A GENERALIZATION OF MITTAG-LEFFLER FUNCTION AND INTEGRAL OPERATOR ASSOCIATED WITH FRACTIONAL CALCULUS

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ABSTRACT. This paper is devoted for the study of a new generalized function of Mittag-Leffler type. Its various properties including differentiation, Laplace transform, Beta transform, Mellin transform, Whittaker transform, generalized hypergeometric series form, Mellin-Barnes integral representation and its relationship with Fox’s H-function and Wright hypergeometric function are investigated and established. Further properties of generalized Mittag-Leffler function associated with fractional differential and integral operators are considered. Also an integral operator associated with fractional calculus operators is studied.

1. INTRODUCTION

The Swedish mathematician Mittag-Leffler [5] introduced the function $E_\alpha(z)$ defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (1)$$

where $z \in \mathbb{C}$ and $\Gamma(s)$ is the Gamma function; $\alpha \geq 0$.

The Mittag-Leffler function is a direct generalization of $\exp(z)$ in which $\alpha = 1$. Mittag-Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations.

A generalization of $E_\alpha(z)$ was studied by Wiman [14] where he defined the function $E_{\alpha,\beta}(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (2)$$

($\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0$) which is also known as Mittag-Leffler function or Wiman’s function.

Prabhakar [6] introduced the function $E_{\alpha,\beta}^{\gamma}(z)$ in the form (see also Kilbas et al. [4]

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\[ E_{\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{\gamma^n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \quad (3) \]

\((\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0)\)

Shukla and Prajapati [10] (see also Srivastava and Tomovski [13]) defined and investigated the function \(E_{\alpha,\beta}^{\gamma,q}(z)\) as

\[ E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{\gamma q^n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \quad (4) \]

where \((\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0)\) and \(q \in (0, 1) \cup \mathbb{N}\) and \(q^n = \prod_{r=1}^{n} \left( \frac{\gamma + q - r}{q} \right) \) if \(q \in \mathbb{N}\)

A new generalization of Mittag-Leffler function was defined by Salim [8] as

\[ E_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{\gamma^n}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_n} \quad (5) \]

where \((\alpha, \beta, \gamma, \delta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0)\)

In this paper, we introduce a new generalization of Mittag-Leffler function defined as

\[ E_{\alpha,\beta}^{\gamma,\delta,q}(z) = \sum_{n=0}^{\infty} \frac{\gamma q^n}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_n} \quad (6) \]

where \((\alpha, \beta, \gamma, \delta \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0; p, q > 0 \text{ and } q \leq \Re(\alpha + p) \quad (7)\)

Equation (6) is a generalization of equations (1) - (5).

- Setting \(p = q = 1\), it reduces to Eq. (5) defined by Salim [8].
- Setting \(\delta = p = 1\), it reduces to Eq. (4) defined by Shukla and Prajapati [10], in addition of that if \(\alpha = 1\), then we get Eq. (3) defined by Prabhakar [6].
- On putting \(\gamma = \delta = p = q = 1\) in (6) it reduces to Wiman’s function, moreover if \(\beta = 1\), Mittag-Leffler function \(E_{\alpha}(z)\) will be the result.

Some recurrence relations, derivation formulas, Laplace transform, Beta transform, Mellin-Barnes integral of \(E_{\alpha,\beta}^{\gamma,\delta,q}(z)\) will be established, also its relationship to Fox’s H-function and Wright hypergeometric function will be established.

The integral operator defined by

\[ E_{\alpha,\beta}^{\gamma,\delta,q, w, a+}(x) = \int_{a}^{x} (x - t)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta,q}(w(x - t)^{\alpha}) \varphi(t) dt \quad (8) \]

which contains the generalized Mittag-Leffler function (6) in its kernel is investigated and its boundedness is proved under certain conditions.

Theorems of composition of fractional calculus operators

\[ (I_{a}^{\lambda} \varphi)(x) = \frac{1}{\Gamma(\lambda)} \int_{a}^{x} (x - t)^{\lambda-1} \varphi(t) dt \quad (\lambda \in \mathbb{C}, \Re(\lambda) > 0) \quad (9) \]
and
\[(D_a^\lambda \varphi)(x) = \left( \frac{d}{dx} \right)^n (I_a^{n-\lambda} \varphi)(x) \quad n = [\Re(\lambda)] + 1 \quad (10)\]

with integral operators defined in (8) are given and proved. As a matter of fact if \(w = 0\), \(q = 1\) and \(p = 1\), then the integral operator corresponds essentially to the Riemann-Liouville fractional integral operator defined in (9). The generalized fractional derivative operator \(D_u,v \varphi\) known as Hilfer’s fractional derivative (see Hilfer [2]) is written as
\[(D_u,v \varphi)(x) = \left( I_v^{(1-u)} \frac{d}{dx} (I_{1-u} \varphi) \right)(x) \quad (11)\]
\(D_u,v\) yields the classical Riemann-Liouville fractional derivative \(D_a^u\) when \(v = 0\); also if \(v = 1\) it reduces to Caputo fractional derivative.

