Fuzzy Topological Properties on Fuzzy Function Spaces

A. I. Aggour$^1$, and F. E. Attounsi$^2$

$^1$Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt.
$^2$Department of Mathematics, Faculty of Science, 7th University, Bani Walid, Libya.

Received: 28 Jan. 2013, Revised: 10 Mar. 2013, Accepted: 19 Mar. 2013

Abstract: In this paper, we study the fuzzy continuous convergence of fuzzy nets on the set $FC(X, Y)$ of all fuzzy continuous functions of a fuzzy topological space $X$ into another $Y$. Also, we introduce fuzzy topologies on fuzzy function spaces.

Keywords: Fuzzy upper limit of fuzzy nets, Fuzzy function spaces, Fuzzy continuously converges nets, Fuzzy jointly continuous topologies, Fuzzy splitting topologies.

1 Introduction and Preliminaries

The notion of convergence is one of the basic notion in analysis. In this paper, fuzzy continuous convergence theory of fuzzy nets on the set $FC(X, Y)$ of fuzzy continuous functions of an fts $X$ into another $Y$ is presented. In 1976, the concept of fuzzy topology was introduced by R. Lowen [4].

In 1980, Pu and Liu introduced the notions of fuzzy nets and $Q$-neighborhoods. The concept of the $Q$-neighborhood reflect the features of the neighborhood structure in fuzzy topological spaces. By this new neighborhood structure the Moore-Smith convergence theory was established [6]. In this paper, we will give new concepts of fuzzy continuous convergence of fuzzy nets on the set $FC(X, Y)$. Also, we introduce the notions of fuzzy splitting topologies and fuzzy jointly continuous topologies on the fuzzy functions spaces.

Let $X$ be an arbitrary nonempty set. A fuzzy set in $X$ is a mapping from $X$ to the closed unit interval $I = [0, 1]$, that is, an element of $I^X$. A fuzzy point $x_t$ is a fuzzy set in $X$ defined by $x_t(x) = t$ and $x_t(y) = 0$ for all $y \neq x$, whose support is the single point $x$ and whose value is $t \in (0, 1)$ [6]. We denote by $FP(X)$ the collection of all fuzzy points in $X$.

Definition 1. [12] Let $\mu, \eta \in I^X$. We define the following fuzzy sets:

i) $\mu \wedge \eta \in I^X$, by $(\mu \wedge \eta)(x) = \min\{\mu(x), \eta(x)\}$, for each $x \in X$.

ii) $\mu \vee \eta \in I^X$, by $(\mu \vee \eta)(x) = \max\{\mu(x), \eta(x)\}$, for each $x \in X$.

iii) $\mu' \in I^X$, by $\mu'(x) = 1 - \mu(x)$, for each $x \in X$.

Definition 2. [4] A fuzzy topology $\mathcal{S}$ on a non empty set $X$ is a family of fuzzy subsets of $X$ such that:

i) $\mathcal{S}$ contains all constant fuzzy subsets of $X$.

ii) For each $\mu, \eta \in \mathcal{S}$, $\mu \wedge \eta \in \mathcal{S}$.

iii) If $\{\mu_a\}_{a \in A}$ is a subfamily of $\mathcal{S}$, then $\bigvee_{a \in A} \mu_a \in \mathcal{S}$.

The pair $(X, \mathcal{S})$ is called a fuzzy topological space denoted by $fxts$. Each member of $\mathcal{S}$ is called fuzzy open set and its complement is called fuzzy closed set.

