Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/070601

On the Periodic Auto–Oscillations of an Electric Circuit with Periodic Imperfections on Its Variables

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Received: 20 Feb. 2013, Revised: 18 Jun. 2013, Accepted: 21 Jun. 2013
Published online: 1 Nov. 2013

Abstract: The aim of the present paper is to study the periodic auto–oscillations of an electric circuit with periodic imperfections on its variables composed by three condensers, one of them without charge, and two bobbins. We model this system by the Lagrangian approach using the morphology of the Hill problem and the main tool used for proving the results is the averaging theory of dynamical systems.

Keywords: Periodic solutions, electric circuit, averaging theory, dynamical systems

1 Introduction and statement of the main results

We consider a dynamical systems consistent in an electric circuit composed by by three condensers and two bobbins such that its variables have periodic imperfections, i.e. our model is a perturbation of the ideal circuit, see Fig. 1. The aim of our work is to study the periodic orbits, i.e. auto–oscillations, produced by the system. For doing this we shall use the averaging theory of dynamical systems, see Appendix for more details on it. We have been inspired by other works where these techniques have been used for studying other perturbed dynamics problems, see for instance [2,3,4,5,6,7,8,9,10].

We consider the Lagrangian formulation of the circuit, using the morphology of the Hill problem,

$$\mathscr{L} = \frac{1}{2}\left(L_1\dot{q}_1^2 + L_2\dot{q}_2^2\right) - \frac{q_1^2}{\bar{c}_1} - \frac{q_2^2}{\bar{c}_2} - \frac{(q_1+q_2)^2}{\bar{c}}$$

where q_i are the charges and c_i are the capacities of the condensers, $i \in \{1,2\}$. The variables L_i, $i \in \{1,2\}$, represent the the auto–inductions of the bobbins. c represents the capacity of a third condenser without charge.

If we make the change of variable:

$$x = \sqrt{L_1}\,q_1, \quad y = \sqrt{L_2}\,q_2,$$

we obtain:

$$\mathscr{L} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{c_1}x^2 - \frac{1}{c_2}y^2 - \frac{1}{c}xy$$

being, for $i = 1,2$,

$$\frac{1}{c_i} = \frac{1}{L_i\bar{c}_i} + \frac{1}{L_i\bar{c}}$$

and

$$\frac{1}{c} = \frac{2}{\bar{c}\sqrt{L_1L_2}}$$

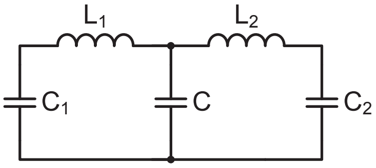

Fig. 1: Circuit

Using the Legendre transformation, we obtain the Hamiltonian:

$$\mathscr{H} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{c_1}x^2 + \frac{1}{c_2}y^2 + \frac{1}{c}xy$$

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Thus, our perturbed model has the following form:

$$\begin{aligned} \dot{x} &= p_1 \\ \ddot{x} + \frac{2}{c_1}x + \frac{1}{c}y &= \varepsilon F_1(t,x,\dot{x},y,\dot{y}) \\ \dot{y} &= p_2 \\ \ddot{y} + \frac{1}{c}x + \frac{2}{c_2}y &= \varepsilon F_2(t,x,\dot{x},y,\dot{y}) \end{aligned} \tag{1}$$

where $p_1 = \frac{\partial \mathscr{L}}{\partial \dot{x}}$ and $p_2 = \frac{\partial \mathscr{L}}{\partial \dot{y}}$. The dot denotes the derivative with respect to the time t, the parameter ε is small and the smooth functions F_1 and F_2, in general, are periodic functions in the variable t and in resonance $p:q$ with some of the periodic solutions for $\varepsilon = 0$, being p and q positive integers relatively prime.

The objective of this paper is to provide, using the averaging theory, a system of nonlinear equations whose simple zeros provide periodic solutions of the differential system (1). In order to present our results we need some preliminary definitions and notation.

