Capacity of Middleton Class-A Impulsive Noise Channel with Binary Input

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Abstract: In many applications, the Middleton Class-A model is used to describe the impulsive noise. A very useful and interesting aspect for a channel affected by such, non-Gaussian noise, is to find an expression for the channel capacity. In this paper we present the calculation of capacity for a channel affected by additive Middleton Class-A noise (AWCN), with binary input. We considered the case when the source is uniform, but also when it is not uniform. The channel capacities for impulsive noise, for various values of the parameters that describe its model, are compared with those for the additive white Gaussian noise (AWGN) channel. The numerical results showed that when the parameters A and T are close to 1, the capacity for impulsive noise channel is equal to that of Gaussian channel. When A and T decrease, the AWCN capacity grows. When the probability p₀, the probability of bit 0, grows or when the encoding rate decreases, each channel capacity decreases. The Shannon limit values are also given for different encoding rates in the case of the two channels. We have shown that Signal-to-Noise Ratio (SNR₀) in dB given only by the Gaussian component of AWCN is closer to SNR₀ in dB of AWGN, as the AWCN capacity increases.

Keywords: capacity channel, impulsive noise, Middleton Class-A model, Shannon limit

1 Introduction

The major concern regarding communication systems is to transmit data at higher speeds and with fewer errors, thus to ensure efficient communication on noisy channels. As a measure for this efficiency, Shannon proposed the notion of channel capacity [1]. He considered that, to ensure efficient communication, the channel transmission rate, has to be lower than its capacity, irrespective of the noise on the channel. Thereby, the meaning of capacity is that of upper limit of the transmission rate for reliable communication on a noisy channel.

Shannon has also mentioned that, by encoding the channel, we can achieve a secure data transmission [1].

The channel capacity thus depends on the code being used, on the number of antennas in the communication system, and also on the noise that affects the channel. Among the error correcting codes, the ones that have remarkable performances and give the systems the possibility to operate close to Shannon limit are the turbo codes [2]. Afterwards, space-time block codes [3] have improved the communication performances by using the advantage of diversity gain and system capacity. In this case, of systems with multiple antennas, the channel capacity depends on the number of emitting and receiving antennas, respectively, increasing linearly with this number [4].

Most of the time, the capacity has been calculated for channels affected by Additive White Gaussian Noise (AWGN), when the channel state is known [5] or not [6], ignoring other sources of noise, like industrial noise, man-made activity such as automobile spark plugs [7], microwave ovens [8] and network interference [9], noises known to be non-Gaussian (or impulse noise).

The Middleton Class-A model is frequently used to describe the impulsive noise. This was used to investigate the performances of turbo codes over an AWCN versus AWGN channel, when the encoder has two identical recursive systematic convolutional encoders with constraint length 5, rate 1/2, generator matrix G=[1,
23/25] and Binary Phase Shift Keying (BPSK) modulation. These are significantly weaker than the ones for Gaussian noise and that is why [10] proposes a decoder that is to be used for eliminating the Middleton Class-A impulsive noise. Most of the systems affected by non-Gaussian noise suffer performance degradation for high Signal-to-Noise Ratio (SNR) values [11].

For a channel affected by impulsive noise of type Middleton Class-A, the channel capacity is obtained considering its model to be a Markov chain [12]. The simulations were done for various values of the parameters that describe the pattern of the impulsive noise, called AWCN, and have shown that for $A \geq 10$, the capacity of the AWCN channel is similar to the one for AWGN. The more impulsive the noise is, the more the channel capacity becomes larger than for an AWGN channel.

To the best of our knowledge, no research results have been published for calculating the capacity of a channel affected by impulsive noise of type Middleton Class-A, with binary input. This is the case when using an antipodal modulation, for example BPSK. To fill this, our paper presents the calculation of capacity for AWCN channel and shows the differences between the AWGN channel capacities and those of the AWCN channel for this type of channel input, for various values of the parameters that describe the model of the impulsive noise.

We considered the case of a uniform source, but also the case when the probabilities of the two symbols, 0 and 1, respectively, differ (the source is not uniform), like in [13].