Definition 3. [6] Let $(X, \mathcal{S})$ be an fts and $\mu, \eta \in I^X$. Then:

i) A fuzzy point $x_t$ is said to be quasi-coincident with $\mu$ denoted by $x_t \approx \mu$ if $\mu(t'(x)) > 1 + \mu(x)$.

ii) $\mu$ is called quasi-coincident with $\eta$, denoted by $\mu \approx \eta$, if there exists $x \in X$ such that $\mu(x) + \eta(x) > 1$. If $\mu$ is not quasi-coincident with $\eta$, then we write $\mu \not\approx \eta$.

iii) A fuzzy subset $\mu$ of $X$ is called a neighborhood (or a nbd, for short) of a fuzzy point $x_t$ iff there exists a fuzzy open set $\nu$ of $X$ such that $x_t \in \nu \subseteq \mu$. The family $N_0$ of all nbd of $x_t$ is called the system of nbd of $x_t$.

iv) $\mu$ is called $Q$-neighborhood of a fuzzy point $x_t \in FP(X)$ if there exists a fuzzy open set $\eta \in \mathcal{S}$ such that $x_t \approx \eta$ and $\eta \leq \mu$. The class of all open $Q$-neighborhoods of $x_t$ is denoted by $N_0$.

Definition 4. [7] A map $f : X \rightarrow Y$ is called fuzzy continuous if the inverse image of every fuzzy open subset of $Y$ is fuzzy open subset of $X$. 

* Corresponding author e-mail: e-mail:atifaggour@yahoo.com
Theorem 1. [7] A map \( f : X \rightarrow Y \) is fuzzy continuous iff for each fuzzy point \( x_0 \) in \( X \) and each fuzzy open nbd \( V \) of \( f(x_0) \), there exists fuzzy open nbd \( U \) of \( x_0 \) such that \( f(U) \subseteq V \).

Definition 5. [6] Let \( \mu \in \mathcal{S}(X) \) and \( \nu \in \mathcal{S}(Y) \). The closure of \( \mu \), denoted by \( \text{cl}(\mu) \), is defined by: \( x \in \text{cl}(\mu) \) iff for each \( \eta \in N_{\mu}^0 \), we have \( \eta \mu \). The fuzzy set \( \mu \) is called closed if \( \mu = \text{cl}(\mu) \).

Definition 6. [5] Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be fuzzy topological spaces, then the fuzzy topology \( \tau = \tau_1 \times \tau_2 \) on the set \( X \times Y \) is defined as the initial fuzzy topology on \( X \times Y \) making the projection mappings \( P_1 : X \times Y \rightarrow X \) and \( P_2 : X \times Y \rightarrow Y \) fuzzy continuous.

Definition 7. [6] A mapping \( S : D \rightarrow FP(X) \) is called a fuzzy net in \( X \) and is denoted by \( \{S(n) : n \in D\} \) or \( \{S_n : n \in D\} \), where \( D \) is a directed set.

Definition 8. [6] A fuzzy net \( \{\xi(m) : m \in M\} \) in \( X \) is called a fuzzy subnet of a fuzzy net \( \{S(n) : n \in D\} \) iff there is a mapping \( f : M \rightarrow D \) such that:

i) \( \xi_{\mu_m} = S_{f(m)} \), for each \( m \in M \).

ii) For each \( n \in D \) there exists some \( m \in M \) such that, if \( \mu \in M \) with \( \mu \geq m \), then \( f(\mu) \geq n \).

Definition 9. [9] A fuzzy net \( \{S(n) : n \in D\} \) in an fts \( X \) is said to be fuzzy converges to \( x_0 \) if for each fuzzy open nbd \( V \) of \( x_0 \) there is some \( m_0 \in D \) such that \( n \geq m_0 \) implies \( S(n) \in V \).

Definition 10. [2] A fuzzy net \( \{f_m : m \in M\} \) in \( FC(X, Y) \) is said to be fuzzy continuously converges to \( f \) in \( FC(X, Y) \) iff for every \( x_0 \) in \( X \) and for every fuzzy open nbd \( V \) of \( f(x_0) \) in \( Y \) there exists an element \( m_0 \in M \) and a fuzzy open nbd \( U \) of \( x_0 \) in \( X \) such that \( f_m(U) \subseteq V \), for every \( m \in M \), \( m \geq m_0 \).

