right)$$
$$+ \sum_{j \in N'} \sum_{k \in P'} w_{jk}^1 \left(q_{jk} - \sum_{i \in M'} f_{ijk} \ x_{ij} - x_{M+k,j} \right)$$

or

$$A = \left[\sum_{i \in M'} u_i^1 a_i + \sum_{j \in N'} v_j^1 b_j + \sum_{j \in N'} \sum_{k \in P'} w_{jk}^1 \ q_{jk} \right] + \sum_{i \in M'} \sum_{j \in N'} \left[\alpha_{ij} - \left(u_i^1 + v_j^1 + \sum_{k \in P'} w_{jk}^1 \ f_{ijk} \right) \right] x_{ij}$$
$$- \sum_{j \in N'} \sum_{k \in P'} w_{jk}^1 \ x_{M+k,j}$$

giving

$$A = \left[\sum_{(i,j) \in G} \alpha'_{ij} x_{ij} - \sum_{(j,k) \in G_1} w_{jk}^1 \ x_{M+k,j} + V_1 \right]$$

where $\sum_{(i,j) \in G}$ and $\sum_{(j,k) \in G_1}$ denote the summation extending over the set of non-basic variables x_{ij} and $x_{M+k,j}$ respectively, and

$$V_1 = \left[\sum_{i \in M'} u_i^1 a_i + \sum_{j \in N'} v_j^1 b_j + \sum_{j \in N'} \sum_{k \in P'} w_{jk}^1 \ q_{jk} \right]$$

Similarly

$$B = \left[\sum_{(i,j) \in G} \beta'_{ij} x_{ij} - \sum_{(j,k) \in G_1} w_{jk}^2 \ x_{M+k,j} + V_2 \right], \quad V_2 = \left[\sum_{i \in M'} u_i^2 a_i + \sum_{j \in N'} v_j^2 b_j + \sum_{j \in N'} \sum_{k \in P'} w_{jk}^2 \ q_{jk} \right]$$

Therefore, the objective function (11) becomes

$$\Im = \frac{A}{B} = \left[\frac{\sum_{(i,j)\in G} \alpha'_{ij} x_{ij} - \sum_{(j,k)\in G_1} w^1_{jk}\ x_{M+k,j} + V_1}{\sum_{(i,j)\in G} \beta'_{ij} x_{ij} - \sum_{(j,k)\in G_1} w^2_{jk}\ x_{M+k,j} + V_2} \right] \quad (12)$$

Differentiating $\Im$ with respect to the non-basic variable x_{ij} (i, j ranging over the set G),

$$\frac{\partial \Im}{\partial x_{ij}} = \frac{\left[\alpha'_{ij} \left\{ V_2 + \sum_{(i,j)\in G} \beta'_{ij} x_{ij} - \sum_{(j,k)\in G_1} w^2_{jk}\ x_{M+k,j} \right\} - \beta'_{ij} \left\{ V_1 + \sum_{(i,j)\in G} \alpha'_{ij} x_{ij} - \sum_{(j,k)\in G_1} w^1_{jk}\ x_{M+k,j} \right\} \right]}{\left[\left\{ V_2 + \sum_{(i,j)\in G} \beta'_{ij} x_{ij} - \sum_{(j,k)\in G_1} w^2_{jk}\ x_{M+k,j} \right\}^2 \right]}$$

Let $\left[\frac{\partial \Im}{\partial x_{ij}} \right]_*$ denote the value of $\left[\frac{\partial \Im}{\partial x_{ij}} \right]$ at the basic feasible solution, then

$$\left[\frac{\partial \Im}{\partial x_{ij}} \right]_* = \frac{V_2\ \alpha'_{ij} - V_1\ \beta'_{ij}}{(V_2)^2}$$

Again from (12), differentiating $\Im$ with respect to the non-basic variables $x_{M+k,j}$ (j,k ranging over the set G_1),

$$\frac{\partial \Im}{\partial x_{M+k,j}} = \frac{\left[(-w^1_{jk}) \left\{ V_2 + \sum_{(i,j)\in G} \beta'_{ij} x_{ij} - \sum_{(j,k)\in G_1} w^2_{jk}\ x_{M+k,j} \right\} - (-w^2_{jk}) \left\{ V_1 + \sum_{(i,j)\in G} \alpha'_{ij} x_{ij} - \sum_{(j,k)\in G_1} w^1_{jk}\ x_{M+k,j} \right\} \right]}{\left[\left\{ V_2 + \sum_{(i,j)\in G} \beta'_{ij} x_{ij} - \sum_{(j,k)\in G_1} w^2_{jk}\ x_{M+k,j} \right\}^2 \right]}$$