The paper is structured as follows. Section 2 describes the Middleton Class-A impulse noise model and Section 3 presents the capacity expressions for the AWGN and AWCN channels and also for Shannon limit. The numerical results are shown in Section 4 and conclusions are highlighted in Section 5.

2 Middleton Class-A Model

In many applications, non-gaussian noise appears in addition to Gaussian noise. Some of its sources are: automotive ignition noise, power transmission lines, devices with electromechanical switches (photocopy machines, printers), microwave ovens etc. [14]. There are many statistical models for impulsive noise; in this study we assume the Middleton Class-A model. This type of noise has two components: a Gaussian one, with variance $\sigma_g^2$, and an impulsive one, with variance $\sigma_i^2$. The probability density function (PDF) of impulsive noise is a Poisson weighted sum of Gaussian distributions and it is given by [10]:

$$p(n) = \sum_{m=0}^{\infty} \frac{A^m e^{-A}}{\sqrt{2\pi} \cdot m! \cdot \sigma_m} \cdot \exp\left(-\frac{n^2}{2\sigma_m^2}\right)$$  

(1)

The significance of quantities in (1) is as follows: $m$ is the number of active interferences (or impulses), $A$ is the impulse index and it indicates the average number of impulses during interference time. This parameter describes the noise as follows: as $A$ decreases, the noise gets more impulsive; conversely, as $A$ increases, the noise tends towards AWGN. $\sigma_m^2$ is given by:

$$\sigma_m^2 = \sigma^2 \cdot \frac{\frac{m}{T} + 1}{1+T}$$  

(2)

where: $\sigma^2 = \sigma_g^2 + \sigma_i^2$ is the total noise power and

$$T = \frac{\sigma_g^2}{\sigma_i^2}$$  

(3)

is the Gaussian factor. We can observe from (3) that for low $T$ values, the impulsive component prevails, and that for high values, the AWGN component.

An impulsive noise sample is given by [15]:

$$n = x_g + \sqrt{K_m} \cdot w$$  

(4)

where $x_g$ is the white Gaussian background noise sequence with zero mean and variance $\sigma_g^2$, $w$ is the white Gaussian sequence with zero mean and variance $\sigma_i^2/A$ and $K_m$ is the Poisson distributed sequence, whose PDF is characterized by the impulse index $A$.

3 Capacity of AWGN and AWCN Channels

3.1 Binary Input AWGN Channel Capacity

The capacity of an AWGN channel with binary input is obtained from the expression for mutual information, when the probabilities of the input symbols are equal. Its expression is:

$$C_{AWGN} = 1 - \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/(2\sigma^2)} \cdot \log_2(1 + e^{y/(2\sigma^2)}) dy$$  

(5)

If we want to express the AWGN channel capacity based on the signal-to-noise ratio (SNR), then we must take into account that for the input symbols with the energy equal to 1:

$$SNR = \frac{1}{2\sigma^2} \Rightarrow \sigma = \sqrt{2 \cdot SNR}$$  

(6)

Replacing (6) into (5), we get:

$$C_{AWGN} = 1 - \int_{-\infty}^{+\infty} \sqrt{\frac{SNR}{\pi}} e^{-y^2/(2\cdot SNR)} \cdot \log_2(1 + e^{y^2/(2\cdot SNR)}) dy$$  

(7)

In [13] the bias of AWGN channel capacity value is presented when the input symbols are not equiprobable.
Noting these probabilities with $p_0$ and $p_1$, respectively, the capacity becomes:

$$C_{\text{AWGN}}^{\text{bias}} = -p_0 \int_{-\infty}^{+\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-y^2/(2\sigma^2)} \cdot \log_2(p_0 + p_1 \cdot e^{2y/\sigma^2}) dy - p_1 \int_{-\infty}^{+\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-(y-1)^2/(2\sigma^2)} \cdot \log_2(p_0 \cdot e^{-2y/\sigma^2} + p_1) dy$$

(8)