2 Fuzzy Continuously Convergence of Fuzzy Nets

Definition 11. Let \( \mathcal{I}^X \) be the set of all fuzzy subsets of the fuzzy topological space \( X \). If \( D \) is a directed set, then by \( \text{lim}(\mu_\lambda) \), \( \mu_\lambda \in \mathcal{I}^X \), where \( \mu_\lambda \in \mathcal{I}^X \), we denote the fuzzy upper limit of the fuzzy net \( \{\mu_\lambda : \lambda \in D\} \) in \( \mathcal{I}^X \), that is \( \mu_\lambda \text{lim}(\mu_\lambda) \) iff for every \( \lambda_0 \in D \) and for every fuzzy open nbd \( V \) of \( x_0 \) in \( X \) there exists an element \( \lambda \in D \) for which \( \lambda \geq \lambda_0 \) and \( \mu_\lambda \text{lim}(\mu_\lambda) \).

Definition 12. Let \( D \) be a directed set, and for each \( m \in D \) there are a directed set \( \mathcal{E}_m \) and fuzzy net \( \{f_m(n) : n \in \mathcal{E}_m\} \) in \( FC(X, Y) \). Then, for the directed set \( T = D \times \prod_{n \in \mathcal{E}_m} \mathcal{E}_m \) (ordered by \( (n_2, g) \geq (n_1, h) \) iff \( n_2 \geq n_1 \) and \( g(n) \geq h(n) \) for each \( n \in D \)), we have a fuzzy net \( f(T) : T \rightarrow FC(X, Y) \) defined by \( f_{(m,g)} = f_m(g(n)) \), \( n \in D, g \in \prod_{n \in \mathcal{E}_m} \mathcal{E}_m \). The fuzzy net \( f(T) \) is called the induced fuzzy net in \( FC(X, Y) \).

Definition 13. Let \( C \) be a class of pairs \( (S, f) \), where \( S \) is a fuzzy net in \( FC(X, Y) \) and \( f \) in \( FC(X, Y) \). We say that \( C \) is a continuously convergence class for \( FC(X, Y) \) iff the following axioms listed below are satisfied. For convenience, we write \( SC \)-converges to \( f \) whenever \( (S, f) \in C \):

1. If \( S = \{f_n : n \in D\} \) is a fuzzy net in \( FC(X, Y) \) such that \( f_n \equiv f \) for each \( n \), then \( (S, f) \in C \);
2. If \( (S, f) \in C \), then for every subnet \( T \) of \( S \), \( (T, f) \in C \);
3. If \( S \) does not \( C \)-converges to \( f \), then there is a subnet of \( S \), no subnet of which \( C \)-converges to \( f \).

Theorem 2. A fuzzy net \( \{f_m : m \in M\} \) in \( FC(X, Y) \) fuzzy continuously converges to \( f \) in \( FC(X, Y) \) iff for every fuzzy net \( \{\eta_m : m \in M\} \) in \( X \) which fuzzy converges to \( x_0 \) in \( X \) we have that the fuzzy net \( \{f_m(\eta_m) : (n, m) \in D \times M\} \) fuzzy converges to \( f(x_0) \) in \( Y \).

Proof. Let \( x_0 \) in \( X \) and let \( V \) be a fuzzy open nbd of \( f(x_0) \) in \( Y \) such that for every \( m \in M \) and for every fuzzy open nbd \( U \) of \( x_0 \) in \( X \) there exists \( m_0 \in M \) and \( m_0 \) in \( D \) such that \( f_m(U) \subseteq V \), for every \( n \geq m_0 \). Since the fuzzy net \( \{S(n) : n \in D\} \) fuzzy converges to \( x_0 \) in \( X \). There exists \( m_0 \in \mathcal{I}^X \) such that \( S(n) \in U \), for every \( n \in A \), \( n \geq m_0 \). Let \( m_0 \in \mathcal{I}^X \) such that \( m_0 \in \mathcal{I}^X \). Hence, the fuzzy net \( \{f_m(S(n)) : (n, m) \in D \times M\} \) fuzzy converges to \( f(x_0) \) in \( Y \).