Suppose $\left[\frac{\partial \Im}{\partial x_{M+k,j}} \right]_*$ denote the value of $\left[\frac{\partial \Im}{\partial x_{M+k,j}} \right]$ at the basic feasible solution, then

$$\left[\frac{\partial \Im}{\partial x_{M+k,j}} \right]_* = \frac{-V_2\ w^1_{jk} + V_1\ w^2_{jk}}{(V_2)^2}$$

Therefore, the optimality criteria are:

$$\Delta_{ij} = [V_2\ \alpha'_{ij} - V_1\ \beta'_{ij}] \underset{\sim}{\geq} 0, \quad \Delta_{M+k,j} = [V_1\ w^2_{jk} - V_2\ w^1_{jk}] \underset{\sim}{\geq} 0$$

where $\alpha'_{ij} = \alpha_{ij} - (u^1_i + v^1_j + \sum_{k\in P'} w^1_{jk}\ f_{ijk})$, $\beta'_{ij} = \beta_{ij} - (u^2_i + v^2_j + \sum_{k\in P'} w^2_{jk}\ f_{ijk})$

$$V_1 = \left[\sum_{i\in M'} u^1_i a_i + \sum_{j\in N'} v^1_j b_j + \sum_{j\in N'} \sum_{k\in P'} w^1_{jk}\ q_{jk} \right], \quad V_2 = \left[\sum_{i\in M'} u^2_i a_i + \sum_{j\in N'} v^2_j b_j + \sum_{j\in N'} \sum_{k\in P'} w^2_{jk}\ q_{jk} \right]$$

4. Altering a Basic Feasible Solution

If a basic feasible solution is to be updated by the introduction of a non-basic variable and the removal of a basic one, then alterations can only be made to the basic variables. To determine the incoming variable, select the minimum

$$\Delta_{i_* j_*} = \min\left\{\Delta_{ij} \middle| \ \Delta_{ij} \underset{\sim}{<} 0\right\}$$

or

(13)

$$\Delta_{M+k_*, j_*} = \min\left\{\Delta_{M+k, j} \middle| \ \Delta_{M+k, j} \underset{\sim}{<} 0\right\}$$

By applying the selection rule (13), the variables $x_{i_* j_*}$ or $x_{M+k_* j_*}$ becomes a basic variable of the new basic feasible solution, and an unknown quantity θ is to be added to this variable while $\theta.\delta_{RS}$ or $\theta.\delta_{M+Y,S}$ is added to all the basic variables x_{RS} or $x_{M+Y,S}$. Then if the new solution satisfies the original constraints, the $\delta's$ must satisfy the equations:

$$\sum_{R=1}^{M} \delta_{RS} = 0 \qquad (S = 1,2,\ldots,N) \tag{14}$$

$$\sum_{S=1}^{N} \delta_{RS} = 0 \qquad (R = 1,2,\ldots,M) \tag{15}$$

$$\sum_{Y=1}^{P} f_{RSY} \ \delta_{RS} + \delta_{M+Y,S} = 0 \qquad (S = 1,2,\ldots,N; \ Y = 1,2\ldots,P) \tag{16}$$

Here, $\delta_{RS} = 0$, if x_{RS} is not in the basis and $\delta_{M+Y,S} = 0$, if $x_{RS} + \delta_{RS}\ \theta,$ is not in the basis. There are $NP + M + N - 1$ independent equations in the set (14), (15) and (16) and $NP + M + N$ unknown $\delta's$. It is therefore possible to solve this set of equations for the ($M + N + NP - 1$) δ`s associated with basic variables in terms of $\delta_{i_* j_*}$ or $\delta_{M+k_* j_*}$. Furthermore, the values of the variables in the updated basic feasible solution are given by $x_{RS} + \delta_{RS}\ \theta$; $x_{M+Y,S} + \delta_{M+Y,S}\ \theta$. By choosing a suitable value of θ from

$$\theta = \min_{\substack{\delta_{RS} < 0 \\ \delta_{M+Y,S} < 0}} \left[-\frac{x_{RS}}{\delta_{RS}}; -\frac{x_{M+Y,S}}{\delta_{M+Y,S}} \right] \tag{17}$$

one of the variables is reduced to zero while the others remain positive and a new updated basic feasible solution is obtained.