In terms of SNR, (8) becomes:

$$C_{\text{AWGN}}^{\text{bias}} = -p_0 \int_{-\infty}^{+\infty} \sqrt{\frac{SNR}{\pi}} e^{-\text{SNR}(y+1)^2} \cdot \log_2(p_0 + p_1 \cdot e^{A\text{SNR}}) dy - p_1 \int_{-\infty}^{+\infty} \sqrt{\frac{SNR}{\pi}} e^{-(y-1)^2/(2\sigma^2)} \cdot \log_2(p_0 \cdot e^{-2y/\sigma^2} + p_1) dy$$

(9)

### 3.2 Binary Input AWCN Channel Capacity

In [12], the AWCN channel capacity for continuous input is given. This is obtained by modelling the AWCN channel through a Markov chain and assuming that both the transmitter and the receiver know the channel state. The probability of the channel's $m$ state is:

$$\pi_m = e^{-A} \cdot \frac{A^m}{m!}, 0 \leq m$$

(10)

and the AWCN channel average capacity is:

$$C = \sum_{m=0}^{\infty} \pi_m \cdot C_m$$

(11)

where $C_m$ is the AWGN channel capacity, with zero mean and dispersion $\sigma_m^2$. Considering (5) and (7), we obtain the capacity for an AWCN channel with binary input, as:

$$C_{\text{AWCN}} = \sum_{m=0}^{\infty} e^{-A} \cdot \frac{A^m}{m!} \cdot (1 -$$

$$- \int_{-\infty}^{+\infty} \frac{1}{\sigma_m \sqrt{2\pi}} e^{-(y+1)^2/(2\sigma_m^2)} \cdot \log_2(1 + e^{2y/\sigma_m^2}) dy)$$

(12)

The capacity in terms of SNR is obtained by substituting (6) into (2):

$$\sigma_m^2 = \frac{1}{2 \cdot \text{SNR}} \frac{m + T}{1 + T}$$

(13)

### 3.3 Shannon Limit

In this subsection, we present the expression for the Shannon limit as a function of probability $p_0$.

Shannon's **Lossy Joint Source-Channel Coding Theorem** states that, for a given memoryless source and channel pair and for sufficiently large source-block lengths, the source can be transmitted via a source-channel code over the channel at a transmission rate of source symbols/channel symbol and reproduced at the receiver end within an end-to-end distortion given by the following condition is satisfied [16]:

$$R_c \cdot R(D) < C$$

(17)

where $C$ is the channel capacity and $R(D)$ is the source rate-distortion function and $D$ is the distortion, generally
taken to be the expected value of a single letter distortion measure [17]. For a discrete binary non-uniform source with the probability of 0 equal to \( p_0 \), we have \( D = P_e \) (that is, the bit error probability (BER) under the Hamming distortion measure). The rate becomes:

\[
R(P_e) = \left\{ \begin{array}{ll}
H_b(p_0) - H_b(P_e), & 0 \leq P_e \leq \min\{p_0, p_1\} \\
0, & P_e > \min\{p_0, p_1\}
\end{array} \right.
\]

where \( p_1 = 1 - p_0 \) and

\[
H_b(x) = -x \cdot \log_2 x - (1 - x) \cdot \log_2 (1 - x)
\]

is the binary entropy function.

The capacity of the AWGN or AWCN channel is a function of \( SNR \). When using an error correcting code with an encoding rate \( R_c \), the \( SNR \) expression is the ratio between the energy of the uncoded bit and the power spectral density of noise, that is:

\[
SNR_b = \frac{1/R_c}{2\sigma^2},
\]

resulting in the noise standard deviation

\[
\sigma = \frac{1}{\sqrt{2 \cdot R_c \cdot SNR_b}}
\]

The optimal \( SNR_b \) value that guarantees a certain \( P_e \) value is called the Shannon limit. This can be found by assuming equality in relation (17). The Shannon limit cannot be explicitly solved for BPSK-modulated channels, due to the lack of a closed-form expression, so it is computed via numerical integration. In the next section, the Shannon limit values are given for the AWGN and AWCN channels, when \( P_e = 10^{-5} \). These values are useful when comparing the performances of an error correcting code with theoretical limit.