Theorem 3. A fuzzy net \( \{f_m : m \in M\} \) in \( FC(X, Y) \) fuzzy continuously converges to \( f \) in \( FC(X, Y) \) iff \( \lim f_m^{-1}(K) \subseteq f^{-1}(K) \), for every fuzzy closed set \( K \) of \( Y \).

Proof. Let \( \{f_m : m \in M\} \) be a fuzzy net in \( FC(X, Y) \), which fuzzy continuously converges to \( f \) and let \( K \) be arbitrary fuzzy closed subset of \( Y \). Let \( \text{lim}(f_m^{-1}(K)) \) and let \( w \) be an arbitrary fuzzy open nbd of \( f(x_0) \) in \( Y \). Since the fuzzy net \( \{f_m : m \in M\} \) fuzzy continuously converges to \( f \), there exists a fuzzy open nbd \( V \) of \( x_0 \) in \( X \) and an element \( m_0 \in M \) such that \( f_m(V) \subseteq w \), for every \( m \in M \), \( m \geq m_0 \). Then \( V \cap f_m^{-1}(K) \). Hence, \( f_m(V) \cap f_m^{-1}(K) \subseteq K \). So, \( w \cap K \). This means that \( f(x_0) \cap K = K \). Thus \( \text{lim}(f_m^{-1}(K)) \).
Conversely, let \( \{ f_m : m \in M \} \) be a fuzzy net in \( FC(X, Y) \) and \( f \in FC(X, Y) \) such that \( \lim(f_m(K)) \subseteq f^{-1}(K) \), for every fuzzy closed subset \( K \) of \( Y \). Let \( x_A \) be a fuzzy point in \( X \) and \( w \) be a fuzzy open nbd of \( f(x_A) \) in \( Y \). Let \( K = \{ w \} \), then \( x_A \notin f^{-1}(K) \). Therefore, \( f_m(K) \nsubseteq \{ w \} \), for every \( m \in M \), \( m \geq m_0 \). Then, we have that \( V \subseteq \lim(f_m(K)) \subseteq f^{-1}(K) \subseteq f_m^{-1}(w) \). Therefore, \( f_m(V) \subseteq w \), for every \( m \in M \), \( m \geq m_0 \). Hence, the fuzzy net \( \{ f_m : m \in M \} \) fuzzy continuously converges to \( f \).

**Theorem 4.** Suppose that \( \{ \eta_n : n \in N \} \) is a fuzzy net in \( FC(X, Y) \) such that \( \eta(n) = \eta \) for every \( n \in N \). Let \( \{ S(e) : e \in E \} \) be a fuzzy net in \( X \). For every \( \eta \in FC(X, Y) \), the fuzzy net \( \{ \eta(S(e)) : e \in E \} \) fuzzy converges to \( \eta(x_A) \) in \( Y \). Therefore, \( \eta(A) = \eta(S) \) fuzzy converges to \( \eta(x_A) \). Hence, \( \eta(n) \) fuzzy continuously converges to \( \eta \in FC(X, Y) \).

**Theorem 5.** If \( \{ \eta_n : n \in N \} \) is a fuzzy net in \( FC(X, Y) \) which fuzzy continuously converges to \( \eta \in FC(X, Y) \) and \( \{ \xi_m(m) : m \in M \} \) is a subnet of \( \{ \eta_n : n \in N \} \), then the fuzzy net \( \{ \xi_m(m) : m \in M \} \) is fuzzy continuously converges to \( \eta \).

**Theorem 6.** Suppose that \( \{ \eta_n : n \in N \} \) is a fuzzy net in \( FC(X, Y) \) and \( \{ \xi_m(m) : m \in M \} \) is a subnet of \( \{ \eta_n : n \in N \} \), then there is a map \( f : M \to D \) such that:

(i) \( \xi_m(m) = \eta(f(m)) \);

(ii) for the element \( n_0 \in D \), there is \( m_0 \in M \) such that if \( m \geq m_0, m \in M \), then \( f(m) \geq n_0 \).