Throughout this paper, we need the following well-known facts and rules.

- Beta transform (Sneddon [11])
\[B\{f(z); a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z) dz, \quad \Re(a) > 0, \Re(b) > 0 \quad (12)\]

- Laplace transform (Sneddon [11])
\[\mathcal{L}\{f(z); s\} = \int_0^\infty e^{-sz} f(z) dz, \quad \Re(s) > 0 \quad (13)\]

- Convolution theorem of Laplace transform (Finney et al. [1])
\[\mathcal{L}\{f \ast g\}(s) = \mathcal{L}\{f(t - \xi) f(\xi) d\xi\} = \mathcal{L}\{f\}(s) \mathcal{L}\{g\}(s) ; \quad \mathcal{L}\left\{ \frac{t^{n-1}}{\Gamma(n)} ; s \right\} = \frac{1}{sn^n}, \quad n > 0 \quad (14)\]

- Mellin transform (Sneddon [11])
\[\mathcal{M}\{f(x); s\} = f^*(s) = \int_0^\infty z^{s-1} f(z) dz \quad (15)\]

and the inverse Mellin transform is given by
\[f(z) = \mathcal{M}^{-1}\{f^*(s); z\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} f^*(s) ds, \quad c \in \Re \quad (16)\]

- Confluent hypergeometric function (Rainville [7])
\[\Phi(a, b, z) =_1 F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \quad (17)\]
• Wright generalized hypergeometric function (Srivastava and Manocha [12]).

\[ p \Psi_q \left( \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} ; z \right) = \sum_{n=0}^{\infty} \prod_{i=1}^{p} \Gamma(a_i + A_i n) \prod_{j=1}^{q} \Gamma(b_j + B_j n) \frac{z^n}{n!} \]  

(18)

• Fox’s H-function (Kilbas and Saigo [3])

\[ H_{m,n}^{p,q} \left[ \begin{array}{c} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array} ; z \right] = \frac{1}{2\pi i} \int_{L} \prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s) \prod_{j=m+1}^{p} \Gamma(a_j + \alpha_j s) \prod_{j=n+1}^{q} \Gamma(1 - b_j - \beta_j s) ds \]  

(19)

• The generalized hypergeometric function (Rainville [7])

\[ p F_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \prod_{i=1}^{p} \Gamma(\alpha_i)_n \frac{z^n}{n!} \]  

(20)

• Whittaker transform (Whittaker and Watson [15])

\[ \int_{0}^{\infty} e^{-t/2} t^{v-1} W_{\lambda, \mu}(t) dt = \frac{\Gamma(\frac{1}{2} + \mu + v) \Gamma(\frac{1}{2} - \mu + v)}{\Gamma(1 - \lambda + v)} \]  

(21)

where \( \Re(\mu \pm v) > -1/2 \) and \( W_{\lambda, \mu}(t) \) is the Whittaker confluent hypergeometric function.

• Fubini’s theorem (Dirichlet formula) (Samko et al. [9])

\[ \int_{a}^{b} dx \int_{a}^{x} f(x, t) dt = \int_{a}^{b} dt \int_{a}^{t} f(x, t) dx; \]  

(22)

\[ \frac{d}{dx} \int_{a}^{x} h(x, t) dt = \int_{a}^{x} \frac{\partial}{\partial x} h(x, t) dt + h(x, x). \]  

(23)

2. Basic properties

**Theorem 2.1** The series in (6) is absolutely convergent for all values of \( z \) provided that \( q < p + \Re(\alpha) \). Moreover if \( q = p + \Re(\alpha) \), then \( E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) \) converges for \( |z| < 1 \).

**Proof.** Rewriting \( E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) \) in the form of power series \( E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) = \sum_{n=0}^{\infty} b_n z^n \) where \( b_n = \frac{(\gamma)_q n}{(\alpha n + \beta)(\delta)_p n} \) and applying \( \frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b} \left[ 1 + \frac{(a-b)(a+b-1)}{2z} + O \left( \frac{1}{z^2} \right) \right] \), we get
Theorem 2.2

\[ \left| \frac{c_{n+1}}{c_n} \right| = \frac{(\gamma)_{q+n} (\delta)_m}{(\gamma)_{q n} (\delta)_{p n}} \frac{\Gamma(\alpha n + \beta)}{\Gamma(\alpha n + \beta + \alpha)} \left| z^n \right| \]