4 Numerical Results

In this section, we present numerical results for the capacity of the channel affected by impulsive noise of type Middleton Class-A, with binary input, and also the optimum values for \( SNR_b \) in dB based on the encoding rate, highlighting the difference from the AWGN channel. The results have been obtained by varying the non-Gaussian noise model parameters \( A \) and \( T \), respectively. The following pairs were considered: \((A = 0.1, T = 0.1)\) and \((A = 0.01, T = 0.01)\). The last set corresponds to a highly impulsive noise. We considered the case of a uniform source (when \( p_0 = 0.5 \)), as well as the case of a non-uniform source, like in [13], with probabilities \( p_0 = 0.8 \) and \( p_0 = 0.9 \), respectively. For the coding rate we used two values: 1/2 and 1/3, respectively.

In [12] it has been shown that, for an AWCN channel modelled as a Markov chain, as the noise gets more impulsive, its capacity is greater than that for AWGN. In Fig.1, we represented the AWCN channel capacity for various values of the \( A \) and \( T \) parameters and AWGN, respectively, for coding rate 1/2. It can be observed that the result is similar with to that obtained by [12], that is, for values greater than or equal to 1 of the Middleton Class-A noise model parameters, the channel capacity is close to that of AWGN, and for lower values, is bigger. So, the more impulsive the noise is, the more the AWCN channel capacity is greater than that of AWGN.

Next, in Figs. 2, 3, 4, 5, the AWGN and AWCN channel capacities are represented for the sets \((A, T)=(0.1, 0.1), (0.01, 0.01)\), cases that describe a highly impulsive noise, with coding rates of 1/2 and 1/3, respectively, and various probabilities for the bit 0, \( p_0 = 0.5, 0.8 \) and \( 0.9 \), respectively. It can be observed that, for all situations, the AWCN channel capacity is greater than that of the AWGN channel. Another aspect is that, for both AWGN and AWCN, the bias of the channel capacity for a uniform source is larger than that of a non-uniform source, and as the probability of the bit 0 is higher, the channel capacity is lower.

The optimum values of \( SNR_b \) calculated for the entire domain of encoding rates \( R_c \in [0, 1] \), according to section 3.3, are represented in Fig. 6 for AWCN, with parameters \( A = 0.1, T = 0.1 \). It can be observed that, for an imposed encoding rate, the optimum \( SNR_b \) value is significantly larger for AWGN, than that of AWCN.

To compare \( SNR_b \) in dB given only by the Gaussian component of AWCN with \( SNR_b \) in dB of AWGN, in Fig. 7 we represented the same simulation as in Fig. 6, only that for AWCN the curves are shifted up with \( 10 \log_{10}((T + 1)/T) \) (lg stands for decimal logarithm). It can be observed that the \( SNR_b \) given by the Gaussian component of AWCN is
greater than $SNR_b$ of AWGN. This shows that if an error correcting code fully eliminates the impulsive component of AWCN, then it will have somewhat lower performances than those of the AWGN channel.

Figs. 8 and 9 show the same simulation as in Figs. 6 and 7, but for AWCN channel with parameters $A = 0.01$, $T = 0.01$.

For $T = 0.1$, $10 \log((T + 1)/T)$ is 10.414 dB, and for $T = 0.01$, it is 20.043 dB. These values explain the difference from Figs. 2, 3, 4 and 5 between the $SNR_b$ values that lead to the same capacity of AWGN and AWCN, of approximately 10 dB when $T = 0.1$ and approximately 20 dB when $T = 0.01$.