5. Change in Time Vectors

To show that the method of altering the solution is valid, consider the introduction of x_{11} using (14), (15), (16), the change in time α_{ij} is:

$$\sum_{i=1}^{M}\sum_{j=1}^{N} \theta \cdot \delta_{ij}\alpha_{ij} = \left[\sum_{i=1}^{M}\sum_{j=1}^{N} \delta_{ij}\left(u_i^1 + v_j^1 + \sum_{k=1}^{P} w_{jk}^1 f_{ijk} \right) - \delta_{11}\left(u_1^1 + v_1^1 + \sum_{k=1}^{P} w_{1k}^1 f_{11k} \right) + \delta_{11}\alpha_{11} \right] \cdot \theta$$

$$= \left[\sum_{i=1}^{M} u_i^1 \left(\sum_{j=1}^{N} \delta_{ij} \right) + \sum_{j=1}^{N} v_j^1 \left(\sum_{i=1}^{M} \delta_{ij} \right) + \sum_{k=1}^{P}\sum_{j=1}^{N} w_{jk}^1 \left(\sum_{i=1}^{M} \delta_{ij} f_{ijk} \right) + \delta_{11}\left(\alpha_{11} - u_1^1 - v_1^1 - \sum_{k=1}^{P} w_{1k}^1 f_{11k} \right) \right] \cdot \theta$$

$$=\left[\sum_{k=1}^{P}\sum_{j=1}^{N} w_{jk}^{1}\left(-\delta_{M+k,j}\right)+\delta_{11}\left(\alpha_{11}-u_{1}^{1}-v_{1}^{1}-\sum_{k=1}^{P} w_{1k}^{1} f_{11k}\right)\right]\cdot\theta$$
$$=\delta_{11}\cdot\theta\left[\left(\alpha_{11}-u_{1}^{1}-v_{1}^{1}-\sum_{k=1}^{P} w_{1k}^{1} f_{11k}\right)\right]$$

Since $w_{jk}^{1}=0$, for those j,k for which $x_{M+k,j}$ is in the basis and $\delta_{M+k,j}=0$, for those j,k for which $x_{M+k,j}$ is not in the basis.

Consider now the introduction of the non-basic variable $x_{M+1,1}$ and then change in time α_{ij} is:

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\theta\cdot\delta_{ij}\alpha_{ij}=\left[\sum_{i=1}^{M}u_{i}^{1}\left(\sum_{j=1}^{N}\delta_{ij}\right)+\sum_{j=1}^{N}v_{j}^{1}\left(\sum_{i=1}^{M}\delta_{ij}\right)+\sum_{k=1}^{P}\sum_{j=1}^{N}w_{jk}^{1}\left(\sum_{i=1}^{M}\delta_{ij}f_{ijk}\right)\right]\cdot\theta$$
$$=\sum_{k=1}^{P}\sum_{j=1}^{N}w_{jk}^{1}\left(-\delta_{M+k,j}\right)\cdot\theta$$
$$=-\theta\cdot w_{11}^{1}\delta_{M+1,1}$$

Since $w_{jk}^{1}=0$, for those j,k for which $x_{M+k,j}$ is in the basis and $\delta_{M+k,j}=0$, for those j,k for which $x_{M+k,j}$ is not in the basis (expect for $x_{M+1,1}$) .

Similarly to show that the method of altering the solution is valid, consider the introduction of x_{11} and the non-basic variable $x_{M+1,1}$ the change in time β_{ij} is:

$\sum_{i=1}^{M}\sum_{j=1}^{N}\theta\cdot\delta_{ij}\beta_{ij}=\delta_{11}\cdot\theta\left[\left(\beta_{11}-u_{1}^{2}-v_{1}^{2}-\sum_{k=1}^{P}w_{1k}^{2}f_{11k}\right)\right]$, $\sum_{i=1}^{M}\sum_{j=1}^{N}\theta\cdot\delta_{ij}\beta_{ij}=-\theta\cdot w_{11}^{2}\delta_{M+1,1}$ respectively.

6. Algorithm

The steps of algorithm, to generate optimal total transportation solution schedules for LFTMTPI in a finite number of iterations, are:

Step 1: Determine the lower bound t_l^a on t^a to reduce the dimension of the vector α_{ij} and lower bound t_l^s on t^s to reduce the dimension of the vector β_{ij} .