\[ = (n q)^q \left[ 1 + \frac{q (2 q + \gamma - 1)}{2 q n} + O \left( \frac{1}{(n q)^2} \right) \right] \]

\[ \times (n p)^p \left[ 1 + \frac{-p (2 \delta + p - 1)}{2 p n} + O \left( \frac{1}{(n p)^2} \right) \right] \]

\[ \times (\alpha n)^{-\alpha} \left[ 1 + \frac{-\alpha (2 \alpha + \alpha - 1)}{2 \alpha n} + O \left( \frac{1}{(\alpha n)^2} \right) \right] \]

\[ \left| z \right| = q^q \frac{n^q}{p^p \alpha^\alpha n^{p+\alpha}}, \]

then \( \left| \frac{c_{n+1}}{c_n} \right| \to 0 \) as \( n \to \infty \) and \( q < p + \Re(\alpha) \),

which means that the function \( E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) \) converges for all \( z \) provided that \( q < p + \Re(\alpha) \). Moreover if \( q = p + \Re(\alpha) \), then \( E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) \) converges for \( \left| z \right| < 1 \).

**Theorem 2.2** If the condition (7) is satisfied, then

\[ E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) - E_{\alpha, \beta, p}^{\gamma, \delta-1, q}(z) = \frac{2 p}{1 - \delta} \frac{d}{dz} E_{\alpha, \beta, p}^{\gamma, \delta, q}(z); \quad \delta \neq 1 \]  

(24)

and

\[ E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) = \beta E_{\alpha, \beta+1, p}^{\gamma, \delta, q}(z) + a z \frac{d}{dz} E_{\alpha, \beta+1, p}^{\gamma, \delta, q}(z) \]  

(25)

**Proof.**

\[ E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) - E_{\alpha, \beta, p}^{\gamma, \delta-1, q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n q}}{\Gamma(\alpha n + \beta)} \frac{1}{(\delta)_m} \left[ 1 + \frac{1}{(\delta-1)_{p n}} \right] z^n \]

\[ = \sum_{n=0}^{\infty} \frac{(\gamma)_{n q}}{\Gamma(\alpha n + \beta) \Gamma(\delta + m n)} \frac{1}{1 - \delta} \frac{2 p}{\Gamma(\alpha n + \beta) (\delta)_{p n}} z^n \]

\[ = \frac{2 p}{1 - \delta} \frac{d}{dz} E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) \]

hence (24) is proved.

\[ E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n q} z^n}{\beta(\gamma)_{n q} + \Gamma(\alpha n + \beta) (\delta)_{p n}} + \sum_{n=0}^{\infty} \frac{(\gamma)_{n q} z^n}{(\gamma)_{n q} + \Gamma(\alpha n + \beta) (\delta)_{p n}} \]

\[ = \beta E_{\alpha, \beta+1, p}^{\gamma, \delta, q}(z) + a z \frac{d}{dz} E_{\alpha, \beta+1, p}^{\gamma, \delta, q}(z) \]

which is (25).

**Theorem 2.3** If the condition (7) is satisfied, then for \( m \in \mathbb{N} \)

\[ \left( \frac{d}{dz} \right)^m E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) = \frac{(\gamma)_{n q}}{(\delta)_m} \sum_{n=0}^{\infty} \frac{(\gamma + q n)_{n q}}{(\delta + p m)_{p n} \Gamma(\alpha n + \beta)} z^n; \]  

(26)

\[ \left( \frac{d}{dz} \right)^m \left[ z^{\beta-1} E_{\alpha, \beta, p}^{\gamma, \delta, q}(w z^\alpha) \right] = z^{\beta-m-1} E_{\alpha, \beta-m, p}^{\gamma, \delta, q}(w z^\alpha) \]  

(27)

**Proof.**

\[ \left( \frac{d}{dz} \right)^m \sum_{n=0}^{\infty} \frac{(\gamma)_{n q} z^n}{\Gamma(\alpha n + \beta) (\delta)_{p n}} = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + q n + q m)}{\Gamma(\gamma) (\delta + p m + q m) \Gamma(n + 1)_{(n + 1) m} \Gamma(n + \alpha m + \beta)} z^n \]
in terms of the generalized hypergeometric function as
E_{\alpha,\beta,p}(wz^{\alpha}) = (\gamma)_{\alpha,\beta,p} \sum_{n=0}^{\infty} \frac{(\gamma+qn)_{\alpha,\beta,p}}{(\delta+n)_\beta (\delta+m)_p} \frac{(n+1)_m}{\Gamma(an+\alpha+\beta)} z^n;

\left( \frac{d}{dz} \right)^m \int \frac{\gamma_{\alpha,\beta,p}(wz^{\alpha})}{\Gamma(\alpha+\beta)(\delta)_p} \frac{d}{dz}(z^{\alpha+\beta-1}) = \int \frac{\gamma_{\alpha,\beta,p}(wz^{\alpha})}{\Gamma(\alpha+\beta)(\delta)_p} \frac{d}{dz}(z^{\alpha+\beta-1}) = z^{\alpha+\beta-1} E_{\alpha,\beta,p}(wz^{\alpha}).