The unperturbed system with four differential equations of second order

$$\begin{aligned} \dot{x} &= p_1 \\ \ddot{x} &= -\frac{2}{c_1}x - \frac{1}{c}y \\ \dot{y} &= p_2 \\ \ddot{y} &= -\frac{1}{c}x - \frac{2}{c_2}y \end{aligned} \tag{2}$$

written as a differential system of first order in the four variables $(X_1 = x, X_2 = \dot{x}, X_3 = y, X_4 = \dot{y})$,

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= -\frac{2}{c_1}X_1 - \frac{1}{c}X_3 \\ \dot{X}_3 &= X_4 \\ \dot{X}_4 &= -\frac{1}{c}X_1 - \frac{2}{c_2}X_3 \end{aligned}$$

has a unique singular point at the origin with eigenvalues

$$\pm\omega_1 i, \quad \pm\omega_2 i$$

which are the roots of the polynomial

$$4c^2 - c_1c_2 + 2c^2c_1\omega^2 + 2c^2c_2\omega^2 + c^2c_1c_2\omega^4$$

where c, c_1 and $c_2 \in \mathbb{R}^+$ and $c_1c_2 < 4c^2$.

The frequencies ω_i are given by

$$\omega_1 = \frac{\sqrt{cc_1c_2\left(c(c_1+c_2) + \sqrt{c^2(c_1-c_2)^2 + c_1^2c_2^2}\right)}}{cc_1c_2},$$

$$\omega_2 = \frac{\sqrt{cc_1c_2\left(c(c_1+c_2) - \sqrt{c^2(c_1-c_2)^2 + c_1^2c_2^2}\right)}}{cc_1c_2}$$

Note that $\omega_1^2 - \omega_2^2 = \frac{2\sqrt{c^2(c_1-c_2)^2+c_1^2c_2^2}}{cc_1c_2}$.

As usual we define that the ratio of the two frequencies ω_i and ω_j is *non-resonant* with π if $\omega_i\pi/\omega_j$ is not a rational number, $i \neq j$.

System (2) in the phase space $(x,\dot{x},y,\dot{y})$ has two planes passing through the origin filled of periodic solutions with the exception of the origin. These periodic solutions have periods $T_1 = 2\pi/\omega_1$ and $T_2 = 2\pi/\omega_2$, according they belong to the plane associated to the eigenvectors with eigenvalues $\pm\omega_1 i$ or $\pm\omega_2 i$, respectively. We shall study which of these periodic solutions persist for the perturbed system (1) when the parameter ε is sufficiently small and the perturbed functions F_i, for $i = 1,2$, have period either pT_1/q or pT_2/q, where p and q are positive integers relatively prime.

We define the constants ϕ and ρ by

$$\phi = \frac{c_2 - c_1}{c_1c_2}, \qquad \rho = \omega_1^2 - \omega_2^2,$$

and the functions:

$$\begin{aligned} \mathscr{G}_1^1(X_1^0,X_2^0) &= \frac{1}{pT_1}\int_0^{pT_1} \cos(\omega_1 t)F_1^*(t)dt, \\ \mathscr{G}_1^2(X_1^0,X_2^0) &= \frac{1}{pT_1}\int_0^{pT_1} \sin(\omega_1 t)F_1^*(t)dt, \end{aligned} \tag{3}$$

where

$$F_1^*(t) = \frac{1}{4c\rho}\left[2F_1 + c(\rho - 2\phi)F_2\right]$$

with

$$F_i = F_i(\sigma_1^1(t),\sigma_1^2(t),\sigma_1^3(t),\sigma_1^4(t)),\ i = 1,2,$$

and

$$\begin{aligned} \sigma_1^1(t) &= \frac{-2cc_2\omega_1}{c_2 + c^2(\rho - 2\phi)}(X_2^0\cos(\omega_1 t) - X_1^0\sin(\omega_1 t)) \\ \sigma_1^2(t) &= c(\rho + 2\phi)(X_1^0\cos(\omega_1 t) + X_2^0\sin(\omega_1 t)) \\ \sigma_1^3(t) &= \frac{-2}{\omega_1}(X_2^0\cos(\omega_1 t) - X_1^0\sin(\omega_1 t)) \\ \sigma_1^4(t) &= 2(X_1^0\cos(\omega_1 t) + X_2^0\sin(\omega_1 t)) \end{aligned}$$

A zero (X_1^{0*}, X_2^{0*}) of the nonlinear system

$$\mathscr{G}_1^1(X_1^0,X_2^0) = 0, \quad \mathscr{G}_1^2(X_1^0,X_2^0) = 0, \tag{4}$$

such that

$$\det\left(\frac{\partial(\mathscr{G}_1^1,\mathscr{G}_1^2)}{\partial(X_1^0,X_2^0)}\right)\Bigg|_{(X_1^0,X_2^0)=(X_1^{0*},X_2^{0*})} \neq 0,$$

is called a *simple zero* of system (4).