Step 2: Determine an initial basic feasible solution X^1 by using method of Saxena [5].

Step 3: From the resulting transportation time t^a and t^s of the initial basic feasible solution X^1, determine an upper bound t_U^a and t_U^s .

Step 4: Partition the set $\xi^a := M\times N$ and $\xi^s := M\times N$ into subset ξ_c^a and ξ_d^s, $(c=1,\ldots,g; d=g+1,\ldots,h)$ respectively and determine the vectors. With the help of vectors $\alpha_{ij} := [e_c]$ and $\beta_{ij} := [e_d]$, obtain the fractional transportation time matrix T.

Step 5: Designate the set of pairs of indices (i,j) of the basic variable by I and using initial basic feasible solution compute recursively the associated vector-valued multipliers u_i^1, u_i^2, v_j^1, v_j^2, w_{jk}^1, w_{jk}^2 defined such that

$$\alpha_{ij} - \left(u_i^1 + v_j^1 + \sum_{k \in P'} w_{jk}^1 \ f_{ijk} \right) = 0\,, \qquad \beta_{ij} - \left(u_i^2 + v_j^2 + \sum_{k \in P'} w_{jk}^2 \ f_{ijk} \right) = 0$$

(for those i, j for which x_{ij} is in the basis) and $w_{jk}^1 = 0$, $w_{jk}^2 = 0$ (for those j, k for which $x_{M+k,j}$ is in the basis)

Step 6: Let $\tilde{U} = (\tilde{u}_i^1, \tilde{u}_i^2, i \in M';\ \tilde{v}_j^1, \tilde{v}_j^2, \ j \in N';\ \tilde{w}_{jk}^1, \tilde{w}_{jk}^2, \ j \in N'; k \in P')$ be the solution of above equations in Step 2. Evaluate the relative criterion vectors

$$\Delta_{ij} = [V_2 \alpha'_{ij} - V_1 \beta'_{ij}] \underset{\sim}{\geq} 0\,, \quad \Delta_{M+k,j} = [V_1 \ \tilde{w}_{jk}^2 - V_2 \ \tilde{w}_{jk}^1] \underset{\sim}{\geq} 0 \ \text{ for all } (i, j) \in J' \setminus I$$

Step 7: If all Δ_{ij} and $\Delta_{M+k,j}$ are lexicographically greater than or equal to the zero vector for all $(i, j) \in J' \setminus I$, the current basic feasible solution is optimal to LLFTMTPI. Go to Step 10, otherwise go to Step 8.

Step 8: By applying the selection rule (13) determine the variable $x_{i_* j_*}$ or $x_{M+k_* j_*}$ which is to be enter. The variable $x_{i_* j_*}$ or $x_{M+k_* j_*}$ then becomes a basic variable of the new basic feasible solution.

Step 9: Change the current basic feasible solution to the new basic feasible solution using equations (14)-(17)
and go to Step 5.

Step 10: If $\tilde{X} = (\tilde{x}_{ij}, \tilde{x}_{M+k,j})$ is optimal total transportation solution schedules for LLFTMTPI denoted by equation (6), then $\tilde{\mathfrak{I}} = \sum_{(i,j) \in J'} \alpha_{ij} \tilde{x}_{ij} \Big/ \sum_{(i,j) \in J'} \beta_{ij} \tilde{x}_{ij}$ and $\dfrac{\tilde{c}}{\tilde{d}}$ is the index of the first positive component of the optimal flow vector $\tilde{\mathfrak{I}}$. Also $\tilde{t} = \dfrac{\tilde{t}_{ij}^{\,a}}{\tilde{t}_{ij}^{\,s}}$ with $(i, j) \in \xi_{\tilde{c}}^{a} \big/ \xi_{\tilde{d}}^{s}$ is the optimal fractional transportation time.

7. Crude-Ore Transportation Problem

The developed algorithm for determining the optimal total transportation solution schedules for the crude-ore transportation problem can be illustrated by considering the following example of SAIL:

SAIL has different type of furnace in each of six work centers (j), situated in Bhilai, Durgapur, Rourkela, Burnpur, Salem and Bhadravati in India. The work centers must receive a fixed weight of crude ore (i) which is available in six different grades. For technical reasons the processing time of crude ore depends on its grades and the work centers to which it is sent.