**Theorem 2.4** If the condition (7) is satisfied, then

\[
\frac{1}{\Gamma(\delta)} \int_0^1 (x-s)^{\delta-1}(s-t)^{\beta-1} E_{\alpha,\beta,p}(\lambda(s-t)^{\alpha}) \, ds = (x-t)^{\delta+\beta-1} E_{\alpha,\beta+\delta,p}(\lambda(s-t)^{\alpha})
\]

(28)

**Proof.** Let \( u = \frac{s-t}{x-t} \), then

\[
\frac{1}{\Gamma(\delta)} \int_0^1 \frac{1}{(x-t)^{\delta+\beta-1}} (1-u)^{\delta-1}(x-t)^{\beta-1} u^{\delta-1} (x-t)^{\alpha} \sum_{n=0}^{\infty} \frac{(\gamma)_{\alpha,\beta,p}(\lambda(x-t)^{\alpha} u^n)}{\Gamma(\alpha+\beta)(\delta)_p} \, du
\]

\[
= \frac{(x-t)^{\delta+\beta-1}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{\alpha,\beta,p}(\lambda(x-t)^{\alpha} u^n)}{\Gamma(\alpha+\beta)(\delta)_p} \frac{\Gamma(n+\beta)}{\Gamma(\alpha+\beta+\delta)}
\]

In particular, setting \( t = 0 \) and \( x = 1 \) in (28), we get

\[
\frac{1}{\Gamma(\delta)} \int_0^1 u^{\beta-1}(1-u)^{\delta-1} E_{\alpha,\beta+\delta,p}(zu^{\alpha}) \, du = E_{\alpha,\beta+\delta,p}(z).
\]

#### 3. \( E_{\alpha,\beta,p}(z) \) in Terms of Other Functions

In this section we write \( E_{\alpha,\beta,p}(z) \) in terms of Wright generalized function, generalized hypergeometric function, Mellin-Barnes integral and Fox's H-function.

\[
E_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\alpha,\beta,p} z^n}{(\delta+n)_\beta (\delta+m)_p} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)} \frac{\Gamma(n+1)}{\Gamma(\delta+n) \Gamma(\alpha+\beta) n!}
\]

hence, we can write \( E_{\alpha,\beta,p}(z) \) in terms of the Wright generalized function as

\[
E_{\alpha,\beta,p}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)}{\Gamma(\delta+n) \Gamma(\alpha+\beta) n!} \frac{z^n}{\Gamma(n+1)} = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \Psi_2 \left( \frac{(\gamma,q),(1,1)}{(\delta,p),(\beta,\alpha)} ; z \right)
\]

(29)

**Theorem 3.1** Let (7) be satisfied with \( \alpha = k \in \mathbb{N} \), then \( E_{\alpha,\beta,p}(z) \) can be written in terms of the generalized hypergeometric function as

\[
E_{\alpha,\beta,p}(z) = \frac{1}{\Gamma(\beta)} z^q+1 F_{p+q} \left[ \frac{1}{\Delta(k,\beta), \Delta(p,\delta)} ; z^q \frac{\Delta(q,\gamma)}{p^q k^q} \right].
\]

(30)
where $\Delta(k, n)$ is $k$-tuple $\frac{n}{k}, \frac{n+1}{k}, \ldots, \frac{n+k-1}{k}$.

**Proof.** Let $\alpha = k \in \mathbb{N}$, then

$$E^{\gamma, \delta, q}_{\alpha, \beta, p}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha n + \beta)(\delta)_{pn}}{\Gamma(\beta) n!} \frac{q^n}{n!} \frac{z^n}{n!}$$

Proof. Let $E^{\gamma, \delta, q}_{\alpha, \beta, p}(z)$ be Mellin-Barnes integral, we get

$$= \frac{1}{\Gamma(\beta) n!} \prod_{j=1}^{p} \left( \frac{\delta + j - 1}{p} \right)^n \frac{z^{kn}}{n!}$$

where $\Delta(k, n) = 1 + \frac{\gamma + i - 1}{q}$.