The statement of our main result on the periodic solutions of the differential system (1) which bifurcate from the periodic solutions of period T_1 of the unperturbed system traveled p times is the following.

Theorem 1.1. Let p and q be positive integers relatively prime and assume that the smooth functions F_1 and F_2 of the equations of motion of (1) are periodic in the variable t of period pT_1/q. We assume that the ratio of the frequencies ω_2/ω_1 is not resonant with π. Then for $\varepsilon \neq 0$ sufficiently small and for every simple zero $(X_1^{0*}, X_2^{0*}) \neq (0,0)$ of the nonlinear system (4), the perturbed system (1) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ tending to the periodic solution $(x(t), y(t)) = (\sigma_1^1(t), \sigma_1^3(t))|_{(X_1^0, X_2^0)=(X_1^{0*}, X_2^{0*})}$ of the unperturbed system (2) traveled p times.

Theorem 1.1 is proved in section 2. Its proof is based in the averaging theory for computing periodic solutions, see Appendix I for more details on this technique.

An application of Theorem 1.1 is presented in the following corollary, which will be proved in section 3.

Corollary 1.2. Let $F_1(t,x,\dot{x},y,\dot{y}) = \dot{x}^2 + \dot{y}^2$ and $F_2(t,x,\dot{x},y,\dot{y}) = \sin(\omega_1 t)(1-\dot{x}^2) + \cos(\omega_1 t)(x - \dot{y})$ be perturbed functions and that the ratio of the frequencies ω_2/ω_1 is not resonant with π. Then the system (1) for $\varepsilon \neq 0$ sufficiently small has two periodic solutions $(x(t,\varepsilon), y(t,\varepsilon))$ tending to the two periodic solutions $(x(t), y(t)) = (\sigma_1^1(t), \sigma_1^3(t))|_{(X_1^0, X_2^0)=(X_1^*, X_2^*)}$ and $(x(t), y(t)) = (\sigma_1^1(t), \sigma_1^3(t))|_{(X_1^0, X_2^0)=(X_3^*, X_4^*)}$ of (2) when $\varepsilon \to 0$, where

$$(X_1^*, X_2^*) = \left(0, \pm\sqrt{\frac{1-c^2\phi(\rho-2\phi)}{3}}\right) \text{ and}$$
$$(X_3^*, X_4^*) = \left(\pm\sqrt{1-c^2\phi(\rho-2\phi)}, 0\right).$$

Corollary 1.2 will be proved in section 3.

Now we define the functions:

$$\mathscr{G}_2^1(X_5^0, X_6^0) = \frac{1}{pT_2}\int_0^{pT_2} \cos(\omega_2 t) F_3^*(t)dt,$$
$$\mathscr{G}_2^2(X_5^0, X_6^0) = \frac{1}{pT_2}\int_0^{pT_2} \sin(\omega_2 t) F_3^*(t)dt,$$

where

$$F_3^*(t) = \frac{1}{4c\rho}[-2F_1 + c(\rho+2\phi)F_2]$$

with

$$F_i = F_i(\sigma_2^1(t), \sigma_2^2(t), \sigma_2^3(t), \sigma_2^4(t)), i = 1,2,$$

and

$$\sigma_2^1(t) = \frac{-2cc_2\omega_2}{c_2 - c^2(\rho-2\phi)}(X_6^0\cos(\omega_2 t) - X_5^0\sin(\omega_2 t))$$
$$\sigma_2^2(t) = -c(\rho-2\phi)(X_5^0\cos(\omega_2 t) + X_6^0\sin(\omega_2 t))$$
$$\sigma_2^3(t) = \frac{-2}{\omega_2}(X_6^0\cos(\omega_2 t) - X_5^0\sin(\omega_2 t))$$
$$\sigma_2^4(t) = 2(X_5^0\cos(\omega_2 t) + X_6^0\sin(\omega_2 t))$$

Consider the nonlinear system

$$\mathscr{G}_2^1(X_5^0, X_6^0) = 0, \quad \mathscr{G}_2^2(X_5^0, X_6^0) = 0. \tag{5}$$

The statement of our main result on the periodic solutions of the differential system (1) which bifurcate from the periodic solutions of period T_2 of the unperturbed system traveled p times is the following.