Hence, we have \( \xi_m(U) = \eta_m(U) \subseteq V \), for every \( m \geq m_0, m \in M \). Thus, the fuzzy net \( \{ \xi_m(m) : m \in M \} \) fuzzy continuously converges to \( \eta \).

**Theorem 7.** Let \( \{ \eta_n : n \in N \} \) be a fuzzy net in \( FC(X, Y) \) continuously converges to \( f \) and \( \{ \xi_m(m) : m \in M \} \) be a fuzzy net in \( FC(Y, Z) \) continuously converges to \( f(n) \). Then, the induced fuzzy net \( \{ \xi_m(m) : m \in M \} \) is a fuzzy net in \( FC(Y, Z) \) continuously converges to \( f \).

In this section, we introduce fuzzy splitting topology and fuzzy jointly continuous topology on the set \( FC(Y, Z) \). Also, we give a necessary and sufficient condition for the existence of the splitting and jointly continuous topology on the set \( FC(Y, Z) \).

**Notation:** By \( FC \) denoted the class of all pairs \( \{ f_n : n \in D \} \) which fuzzy continuously converges to \( f \). If \( \mathcal{S} \) is a fuzzy topology on \( FC(Y, Z) \), then by \( FC(\mathcal{S}) \) we denote the class of all pairs \( \{ f_n : n \in D \} \) which fuzzy continuously converges to \( f \) in \( FC(Y, Z) \) in the fuzzy topology \( \mathcal{S} \).

**Definition 14.** A fuzzy topology \( \mathcal{S} \) on \( FC(Y, Z) \) is called fuzzy splitting if for every \( \mathcal{S} \)-fuzzy open \( f \), the fuzzy continuity of the map \( F : X \times Y \to Z \) implies that of the map \( \widehat{F} : X \to FC_3(Y, Z) \) for which \( \hat{F}(x, y_m) = F(x)(y_m) \).
Theorem 8. There exists the greatest splitting topology on the set \( FC(Y,Z) \).

Proof. Suppose that \( \{ \tau_i \}_{i \in A} \) be a family of fuzzy splitting topologies on \( FC(Y,Z) \) and let \( \mathcal{F} = \sup \{ \tau_i \} \). For any fuzzy topological space \( X \), let \( \hat{F} : X \times Y \to Z \) be a fuzzy continuous map. Consider the map \( \hat{F} : X \to FC_2(Y,Z) \). Let \( x_i \in X \) and let \( U \) be a fuzzy open nbd of \( \hat{F}(x_i) \) in \( FC(Y,Z) \). Since \( \mathcal{F} \supseteq \tau_i \), we have that \( U \in \tau_i \) for some \( i \). Also, since \( \hat{F} : X \to FC_2(Y,Z) \) is fuzzy continuous, there exists a fuzzy open nbd \( V \) of \( x_i \) such that \( \hat{F}(V) \subseteq U \). Thus, the map \( \hat{F} \) is fuzzy continuous and the fuzzy topology \( \mathcal{F} \) is fuzzy splitting.

Theorem 9. A fuzzy topology \( \mathcal{F} \) on \( FC(Y,Z) \) is fuzzy splitting topology iff \( FC^* \subseteq FC(\mathcal{F}) \).