Now in order to write $E^{\gamma, \delta, q}_{\alpha, \beta, p}(z)$ in terms of Fox’s H-function, we first express $E^{\gamma, \delta, q}_{\alpha, \beta, p}(z)$ as Mellin-Barnes type integral

**Theorem 3.2** Let (7) be satisfied, then $E^{\gamma, \delta, q}_{\alpha, \beta, p}(z)$ is represented in the Mellin-Barnes type integral as

$$E^{\gamma, \delta, q}_{\alpha, \beta, p}(z) = \frac{1}{2\pi i} \int_{L} \frac{\Gamma(s) \Gamma(1 - s) \Gamma(\gamma - qs)(-z)^{-s}}{\Gamma(\beta - \alpha s) \Gamma(\delta - ps)} ds,$$

where $|\arg(z)| < \pi$; the contour of integration begins at $-i \infty$ and ending at $i \infty$, and intended to separate the poles of the integrand at $s = -n$ for all $n \in \mathbb{N}$ (to the left) from those at $s = n + 1$ and at $s = \gamma + \frac{n}{q}$ for all $n \in \mathbb{N} \cup \{0\}$ (to the right).

**Proof.** Simply, by writing the Wright generalized function in (29) in terms of Mellin-Barnes integral, we get

$$E^{\gamma, \delta, q}_{\alpha, \beta, p}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + qn)}{\Gamma(\delta + pn)} \frac{(n + 1). z^n}{n!} = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \cdot \mathbf{2}_{2} \left[ \frac{(\gamma, q), (1, 1)}{(\delta, p), (\beta, \alpha)} ; z \right]$$

$$= \frac{1}{2\pi i} \int_{L} \frac{\Gamma(s) \Gamma(1 - s) \Gamma(\gamma - qs)(-z)^{-s}}{\Gamma(\beta - \alpha s) \Gamma(\delta - ps)} ds$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)} \mathbf{H}_{2,3}^{1,2} \left[ -z \middle| \begin{array}{c} (0, 1), (1 - \gamma, q) \\ (0, 1), (1 - \beta, \alpha), (1 - \delta, p) \end{array} \right].$$

The last equation is just a representation of $E^{\gamma, \delta, q}_{\alpha, \beta, p}(z)$ in terms of Fox’s H-function.

**4. Integral Transforms of $E^{\gamma, \delta, q}_{\alpha, \beta, p}(z)$**

In this section, the image of $E^{\gamma, \delta, q}_{\alpha, \beta, p}(z)$ under Beta, Laplace, Mellin and Whittaker transforms with some special cases are proved in the following theorems

**Theorem 4.1** (Beta Transform)

$$B \left\{ E^{\gamma, \delta, q}_{\alpha, \beta, p}(xz^{\sigma}) ; a, b \right\} = \frac{\Gamma(b) \Gamma(\delta)}{\Gamma(\gamma)} 3_{\Psi_{3}} \left[ \frac{(\gamma, q), (a, \sigma), (1, 1)}{(\beta, \alpha), (\delta, p), (a + b, \sigma)} ; z \right],$$

where (7) is satisfied and $\Re(a) > 0, \Re(b) > 0$.

**Proof.**
Proof. Setting \( \frac{\partial}{\partial t} \gamma \rightarrow 1 \), \( (\beta, \alpha, (\delta, p), (a + b, \sigma); x) \); \( \gamma \rightarrow 0 \), \( (\alpha, \beta, p) \) and \( \alpha, \beta, p \) is satisfied and \( \Re(\zeta) > 0, \Re(\phi) > 0 \).

Theorem 4.3 ( Mellin Transform)

\[
\mathcal{M} \left\{ E_{\alpha, \beta, p}^{\gamma, \delta, q}(-wz); s \right\} = \frac{\Gamma(\delta) \Gamma(s) \Gamma(1 - s) \Gamma(\gamma - qs)}{\Gamma(\beta - as) \Gamma(\delta - ps)} w^{-s}
\]

Proof. According to Theorem 3.2 and using (31), \( E_{\alpha, \beta, p}^{\gamma, \delta, q}(-wz) \) can be written as

\[
E_{\alpha, \beta, p}^{\gamma, \delta, q}(-wz) = \frac{1}{2\pi i} \frac{\Gamma(\delta)}{\Gamma(\gamma)} \int L s \Gamma(s) \Gamma(1 - s) \Gamma(\gamma - qs) (wz)^{-s} ds = \frac{1}{2\pi i} \frac{\Gamma(\delta)}{\Gamma(\gamma)} \int L f^*(s) z^{-s} ds
\]

where \( f^*(s) = \frac{\Gamma(s) \Gamma(1 - s) \Gamma(\gamma - qs)}{\Gamma(\beta - as) \Gamma(\delta - ps) w^s} \) and \( L \) is the contour of integration that begins at \( c - i\infty \) and ends at \( c + i\infty; c \in \mathbb{R} \).