The problem is to generate total transportation solution schedules which minimizes the total fractional transportation-processing time t of transporting crude ore while satisfying the extra requirement that the amount of phosphorus is less than a certain critical level. In Table 1, the total fractional transportation-processing time (in hours) required for transporting-processing the crude-ore from i^{th} source to j^{th} destination are displayed.

Let x_{ij} be the tonnage sent from i to j then it is required to:

$$\min \ t = \max_{(i,j)} \left\{ \frac{t_{ij}^{a}}{t_{ij}^{s}} \,\middle|\ x_{ij} > 0 \right\}$$

subject to $\sum_{j=1}^{6} x_{ij} = a_i$, $(i = 1,2,\ldots,6)$, $\sum_{i=1}^{6} x_{ij} = b_j$, $(j = 1,2,\ldots,6)$, $\sum_{i=1}^{6} p_i . x_{ij} \le L_j b_j$, $x_{ij} \ge 0$
$(i = 1,2,\ldots,6;\ j = 1,2,\ldots,6)$

		Work Centers j						Tons aval. a_i	Phos. Cont. p_i
		1	2	3	4	5	6		
Crude -Ore i	1	$\left[\frac{5:35}{4:25}\right]$	$\left[\frac{5:30}{4:20}\right]$	$\left[\frac{5:35}{4:25}\right]$	$\left[\frac{5:10}{4:20}\right]$	$\left[\frac{5:50}{4:50}\right]$	$\left[\frac{5:35}{4:25}\right]$	6	0.4
	2	$\left[\frac{5:45}{4:25}\right]$	$\left[\frac{5:20}{4:20}\right]$	$\left[\frac{5:35}{4:25}\right]$	$\left[\frac{5:00}{4:25}\right]$	$\left[\frac{5:40}{4:25}\right]$	$\left[\frac{5:35}{4:20}\right]$	11	0.8
	3	$\left[\frac{5:15}{4:20}\right]$	$\left[\frac{5:35}{4:25}\right]$	$\left[\frac{5:35}{4:20}\right]$	$\left[\frac{5:35}{4:20}\right]$	$\left[\frac{5:20}{4:25}\right]$	$\left[\frac{5:40}{4:00}\right]$	8	0.6
	4	$\left[\frac{5:40}{4:25}\right]$	$\left[\frac{5:00}{4:50}\right]$	$\left[\frac{5:10}{4:20}\right]$	$\left[\frac{5:35}{4:45}\right]$	$\left[\frac{5:40}{4:25}\right]$	$\left[\frac{5:35}{4:25}\right]$	5	0.4
	5	$\left[\frac{5:35}{4:30}\right]$	$\left[\frac{5:40}{4:40}\right]$	$\left[\frac{5:35}{4:25}\right]$	$\left[\frac{5:20}{4:15}\right]$	$\left[\frac{5:30}{4:20}\right]$	$\left[\frac{5:35}{4:45}\right]$	1	0.6
	6	$\left[\frac{5:35}{4:25}\right]$	$\left[\frac{6:00}{4:45}\right]$	$\left[\frac{5:10}{4:35}\right]$	$\left[\frac{5:35}{4:05}\right]$	$\left[\frac{5:25}{4:20}\right]$	$\left[\frac{5:35}{4:25}\right]$	3	0.4
Tons Reqd. b_j		7	10	9	4	1	3		
Max Phos. L_j		0.7	0.7	0.7	0.7	0.7	0.7		

Table 1: Total Fractional Transportation-processing Time

The initial basic feasible solution X^1 is:

$$x_{11} = 2,\ x_{12} = \frac{5}{2},\ x_{14} = 1,\ x_{16} = \frac{1}{2},\ x_{22} = \frac{15}{2},\ x_{23} = \frac{7}{2},\ x_{33} = \frac{11}{2},\ x_{36} = \frac{5}{2},\ x_{41} = 5,$$

$$x_{54} = 1,\ x_{64} = 2,\ .x_{65} = 1,\ x_{71} = 21,\ x_{73} = 2,\ x_{74} = 10,\ x_{75} = 3,\ x_{76} = 4$$