Theorem 1.3. Let p and q be positive integers relatively prime and assume that the smooth functions F_1 and F_2 of the equations of motion of (1) are periodic in the variable t of period pT_2/q. We assume that the ratio of the frequencies ω_2/ω_1 is not resonant with π. Then for $\varepsilon \neq 0$ sufficiently small and for every simple zero $(X_5^{0*}, X_6^{0*}) \neq (0,0)$ of the nonlinear system (5), the perturbed system (1) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ tending to the periodic solution $(x(t), y(t)) = (\sigma_2^1(t), \sigma_2^3(t))|_{(X_5^0, X_6^0)=(X_5^{0*}, X_6^{0*})}$ of the unperturbed system (2) traveled p times.

Theorem 1.3 is proved in section 2.

In the next corollary an application of Theorem 1.3 is given.

Corollary 1.4. Let $F_1(t,x,\dot{x},y,\dot{y}) = \sin(\omega_2 t)(1+x+y)$ and $F_2(t,x,\dot{x},y,\dot{y}) = 1+\dot{x}$ be perturbed functions and that the ratio of the frequencies ω_2/ω_1 is not resonant with π. Then the system (1) for $\varepsilon \neq 0$ sufficiently small has one periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ tending to the periodic solution $(x(t), y(t)) = (\sigma_2^1(t), \sigma_2^3(t))|_{(X_5^0, X_6^0)=(X_5^{0*}, X_6^{0*})}$ of (2) when $\varepsilon \to 0$, given by $(X_5^{0*}, X_6^{0*}) = \left(0, \frac{-1}{2}\right)$.

Corollary 1.4 will be proved in section 3.

2 Proof of the Theorems 1.1 and 1.3

Introducing the variables $(X_1, X_2, X_3, X_4) = (x, \dot{x}, y, \dot{y})$ we can write the differential system (1) as a first-order differential system defined in $\mathbb{R}^4$ in the following form

$$\begin{aligned}
\dot{X}_1 &= X_2\\
\dot{X}_2 &= -\frac{2}{c_1}X_1 - \frac{1}{c}X_3 + \varepsilon F_1(X_1, X_2, X_3, X_4)\\
\dot{X}_3 &= X_4\\
\dot{X}_4 &= -\frac{1}{c}X_1 - \frac{2}{c_2}X_3 + \varepsilon F_2(X_1, X_2, X_3, X_4)
\end{aligned} \tag{6}$$

Note that the differential system (6) when $\varepsilon = 0$ is equivalent to the differential system (2), called simply in what follows the *unperturbed system*. When $\varepsilon \neq 0$ we called it the *perturbed system*.

The change of variables

$$\mathrm{x} = (x_1, x_2, x_3, x_4) \to \mathrm{X} = (X_1, X_2, X_3, X_4)$$

given by

$$\mathrm{X} = B\mathrm{x}, \tag{7}$$

with

$$B = \begin{pmatrix} 0 & \frac{-2cc_2\omega_1}{c_2+c^2(\rho-2\phi)} & 0 & \frac{-2cc_2\omega_2}{c_2-c^2(\rho+2\phi)} \\ c(\rho+2\phi) & 0 & -c(\rho-2\phi) & 0 \\ 0 & \frac{-2}{\omega_1} & 0 & \frac{-2}{\omega_2} \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

writes the linear part of the differential system (6) in its real Jordan normal form, and this system in the new variables (x_1, x_2, x_3, x_4) becomes

$$\begin{aligned} \dot{x}_1 &= \omega_1 x_2 + \varepsilon F_1^* \\ \dot{x}_2 &= -\omega_1 x_1 + \varepsilon F_2^* \\ \dot{x}_3 &= \omega_2 x_4 + \varepsilon F_3^* \\ \dot{x}_4 &= -\omega_2 x_3 + \varepsilon F_4^* \end{aligned} \tag{8}$$

where

$$\begin{aligned} F_1^* &= \frac{1}{4c\rho}\left[2F_1 + c(\rho-2\phi)F_2\right] \\ F_2^* &= 0 \\ F_3^* &= \frac{1}{4c\rho}\left[-2F_1 + c(\rho+2\phi)F_2\right] \\ F_4^* &= 0 \end{aligned}$$