Hence

\[
E_{\alpha, \beta, p}^{\gamma, \delta, q}(-wz) = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \mathcal{M}^{-1} \left\{ f^*(s); z \right\}
\]

Now applying Mellin transform to both sides, we obtain

\[
\mathcal{M} \left\{ E_{\alpha, \beta, p}^{\gamma, \delta, q}(-wz); s \right\} = \frac{\Gamma(\delta) \Gamma(s) \Gamma(1 - s) \Gamma(\gamma - qs)}{\Gamma(\beta - as) \Gamma(\delta - ps)} w^{-s}
\]

which proves (35).

Theorem 4.4 (Whittaker Transform)

\[
\int_0^\infty e^{-\frac{\gamma}{\phi}t} t^{\frac{\gamma - 1}{\phi}} W_{\lambda, \mu}(\phi t) E_{\alpha, \beta, p}^{\gamma, \delta, q}(wt^\sigma) dt
\]

\[
= \frac{\Gamma(\delta) \phi^{-\zeta}}{\Gamma(\gamma)} \psi_3 \left[ \begin{array}{c} (\gamma, q), (1, 1), (\frac{1}{2} + \mu + \zeta, \sigma), (\frac{1}{2} - \mu + \zeta, \sigma) \\ (\beta, \alpha), (\delta, p), (1 - \lambda + \zeta, \sigma) \\ w \end{array} \right] \phi^{\sigma}
\]

where \( (\zeta) \) is satisfied and \( \Re(\zeta) > 0, \Re(\phi) > 0 \).

Proof. Setting \( v = \phi t \), then we get
\[
\int_0^\infty e^{-\frac{\pi}{2}t^\zeta(w)}E_{\alpha,\beta,q}(wt^\sigma)dt = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(qn+\gamma)}{\Gamma(\alpha n + \beta + \delta)} \frac{(w)}{\phi^n} \int_0^\infty e^{-\frac{\pi}{2}t^\zeta(w)} \frac{\phi}{\phi^n} \frac{1}{\phi} dv
\]

\[
= \frac{\Gamma(\delta)\phi^{-\zeta}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(qn+\gamma)}{\Gamma(\alpha n + \beta + \delta)} \frac{(w)}{\phi^n} \int_0^\infty e^{-\frac{\pi}{2}t^\zeta(w)} \frac{\phi}{\phi^n} \frac{1}{\phi} dv
\]

which directly yields (36).

5. Integral Operators with Generalized Mittag-Leffler Function in the Kernel

In this section, we consider composition of the Riemann-Liouville fractional integral and derivative and Hilfer's fractional derivative (9) - (11) with Mittag-Leffler function defined by (7).

Theorem 5.1 Let \( a \in \mathbb{R}^+, \alpha, \beta, \gamma, \delta, \lambda, w \in \mathbb{C}, \min \{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\lambda)\} > 0 \) and \( p, q > 0 \), then for \( x > a \) we have

\[
D_{a+}^\lambda [(t-a)^{\beta-1}E_{\alpha,\beta,q}[w(t-a)^\alpha]](x) = (x-a)^{\beta-\lambda-1}E_{\alpha,\beta-\lambda,p}^{\gamma,\delta,q}[w(x-a)^\alpha] \tag{37}
\]

Proof. Beginning with \( I_{a+}^\lambda [(t-a)^{\beta-1}] = \frac{\Gamma(\beta)}{\Gamma(\beta + \lambda)} (x-a)^{\beta + \lambda - 1} \), then

\[
I_{a+}^\lambda [(t-a)^{\beta-1}E_{\alpha,\beta,q}[w(t-a)^\alpha]](x) = I_{a+}^\lambda \left[ \sum_{n=0}^{\infty} \frac{(\gamma)_n w^n (t-a)^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)(\delta)_{pn}} \right](x)
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)_n w^n (t-a)^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)(\delta)_{pn}} \frac{\Gamma(\alpha n + \beta)}{\Gamma(\alpha n + \beta + \lambda)} (x-a)^{\alpha n + \beta + \lambda - 1} \tag{38}
\]

Now making use of (9), (27) and (38) yields

\[
D_{a+}^\lambda [(t-a)^{\beta-1}E_{\alpha,\beta,q}^{\gamma,\delta,q}[w(t-a)^\alpha]](x) = \left( \frac{d}{dx} \right)^m \left[ I_{a+}^m(t-a)^{\beta-1}E_{\alpha,\beta,q}^{\gamma,\delta,q}[w(t-a)^\alpha] \right](x)
\]

\[
= \left( \frac{d}{dx} \right)^m [(x-a)^{\beta+m-\lambda-1}E_{\alpha,\beta,q}^{\gamma,\delta,q}[w(x-a)^\alpha]] = (x-a)^{\beta-\lambda-1}E_{\alpha,\beta-\lambda,p}^{\gamma,\delta,q}[w(x-a)^\alpha].
\]