Proof. Let \( \mathcal{F} \) be a fuzzy splitting topology on \( FC(Y,Z) \) and let \( \{ (f_j : \lambda \in \Lambda) \} \subseteq FC^* \). Consider the set \( X = \Lambda \times \{ \} \), where \( \in \Lambda \) is a symbol such that \( \in \Lambda \), for every \( \lambda \in \Lambda \). Then, we define a fuzzy topology on \( X \) by defining any singleton \( \{x_\lambda\} \) where \( x_\lambda \) to be fuzzy open and a fuzzy nbd of \( x_\lambda \) are the fuzzy sets \( \{x_\lambda : \lambda \in \Lambda \} \subseteq \mathcal{F} \) for some \( \lambda_0 \in \Lambda \). Let \( \mathcal{F} : X \times Y \to Z \) be a map, for which \( \mathcal{F}(\lambda,Y) = f_j(\lambda,Y) \) and \( \mathcal{F}(\lambda,Y) = f_j(\lambda,Y) \) for every \( \lambda \in \Lambda \). The map \( \mathcal{F} \) is fuzzy continuous. Also, \( \mathcal{F}(\lambda) = f_j(\lambda) \). Since, the fuzzy topology \( \mathcal{F} \) is fuzzy splitting, the map \( \hat{F} : X \to FC_2(Y,Z) \) is fuzzy continuous. Then, for every fuzzy open nbd \( U \in FC_2(Y,Z) \), there exists a fuzzy open nbd \( V \) of \( \hat{F}(\lambda) \) in \( Y \) such that \( \hat{F}(V) \subseteq U \). Hence, there exists \( \lambda_0 \in \Lambda \) such that \( \lambda \in \Lambda \). For every \( \lambda \in \Lambda \), \( \lambda \geq \lambda_0 \), and \( \mathcal{F}(\lambda) \). Therefore, \( \mathcal{F}(\lambda) = f_j(\lambda) \). For every \( \lambda \in \Lambda \), \( \lambda \geq \lambda_0 \), which means that the fuzzy nets \( \{ f_j : \lambda \in \Lambda \} \) converge to \( f \) in the fuzzy topology \( \mathcal{F} \). Thus \( FC^* \subseteq FC(\mathcal{F}) \).

Conversely, let \( \mathcal{F} \) be a fuzzy topology on \( FC(Y,Z) \) such that \( FC^* \subseteq FC(\mathcal{F}) \). We aim to prove that the fuzzy topology \( \mathcal{F} \) is fuzzy splitting. Let \( f \) be any fts and \( \lambda \), let \( \hat{F} : X \times Y \to Z \) be a fuzzy continuous map. Consider the map \( \hat{F} : X \to FC_2(Y,Z) \). Let \( \{ S(n) : n \in D \} \subseteq FC^* \) be a fuzzy net in \( X \) which fuzzy converges to \( x_i \) in \( X \). We prove that the fuzzy net \( \{ F(S(n)) : n \in D \} \subseteq FC^* \) fuzzy converges to \( F(x_i) \). Let \( \{ \eta(m) : m \in M \} \subseteq FC^* \) be a fuzzy net in \( Y \) which fuzzy converges to \( y_i \) in \( Y \). Since the map \( \hat{F} \) is fuzzy continuous and the fuzzy net \( \{ S(n) : n \in D \} \subseteq FC^* \) in \( X \times Y \) fuzzy converges to \( (x_i,y_i) \) in \( X \times Y \), we have that \( \{ F(S(n)) : n \in D \} \subseteq FC^* \) fuzzy converges to \( F(x_i,y_i) \). Therefore, the fuzzy nets \( \{ F(S(n)) : n \in D \} \subseteq FC^* \) fuzzy continuously converges to \( F(x_i) \). Since \( FC^* \subseteq FC(\mathcal{F}) \), then the fuzzy net \( \{ F(S(n)) : n \in D \} \subseteq FC^* \) fuzzy converges to \( \hat{F}(x_i) \). Hence, the map \( \hat{F} \) is fuzzy continuous and the fuzzy topology \( \mathcal{F} \) is fuzzy splitting.

Theorem 10. A subset \( U \) of \( FC(Y,Z) \) is fuzzy open in the finest splitting topology iff for every \( f \in U \) and for every fuzzy net \( \{ f_j : n \in D \} \subseteq FC(Y,Z) \) such that \( \liminf \langle f_j \rangle (K) \subseteq \lim\langle f \rangle (K) \) for each fuzzy closed subset \( K \) of \( Z \), there exists \( n_0 \in D \) such that \( f_n \in U \) for every \( n \geq n_0 \).