with $F_i = F_i(\sigma^1, \sigma^2, \sigma^3, \sigma^4)$, and

$$\begin{aligned} \sigma^1 &= -\frac{2cc_2\omega_1}{c_2+c^2(\rho-2\phi)}x_2 - \frac{2cc_2\omega_2}{c_2-c^2(\rho+2\phi)}x_4 \\ \sigma^2 &= c(\rho+2\phi)x_1 - c(\rho-2\phi)x_3 \\ \sigma^3 &= -\frac{2}{\omega_1}x_2 - \frac{2}{\omega_2}x_4 \\ \sigma^4 &= 2x_1 + 2x_3 \end{aligned}$$

Now, in the following lemma we characterize the periodic orbits of the unperturbed system as a first step for proving Theorems 1.1 and 1.3.

Lemma 2.1. The periodic solutions $(x_1(t), x_2(t), x_3(t), x_4(t))$ of the differential system (8) with $\varepsilon = 0$ are

$$(X_1^0\cos(\omega_1 t) + X_2^0\sin(\omega_1 t), X_2^0\cos(\omega_1 t) - X_1^0\sin(\omega_1 t), 0, 0), \tag{9}$$

of period T_1,

$$(0, 0, X_5^0\cos(\omega_2 t) + X_6^0\sin(\omega_2 t), X_6^0\cos(\omega_2 t) - X_5^0\sin(\omega_2 t)),$$

of period T_2.

Proof. Since (8) for $\varepsilon = 0$ is a linear differential system the proof follows easily. □

Proof of Theorem 1.1. Assume that the functions F_1 and F_2 of (1) are periodic in t of period pT_1/q with p and q positive integers relatively prime. Then, we can consider that the differential system (8) and the periodic solutions (9) have the same period pT_1.

We apply Theorem 4.1 of Appendix I to the differential system (8), and we use the notation introduced there. Note that system (8) can be written in the form of system (10) taking

$$\mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, G_0(t, \mathrm{x}) = \begin{pmatrix} \omega_1 x_2 \\ -\omega_1 x_1 \\ \omega_2 x_4 \\ -\omega_2 x_3 \end{pmatrix},$$

$$G_1(t, \mathrm{x}) = \begin{pmatrix} F_1^* \\ F_2^* \\ F_3^* \\ F_4^* \end{pmatrix}, G_2(t, \mathrm{x}, \varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now we shall study what periodic solutions of the unperturbed system (8) with $\varepsilon = 0$ of the type (9) persist as periodic solutions for the perturbed one for $\varepsilon \neq 0$ sufficiently small.

We start with the description of the different elements which appear in the statement of Theorem 4.1 for the particular case of the differential system (8). Thus, we have that $\Omega = \mathbb{R}^4$, $k = 2$ and $n = 4$. Now, let $r_1 > 0$ be arbitrarily small and let $r_2 > 0$ be arbitrarily large. Let V be the open and bounded subset of the plane $x_3 = x_4 = 0$ of the form

$$V = \{(X_1^0, X_2^0, 0, 0) \in \mathbb{R}^4 \ : \ r_1 < \sqrt{(X_1^0)^2 + (X_2^0)^2} < r_2\}.$$

As usual $\mathrm{Cl}(V)$ denotes the closure of V. If $\alpha = (X_1^0, X_2^0)$, then we identify V with the set $\{\alpha \in \mathbb{R}^2 \ : \ r_1 < ||\alpha|| < r_2\}$, being $||\cdot||$ the Euclidean norm in $\mathbb{R}^2$. The function $\beta\colon \mathrm{Cl}(V) \to \mathbb{R}^2$ is $\beta(\alpha) = (0, 0)$. Therefore, for our system we have

$$\begin{aligned} \mathscr{Z} &= \{\mathrm{z}_\alpha = (\alpha, \beta(\alpha)), \ \alpha \in \mathrm{Cl}(V)\} = \\ &= \{(X_1^0, X_2^0, 0, 0) \in \mathbb{R}^4 \ : \ r_1 \leq \sqrt{(X_1^0)^2 + (X_2^0)^2} \leq r_2\}. \end{aligned}$$

We are going to consider now, for each $\mathrm{z}_\alpha \in \mathscr{Z}$, the periodic solution $\mathrm{x}(t, \mathrm{z}_\alpha) = (X_1(t), X_2(t), 0, 0)$ given by (9) of period pT_1.