Now, making use of the formulas in (27) and (38), we can get the following result contained in
**Theorem 5.2** Let \(a \in \mathbb{R}_+, \alpha, \beta, \gamma, \delta, w \in \mathbb{C}, \min \{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0, u < 1, 0 \leq v \leq 0, \Re(\beta) > u + v - uw \) and \(p,q > 0\), then for \(x > 0\) we have

\[
D^{u,v}_+ \left[(t - a)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta, p}(w(t - a)^\alpha) \right](x) = (x - a)^{\beta-u-1} E^{\gamma,\delta,q}_{\alpha,\beta, p-u}(w(x - a)^\alpha). \quad (39)
\]

Consider the integral operator defined in (8) containing the Mittag-Leffler function \(E^{\gamma,\delta,q}_{\alpha,\beta, p}(z)\) in the kernel. First of all we will prove that the operator \(E^{\gamma,\delta,q}_{\alpha,\beta, p, w,a+}\) is bounded on \(L(a,b)\).

**Theorem 5.3** Let \(\alpha, \beta, \gamma, \delta, w \in \mathbb{C}, \min \{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0, b > a \) and \(p,q > 0\), then the operator \(E^{\gamma,\delta,q}_{\alpha,\beta, p,w,a+}\) is bounded on \(L(a,b)\) and

\[
\|E^{\gamma,\delta,q}_{\alpha,\beta, p,w,a+} \|_1 \leq B \|\varphi\|_1
\]

where

\[
B = (b - a)^{\Re(\beta)} \sum_{n=0}^{\infty} \frac{|(\gamma)_{qn}| |w(b - a)^{\Re(\alpha)}|}{\Gamma(\alpha + \beta)} \frac{|(\delta)_{pn}| |\Re(\alpha)n + \Re(\beta)|}{\Re(\alpha)n + \Re(\beta)} \quad (41)
\]

**Proof.** First of all, let \(C_n\) denote the \(n^{th}\) term of (41), then

\[
\frac{c_{n+1}}{c_n} = \frac{|(\gamma)_{qn + p}|}{|\alpha|^n} \lim_{n \to \infty} \frac{|(\delta)_{pn + p}|}{|\beta|^n} \frac{|\Re(\alpha)n + \Re(\beta)|}{\Re(\alpha)n + \Re(\beta)} \quad (39)
\]

as \(n \to \infty\), provided that \(q < p + \Re(\alpha)\). Hence

\[
\frac{c_{n+1}}{c_n} \to 0 \quad \text{as} \quad n \to \infty,
\]

which means that the right hand side of (41) is convergent and finite under the given condition.

Now according to (8) and (22)

\[
\|E^{\gamma,\delta,q}_{\alpha,\beta, p,w,a+} \|_1 = b \int_a^b \left(\int_0^{x-t} t^\beta E^{\gamma,\delta,q}_{\alpha,\beta, p}(w(t - x)^\alpha) \varphi(t) dt \right) dx
\]

\[
\leq b \int_a^b \left(\int_0^{x-t} E^{\gamma,\delta,q}_{\alpha,\beta, p}(w(t - x)^\alpha) dx \right) |\varphi(t)| dt = b \int_a^b \left(\int_0^{x-t} u^{\Re(\beta) - 1} E^{\gamma,\delta,q}_{\alpha,\beta, p}(wu^\alpha) du \right) |\varphi(t)| dt
\]

\[
\leq b \int_a^b \left(\int_0^{x-t} E^{\gamma,\delta,q}_{\alpha,\beta, p}(wu^\alpha) du \right) |\varphi(t)| dt.
\]

But we have

\[
\int_0^{b-a} u^{\Re(\beta) - 1} E^{\gamma,\delta,q}_{\alpha,\beta, p}(wu^\alpha) du = \sum_{n=0}^{\infty} \frac{|(\gamma)_{qn}| |w|^n}{\Gamma(\alpha + \beta)} \int_0^{b-a} u^{\Re(\alpha)n + \Re(\beta) - 1} du = B
\]

so that \(B = (b - a)^{\Re(\beta)} \sum_{n=0}^{\infty} \frac{|(\gamma)_{qn}| |w(b - a)^{\Re(\alpha)}|}{\Gamma(\alpha + \beta)} \frac{|(\delta)_{pn}| |\Re(\alpha)n + \Re(\beta)|}{\Re(\alpha)n + \Re(\beta)} \quad (41)
\]

Hence

\[
\|E^{\gamma,\delta,q}_{\alpha,\beta, p,w,a+} \|_1 \leq b \int_a^b |\varphi(t)| dt = B \|\varphi\|_1.
\]
Corollary 5.4 Let \( \alpha, \beta, \gamma, \delta, \zeta, w \in \mathbb{C}, \) \( \min \{ \Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\lambda) \} > 0 \) and \( p, q > 0, \) then
\[
\left[ E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} (t-a)^{\zeta-1} \right] (x) = \Gamma(\zeta) (x-a)^{\beta+\zeta-1} E^{\gamma, \delta, q}_{\alpha, \beta+\zeta, p, w, a^+} [w(x-a)^{\alpha}].
\] (42)