Proof. It is clear that the set \( \mathcal{F} \) of all subsets \( U \) of \( FC(Y,Z) \) satisfy the condition (\( * \)) is a fuzzy topology on \( FC(Y,Z) \). Also, we prove that this fuzzy topology is splitting. For any fuzzy topological space \( X \), let \( \hat{F} : X \times Y \to Z \) be a fuzzy continuous map. Consider the map \( \hat{F} : X \to FC_3(Y,Z) \), let \( \{ S(n) : n \in D \} \subseteq FC_3(Y,Z) \) be a fuzzy net in \( X \) which fuzzy converges to \( x_i \) in \( X \). We prove that the fuzzy net \( \{ F(S(n)) : n \in D \} \subseteq FC(Y,Z) \) fuzzy converges to \( \hat{F}(x_i) \). Let \( \{ \eta(m) : m \in M \} \subseteq FC(Y,Z) \) be a fuzzy net in \( Y \) which fuzzy converges to \( y_i \) in \( Y \). Since the map \( \hat{F} \) is fuzzy continuous and the fuzzy net \( \{ F(S(n)) : n \in D \} \subseteq FC(Y,Z) \) fuzzy converges to \( \hat{F}(x_i) \). Hence, the map \( \hat{F} \) is fuzzy continuous and the fuzzy topology \( \mathcal{F} \) is fuzzy splitting. Now, we prove that \( \mathcal{F} \) is the finest splitting topology on \( FC(Y,Z) \). Let \( \mathcal{F} \) be a fuzzy splitting topology on \( FC(Y,Z) \) and let \( V \subseteq \mathcal{F} \). Suppose that \( f \in V \) and \( \{ f_j : \lambda \in \Lambda \} \subseteq FC(Y,Z) \) be a fuzzy net in \( FC(Y,Z) \) such that the condition (\( * \)) is satisfied, for every fuzzy closed subset \( K \) of \( Z \). Then, \( \{ f : \lambda \in \Lambda \} \subseteq FC(\mathcal{F}) \) since \( \{ f_j : \lambda \in \Lambda \} \subseteq FC(\mathcal{F}) \). Therefore, there exists \( \lambda_0 \in D \) such that \( f_j \subseteq V \) for each \( \lambda \geq \lambda_0 \). Since \( V \subseteq \mathcal{F} \). Hence \( \mathcal{F} \) is the finest splitting topology.

Definition 11. A fuzzy topology \( \mathcal{F} \) on \( FC(Y,Z) \) is called fuzzy jointly continuous iff for any fts \( X \), the fuzzy continuity of the map \( \hat{G} : X \to FC_3(Y,Z) \) implies the fuzzy continuity of the map \( \hat{G} : X \times Y \to Z \) for which \( G(x,y,m) = G(x,m)(y) \).

Theorem 11. A fuzzy topology \( \mathcal{F} \) on \( FC(Y,Z) \) is fuzzy jointly continuous iff the mapping evaluation map \( e : FC_3(Y,Z) \times Y \to Z \) defined by \( e(f,y) = f(y) \) is fuzzy continuous.

Proof. Obviously, the identity map \( \hat{G} : 1 : FC_3(Y,Z) \to FC_3(Y,Z) \) is fuzzy continuous. Since, the fuzzy topology \( \mathcal{F} \) is fuzzy jointly continuous. Then, the map \( \hat{G} = e : FC_3(Y,Z) \times Y \to Z \) is fuzzy continuous. Conversely, let \( X \) be any fts, \( \hat{G} : X \to FC_3(Y,Z) \) be a fuzzy continuous map and \( Y \) be the identity map. The map \( \hat{G} \times 1 : X \times Y \to FC_3(Y,Z) \) is fuzzy continuous. Hence, the map \( e \circ (\hat{G} \times 1) : X \times Y \to Z \) is fuzzy continuous.
Theorem 12. A fuzzy topology $\mathcal{S}$ on $FC(Y, Z)$ is fuzzy jointly continuous iff $FC(\mathcal{S}) \subseteq FC^*$. 