Table 1 shows the values of the Shannon limit in the case of AWGN and AWCN (with $A = 0.1$, $T = 0.1$ and $A = 0.01$, $T = 0.01$) channels, for the encoding rates $1/2$ and $1/3$ and for the values of $p_0$ equal to 0.5, 0.8, 0.9, respectively. Obviously, the values are smaller when $p_0$ grows or when the encoding rate decreases. We can
observe the difference of approximately 10 dB (that is $10 \log((T+1)/T)$ for $T=0.1$) and of approximately 20 dB (that is $10 \log((T+1)/T)$ for $T=0.01$) between the Shannon limit for AWGN and that of AWCN. In the fifth and seventh columns, in paranthesis, are given the difference values in dB between $SNR_b$ for the Gaussian component of AWCN and $SNR_b$ in dB of AWGN. It can be observed that when $A=0.01$ and $T=0.01$, the Shannon limit, for AWCN expressed as $SNR_b$ of gaussian component is very close to that for AWGN (the same thing can be seen in Fig. 9 for the full range of coding rates). This shows that if an error correcting code fully eliminates the impulsive component of AWCN, then it will have performances closer to those of the AWGN channel. This is shown through simulations in [10], when a turbo code is used on the AWCN channel.
Table 1: Shannon Limit in $SNR_b$ [dB] for AWGN and AWCN (with $A=0.1, T=0.1$ and $A=0.01, T=0.01$) Channels at $P_e = 10^{-5}$.

<table>
<thead>
<tr>
<th>$R_c$</th>
<th>$p_0$</th>
<th>AWGN</th>
<th>Shannon limit</th>
<th>AWCN</th>
<th>Difference for AWCN</th>
<th>Difference for AWCN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$A=0.1$</td>
<td>$A=0.1$</td>
<td>$A=0.01$</td>
<td>$A=0.01$</td>
</tr>
<tr>
<td>1/3</td>
<td>0.5</td>
<td>-0.498</td>
<td>-10.352</td>
<td>9.854</td>
<td>-20.484</td>
<td>19.986</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>-2.244</td>
<td>-12.142</td>
<td>9.898</td>
<td>-22.235</td>
<td>19.991</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>-4.398</td>
<td>-14.398</td>
<td>9.931</td>
<td>-24.392</td>
<td>19.994</td>
</tr>
<tr>
<td>1/2</td>
<td>0.5</td>
<td>0.185</td>
<td>-9.557</td>
<td>9.742</td>
<td>-19.791</td>
<td>19.976</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>-1.808</td>
<td>-11.648</td>
<td>9.840</td>
<td>-21.793</td>
<td>19.985</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>-4.138</td>
<td>-14.040</td>
<td>9.902</td>
<td>-24.129</td>
<td>19.991</td>
</tr>
</tbody>
</table>

5 Conclusion

This paper has established a mathematical expression for the AWCN channel capacity with binary input. The expression was obtained by modeling the AWCN channel through a Markov chain and assuming that both the emitter and the receiver know the channel state, similar to [12]. Likewise, the expression for the capacity bias was given when the input is non-uniform (that is the input symbols probabilities are not equal).

Then, we made an analysis of the AWGN and AWCN channel capacity values for various parameters. When $A$ and $T$ are close to 1 the AWCN capacity is equal to that of AWGN. When $A$ and $T$ decrease, the AWCN capacity grows.

When the probability $p_0$ grows or when the encoding rate decreases, each channel capacity decreases. This is obvious, because when $p_0$ grows, the source redundancy grows, and when the encoding rate drops, imposing a constant energy for the encoded bit, the uncoded bit energy becomes higher.

The Shannon limit values are also given for different encoding rates in the case of the two channels. We have shown that $SNR_b$ in dB given only by the Gaussian component of AWCN is closer to $SNR_b$ in dB of AWGN, as the AWCN capacity increases. This explains the differences between the $SNR_b$ values that lead to the same AWGN and AWCN capacity, with parameter $T$ of approximately $10\log((T+1)/T)$ dB. When $T$ is small enough, this shows that if an error correcting code fully eliminates the impulsive component of AWCN, then it will have performances in terms of BER closer to those of the AWGN channel. In [10] is shown by simulations that when a turbo code is used on the AWCN channel with a matched turbo decoder at reception, the BER performances are closer to those of AWGN channel when $T = 0.01$ compared to $T = 0.1$ case, confirming the previous mentioned result.

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References


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