The lower and upper bounds are:

$$t_l^a = 5.58, \qquad t_l^s = 4.33, \qquad t_U^a = 5.67, \qquad t_U^s = 4.42$$

Hence $g = 4$ and $h = 4$ so ξ^a and ξ^s has four subsets:

$$\xi_1^a := \{(i,j) \in \xi^a \mid t_{ij}^a > 5.67\}, \qquad \xi_2^a := \{(i,j) \in \xi^a \mid t_{ij}^a = 5.67\},$$

$$\xi_3^a := \{(i,j) \in \xi^a \mid t_{ij}^a = 5.58\}, \qquad \xi_4^a := \{(i,j) \in \xi^a \mid t_{ij}^a < 5.58\},$$

and

$$\xi_5^s := \{(i,j) \in \xi^s \mid t_{ij}^s > 4.42\}, \qquad \xi_6^s := \{(i,j) \in \xi^s \mid t_{ij}^s = 4.42\},$$

$$\xi_7^s := \{(i,j) \in \xi^s \mid t_{ij}^s = 4.33\}, \qquad \xi_8^s := \{(i,j) \in \xi^s \mid t_{ij}^s < 4.33\},$$

The fractional transportation-processing time matrix T of the following related lexicographic linear fractional time minimizing crude-ore transportation problem:

$$\text{lexmin}\left[\Im=\frac{\sum_{i=1}^{6}\sum_{j=1}^{6}\alpha_{ij}x_{ij}}{\sum_{i=1}^{6}\sum_{j=1}^{6}\beta_{ij}x_{ij}}\;\middle|\;\begin{array}{l}\sum_{j=1}^{6}x_{ij}=a_i,\ (i=1,2,\ldots,6)\\ \sum_{i=1}^{6}x_{ij}=b_j,\ (j=1,2,\ldots,6)\\ \sum_{i=1}^{6}p_i.x_{ij}\le L_j b_j\\ x_{ij}\ge 0\end{array}\right]$$

can be written as:

$$T=\begin{bmatrix}\begin{bmatrix}\frac{e_3}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_7}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_7}\end{bmatrix}&\begin{bmatrix}\frac{e_1}{e_5}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_6}\end{bmatrix}\\ \begin{bmatrix}\frac{e_1}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_7}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_2}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_7}\end{bmatrix}\\ \begin{bmatrix}\frac{e_4}{e_7}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_7}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_7}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_2}{e_8}\end{bmatrix}\\ \begin{bmatrix}\frac{e_2}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_5}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_7}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_5}\end{bmatrix}&\begin{bmatrix}\frac{e_2}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_6}\end{bmatrix}\\ \begin{bmatrix}\frac{e_3}{e_5}\end{bmatrix}&\begin{bmatrix}\frac{e_2}{e_5}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_8}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_7}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_5}\end{bmatrix}\\ \begin{bmatrix}\frac{e_3}{e_6}\end{bmatrix}&\begin{bmatrix}\frac{e_1}{e_5}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_5}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_8}\end{bmatrix}&\begin{bmatrix}\frac{e_4}{e_7}\end{bmatrix}&\begin{bmatrix}\frac{e_3}{e_6}\end{bmatrix}\end{bmatrix}$$

Using initial basic feasible solution X^1, the vector-valued multipliers u_i^1, u_i^2, v_j^1, v_j^2 and w_{jk}^1, w_{jk}^2 $(i=1,2,\ldots,6;\ j=1,2,\ldots,6;\ k=1)$ are calculated as explained in Step 5 and then relative criterion vectors Δ_{ij} and $\Delta_{M+k,j}$ are computed. The flow vector $\Im(X^1)=(0,5/2,0,2,0,0,1,11/2,5,0,12,6,0,0,0,0)^{\mathrm{T}}$ indicates that fractional transportation-processing time = 1.334 and bottleneck flow = 5/2. As Δ_{ij} and $\Delta_{M+k,j}$ are not lexicographically greater than or equal to zero vector, therefore applying the selection rule of equation (13), the variable x_{42} becomes an entering basic variable and so δ_{42} is added to this variable and δ_{RS}, $\delta_{M+Y,S}$ is added to all the basic variables x_{RS}, $x_{M+Y,S}$. Change the current basic feasible solution to the new basic feasible solution using equations (14)-(17).

The new basic feasible solution X^2 is:

$$X^2=\begin{bmatrix}6&0&0&0&0&0\\0&15/2&7/2&0&0&0\\1/2&0&11/2&0&0&2\\1/2&5/2&0&2&0&0\\0&0&0&0&0&1\\0&0&0&2&1&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\20&0&2&12&3&3\end{bmatrix}$$

Fractional Transportation-processing Time: 1.334. Bottleneck Flow: 2.