Proof. Let $\mathcal{S}$ be a fuzzy jointly continuous topology on $FC(Y, Z)$, $X$ be the space which was defined in the proof of theorem 9 and $(\{f_\lambda : \lambda \in \Lambda\}, f) \in FC(\mathcal{S})$. The map $\hat{G} : X \to FC_2(Y, Z)$, where $\hat{G}(\lambda) = f_\lambda$ and $\hat{G}(z) = f$ is fuzzy continuous. Thus, the map $\hat{G} : X \times Y \to Z$ is fuzzy continuous. Let $\{S(n) : n \in D\}$ be a fuzzy net in $Y$ fuzzy converges to $y_r$, in $Y$. So, the fuzzy net $\{\hat{G}(z) : z \in \Lambda\}$ in $X$ fuzzy converges to $z$. Hence, the fuzzy net $(\hat{G}(z), S(n)) = f_\lambda(S(n)) = (\lambda, n) \in \Lambda \times D$ fuzzy converges to $(z, y_r)$. Since the map $\hat{G}$ is fuzzy continuous, the fuzzy net $(\hat{G}(z), S(n)) = f_\lambda(S(n)) = (\lambda, n) \in \Lambda \times D$ fuzzy converges to $\hat{G}(z, y_r) = f(y_r)$ in $Y$.

Conversely, let $\mathcal{S}$ be a fuzzy topology on $FC(Y, Z)$ such that $FC(\mathcal{S}) \subseteq FC^*$. Our aim is to show that the fuzzy topology $\mathcal{S}$ is fuzzy jointly continuous. Let $X$ be arbitrary its and let $G : X \to FC_2(Y, Z)$ be a fuzzy continuous map. We shall prove that the map $G : X \times Y \to Z$ is fuzzy continuous. Let $(\{S(n), \eta(m) : (n, m) \in D \times M\}$ be a fuzzy net in $X \times Y$ fuzzy converges to $(x_r, y_r)$. Since the fuzzy net $\{S(n) : n \in D\}$ fuzzy converges to $x_r$ in $X$ and the map $G$ is fuzzy continuous. The fuzzy net $\{G(S(n)) : n \in D\}$ fuzzy continuously converges to $G(x_r)$. By the hypothesis the fuzzy net $\{G(S(n)) : n \in D\}$ fuzzy continuously converges to $G(x_r)$. Since $FC(\mathcal{S}) \subseteq FC^*$. Therefore, the fuzzy net $\{G(S(n)) : n \in D\}$ fuzzy continuously converges to $G(x_r)$. Hence, the fuzzy topology $\mathcal{S}$ is fuzzy jointly continuous.

Acknowledgements

The authors are grateful to the referees for their fruitful comments for the improvement of this paper.

References


A. I. Aggour is a leading world-known figure in mathematics and is presently employed as Assist. Prof. at Mathematics Department, Faculty of Sciences, Al-Azhar University. He obtained Ph D from Al-Azhar University. He introduced a new types of convergence on fuzzy function spaces by using fuzzy nets he is an active researcher, teaching experience in various countries of the arabic world. He has published more than 15 papers.

F. E. Attounsi is studied mathematics for five in a secondary school, she obtained her B.S.C. (Mathematics) Faculty of Science, 7th October University, Libya-Misurata, she studied pre-Masters courses of Ani Shams University (Egypt): Functional Analysis, Abstract algebra, Topology. Partial differential equations, Mathematical logic, specialized this manuscript in fuzzy topology. The notion of convergence is one of the basic notion in analysis, we introduce and study the fuzzy continuously converges theory of fuzzy nets on the set $FC(X, Y)$, we introduce and study of the important and properties of fuzzy nets on the set $FC(X, Y)$. Also, we introduce new fuzzy topologies on fuzzy function spaces and then introduce some results properties of the above concepts.