Computing the fundamental matrix $M_{\mathrm{z}_\alpha}(t)$ of the linear differential system (8) with $\varepsilon = 0$ associated to the

pT_1-periodic solution $z_\alpha = (X_1^0, X_2^0, 0, 0)$ such that $M_{z_\alpha}(0)$ be the identity of $\mathbb{R}^4$, we get

$$M_{z_\alpha}(t) = M(t) = \\ = \begin{pmatrix} \cos(\omega_1 t) & \sin(\omega_1 t) & 0 & 0 \\ -\sin(\omega_1 t) & \cos(\omega_1 t) & 0 & 0 \\ 0 & 0 & \cos(\omega_2 t) & \sin(\omega_2 t) \\ 0 & 0 & -\sin(\omega_2 t) & \cos(\omega_2 t) \end{pmatrix}.$$

Note that the matrix $M_{z_\alpha}(t)$ does not depend on the particular periodic solution $\mathrm{x}(t, z_\alpha, 0)$. Since the matrix

$$M^{-1}(0) - M^{-1}(pT_1) = \\ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) & \sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) \\ 0 & 0 & -\sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) & 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) \end{pmatrix}$$

satisfies the assumptions of statement (ii) of Theorem 4.1 because the determinant

$$\begin{vmatrix} 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) & \sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) \\ -\sin\left(\frac{2p\pi\omega_2}{\omega_1}\right) & 2\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) \end{vmatrix} = 4\sin^2\left(\frac{p\pi\omega_2}{\omega_1}\right) \neq 0,$$

because the ratio of the frequencies is non-resonant with π. In short, all the assumptions of Theorem 4.1 are satisfied by the system (8).

For our system the map $\xi : \mathbb{R}^4 \to \mathbb{R}^2$ has the form $\xi(x_1, x_2, x_3, x_4) = (x_1, x_2)$. Calculating the function

$$\mathscr{G}_1(X_1^0, X_2^0) = \mathscr{G}(\alpha) = \\ = \xi\left(\frac{1}{pT_1}\int_0^{pT_1} M_{z_\alpha}^{-1}(t) G_1^1(t,\ \mathrm{x}(t, z_\alpha, 0))dt\right),$$

we obtain that

$$\mathscr{G}_1(X_1^0, X_2^0) = (\mathscr{G}_1^1(X_1^0, X_2^0), \mathscr{G}_1^2(X_1^0, X_2^0)),$$

where the functions $\mathscr{G}_1^k$, for $k = 1, 2$, are the ones given in (3). Then, by Theorem 4.1 we have that for every simple zero $(X_1^{0*}, X_2^{0*}) \in V$ of the system of nonlinear functions (4) we have a periodic solution $(x_1, x_2, x_3, x_4)(t, \varepsilon)$ of system (8) such that

$$(x_1, x_2, x_3, x_4)(0, \varepsilon) \to (X_1^{0*}, X_2^{0*}, 0, 0) \text{ when } \varepsilon \to 0.$$

Going back through the change of coordinates (7) we get a periodic solution $(x_1, x_2, x_3, x_4)(t, \varepsilon)$ of system (8) such that

$$\begin{pmatrix} x_1(t,\varepsilon) \\ x_2(t,\varepsilon) \\ x_3(t,\varepsilon) \\ x_4(t,\varepsilon) \end{pmatrix} \to \begin{pmatrix} \frac{-2cc_2\omega_1}{c_2+c^2(\rho-2\phi)}(X_2^{0*}\cos(\omega_1 t) - X_1^{0*}\sin(\omega_1 t)) \\ c(\rho+2\phi)(X_1^{0*}\cos(\omega_1 t) + X_2^{0*}\sin(\omega_1 t)) \\ \frac{-2}{\omega_1}(X_2^{0*}\cos(\omega_1 t) - X_1^{0*}\sin(\omega_1 t)) \\ 2(X_1^{0*}\cos(\omega_1 t) + X_2^{0*}\sin(\omega_1 t)) \end{pmatrix}$$

when $\varepsilon \to 0$.