As all the values of Δ_{ij} and $\Delta_{M+k,j}$ are not lexicographically greater than or equal to the zero vector, the current basic feasible solution is not optimal. Proceeding in the manner describes above, the further solutions are:

$$X^3 = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 15/2 & 7/2 & 0 & 0 & 0 \\ 1 & 0 & 11/2 & 0 & 0 & 3/2 \\ 0 & 5/2 & 0 & 5/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3/2 & 1 & 1/2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 19 & 0 & 2 & 12 & 3 & 4 \end{bmatrix}$$

Fractional Transportation-processing Time: 1.334. Bottleneck Flow: 3/2.

$$X^4 = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 15/2 & 2 & 0 & 0 & 3/2 \\ 1 & 0 & 7 & 0 & 0 & 0 \\ 0 & 5/2 & 0 & 5/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3/2 & 1 & 1/2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 19 & 0 & 5 & 12 & 3 & 1 \end{bmatrix}$$

Fractional Transportation-processing Time: 1.313. Bottleneck Flow: 3/2.

$$X^5 = \begin{bmatrix} 9/2 & 0 & 0 & 3/2 & 0 & 0 \\ 0 & 15/2 & 7/4 & 0 & 0 & 7/4 \\ 5/2 & 0 & 11/2 & 0 & 0 & 0 \\ 0 & 5/2 & 0 & 5/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 7/4 & 0 & 1 & 1/4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 16 & 0 & 9 & 12 & 3 & 0 \end{bmatrix}$$

Fractional Transportation-processing Time: 1.289. Bottleneck Flow: 29/4.

$$X^6 = \begin{bmatrix} 17/4 & 0 & 0 & 3/2 & 0 & 1/4 \\ 0 & 15/2 & 7/4 & 0 & 0 & 7/4 \\ 11/4 & 0 & 21/4 & 0 & 0 & 0 \\ 0 & 5/2 & 0 & 5/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 31/2 & 0 & 19/2 & 12 & 3 & 0 \end{bmatrix}$$

Fractional Transportation-processing Time: 1.289. Bottleneck Flow: 7

$$X^7 = \begin{bmatrix} 17/4 & 0 & 0 & 3/2 & 0 & 1/4 \\ 0 & 15/2 & 7/4 & 0 & 0 & 7/4 \\ 11/4 & 0 & 17/4 & 0 & 1 & 0 \\ 0 & 5/2 & 0 & 5/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 31/2 & 0 & 23/2 & 12 & 1 & 0 \end{bmatrix}$$

Fractional Transportation-processing Time: 1.289. Bottleneck Flow: 6

$$X^8 = \begin{bmatrix} 7/4 & 0 & 0 & 4 & 0 & 1/4 \\ 0 & 15/2 & 7/4 & 0 & 0 & 7/4 \\ 21/4 & 0 & 7/4 & 0 & 1 & 0 \\ 0 & 5/2 & 5/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 21/2 & 0 & 33/2 & 12 & 1 & 0 \end{bmatrix}$$

Fractional Transportation-processing Time: 1.289. Bottleneck Flow: 7/2

$$X^9 = \begin{bmatrix} 0 & 0 & 19/4 & 1 & 0 & 1/4 \\ 0 & 5 & 5/4 & 3 & 0 & 7/4 \\ 7 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 7 & 10 & 22 & 0 & 1 & 0 \end{bmatrix}$$

Fractional Transportation-processing Time: 1.289. Bottleneck Flow: 7/4

$$X^{10} = \begin{bmatrix} 0 & 0 & 3 & 1 & 0 & 2 \\ 0 & 5 & 3 & 3 & 0 & 0 \\ 7 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 7 & 10 & 15 & 0 & 1 & 7 \end{bmatrix}$$

The feasible solution X^{10} is optimal to lexicographic linear fractional time minimizing crude-ore transportation problem having optimal total transportation solution schedules with optimal fractional transportation-processing time = 1.270 and bottleneck flow = 13.

8. Concluding Remarks

An algorithm has been developed in this paper for solving linear fractional time minimizing transportation problems with impurities. The algorithm minimizes the vector of partial flows in a lexicographic order on the feasible set. This lexicographic approach based algorithm will prove to be useful for transportation system decision makers to solve fractional decision priority problems for management of transportation system.

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