6. COMPOSITION OF FRACTIONAL CALCULUS OPERATORS AND INTEGRAL OPERATOR WITH GENERALIZED MITTAG-LEFFLER FUNCTION IN THE KERNEL

We consider now composition of the Riemann-Liouville fractional integration operator \( I_a^\lambda \) with the operator \( E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} \)

**Theorem 6.1** Let \( \alpha, \beta, \gamma, \delta, \lambda, w \in \mathbb{C}, \) \( \min \{ \Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\lambda) \} > 0 \) and \( p, q > 0, \) then
\[
I_a^\lambda E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} \varphi = E^{\gamma, \delta, q}_{\alpha, \beta+\lambda, p, w, a^+} I_a^\lambda \varphi
\] (43)
holds for any summable function \( \varphi \in L(a, b). \)

**Proof.**
\[
\left( I_a^\lambda E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} \varphi \right) (x) = \frac{1}{\Gamma(\lambda)} \int_a^x (x-u)^{\lambda-1} \left[ \int_u^x (u-t)^{\beta-1} E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} [w(u-t)^{\alpha}] \varphi(t) dt \right] du
\]
letting \( \tau = u-t \) implies
\[
\left( I_a^\lambda E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} \varphi \right) (x) = \frac{1}{\Gamma(\lambda)} \int_a^x (x-t-\tau)^{\lambda-1} \varphi(\tau) \left[ \frac{1}{\Gamma(\beta+\lambda)} \int_0^{\tau} (\tau-t)^{\beta-1} E^{\gamma, \delta, q}_{\alpha+\lambda, \beta, p, w, a^+} [w(x-t)^{\alpha}] \varphi(t) dt \right] d\tau
\]
Similarly, we can prove the other side.

**Theorem 6.2** If the conditions of Theorem 6.1 is satisfied, then
\[
\left( D_a^\lambda E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} + \varphi \right) (x) = \left( E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} + \varphi \right) (x).
\] (44)

**Proof.** Let \( n = \lfloor \Re(\lambda) \rfloor + 1 \) and using (9), we get
\[
\left( D_a^\lambda E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} + \varphi \right) (x) = \left( \frac{d}{dx} \right)^n \left( I_a^{n-\lambda} E^{\gamma, \delta, q}_{\alpha, \beta, p, w, a^+} + \varphi \right) (x)
\]
\[
= \left( \frac{d}{dx} \right)^n \left( E^{\gamma, \delta, q}_{\alpha, \beta+n-\lambda, p, w, a^+} + \varphi \right) (x)
\]
\[
= \left( \frac{d}{dx} \right)^n \int_a^x (x-t)^{\beta+n-\lambda-1} E^{\gamma, \delta, q}_{\alpha+\lambda+n-\beta, p} [w(x-t)^{\alpha}] \varphi(t) dt
\]
Since the integral is continuous, (23) yields
\[
\left( D^\lambda_{a+} e^{\gamma x} \right) (x) = \left( \frac{d}{dx} \right)^{n-1} \int_a^x \frac{\partial}{\partial x} \left[ (x-t)^{\beta+n-\lambda-1} E^{\gamma,\delta,q}_{\alpha,\beta+n-\lambda,p} [w(x-t)^\alpha] \right] \varphi(t) dt
\]

Repeating this process \((n-1)\) times, then we get

\[
\left( D^\lambda_{a+} e^{\gamma x} \right) (x) = e^x \left[ (x-t)^{\beta+n-\lambda-1} E^{\gamma,\delta,q}_{\alpha,\beta+n-\lambda,p} [w(x-t)^\alpha] \right] \varphi(t) dt
\]

\[
= \left( \mathcal{E}_{\alpha,\beta+n-\lambda,p,\varphi}^{\gamma,\delta,q} \right) (x).
\]

**Theorem 6.3** Let \(\alpha, \beta, \gamma, \delta, w \in \mathbb{C}\), \(\min \{\mathbb{R}(\alpha), \mathbb{R}(\beta), \mathbb{R}(\gamma), \mathbb{R}(\delta)\} > 0\) \(0 < u < 1\), \(0 \leq v \leq 1\), \(\mathbb{R}(\beta) > u + v - uv\) and \(p, q > 0\), then

\[
\left( D_{a+}^{u,v} e^{\gamma x} \right) (x) = \left( \mathcal{E}_{\alpha,\beta+n-\lambda,p,\varphi}^{\gamma,\delta,q} \right) (x).
\]

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**References**


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