Consequently we obtain a periodic solution $(x,y)(t,\varepsilon)$ of system (1) such that

$$(x,y)(t,\varepsilon) \to \begin{pmatrix} \frac{-2cc_2\omega_1}{c_2+c^2(\rho-2\phi)}(X_2^{0*}\cos(\omega_1 t) - X_1^{0*}\sin(\omega_1 t)) \\ \frac{-2}{\omega_1}(X_2^{0*}\cos(\omega_1 t) - X_1^{0*}\sin(\omega_1 t)) \end{pmatrix}$$

when $\varepsilon \to 0$. This completes the proof of the theorem. □

Proof of Theorem 1.3. The proof is analogous to the proof of Theorem 1.1 changing the roles of T_1 for T_2. □

3 Proof of the two corollaries

Proof of Corollary 1.2. Under the assumptions of Corollary 1.2, the nonlinear system (4) becomes

$$\mathscr{G}_1^1(X_1^0, X_2^0) = \frac{-X_1^0 X_2^0(\rho + 2\phi)}{4\rho}$$
$$\mathscr{G}_1^2(X_1^0, X_2^0) = \frac{-2(1 + (X_1^0)^2 + 3(X_2^0)^2)\phi}{8\rho} - \\ - \frac{(X_1^0)^2 + 3(X_2^0)^2 - 1}{8}.$$

This system has the following four solutions

$$(X_1^{0*}, X_2^{0*}) = \left(0, \pm\sqrt{\frac{1 - c^2\phi(\rho - 2\phi)}{3}}\right),$$
$$(X_3^{0*}, X_4^{0*}) = \left(\pm\sqrt{1 - c^2\phi(\rho - 2\phi)}, 0\right).$$

Note that the solutions which differs in a sign are different initial conditions of the same periodic solution of the system (2). Moreover, since

$$\det\left(\frac{\partial(\mathscr{G}_1^1, \mathscr{G}_1^2)}{\partial(X_1^0, X_2^0)}\right)\bigg|_{(X_1^{0*}, X_2^{0*})} = \frac{1}{4c^2\rho^2} \neq 0,$$

and

$$\det\left(\frac{\partial(\mathscr{G}_1^1, \mathscr{G}_1^2)}{\partial(X_1^0, X_2^0)}\right)\bigg|_{(X_3^{0*}, X_4^{0*})} = \frac{-1}{4c^2\rho^2} \neq 0,$$

these solutions are simple. Finally, by Theorem 1.1 we only have two periodic solutions for the system of this corollary. □

Proof of Corollary 1.4. Under the assumptions of Corollary 1.4, the nonlinear system (5) becomes

$$\mathscr{G}_2^1(X_5^0, X_6^0) = -\frac{X_5^0}{2c\rho},$$
$$\mathscr{G}_2^2(X_5^0, X_6^0) = -\frac{1 + 2X_6^0}{4c\rho}.$$

This system has the following solution

$$(X_5^{0*}, X_6^{0*}) = (0, \frac{-1}{2}).$$

Moreover, since

$$\det\left(\frac{\partial(\mathscr{G}_2^1,\mathscr{G}_2^2)}{\partial(X_5^0,X_6^0)}\right)\Bigg|_{(X_5^{0*},X_6^{0*})} = \frac{1}{4c^2\rho^2}\neq 0$$

this solution is simple. Finally, by Theorem 1.3 we only have one periodic solution for the system of this corollary. □

4 Appendix: Basic results on averaging theory

In this appendix we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of $T-$periodic solutions from a differential system of the form

$$\dot{\mathrm{x}}(t) = G_0(t,\mathrm{x}) + \varepsilon G_1(t,\mathrm{x}) + \varepsilon^2 G_2(t,\mathrm{x},\varepsilon), \qquad (10)$$

with $\varepsilon \neq 0$ sufficiently small. Here the functions $G_0, G_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $G_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ are $\mathscr{C}^2$ functions, $T-$periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathrm{x}}(t) = G_0(t,\mathrm{x}), \qquad (11)$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathrm{x}(t,\mathrm{z},\varepsilon)$ be the solution of the system (11) such that $\mathrm{x}(0,\mathrm{z},\varepsilon)=\mathrm{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathrm{x}(t,\mathrm{z},0)$ as

$$\dot{\mathrm{y}} = D_{\mathrm{x}} G_0(t,\mathrm{x}(t,\mathrm{z},0))\mathrm{y}. \qquad (12)$$

In what follows we denote by $M_{\mathrm{z}}(t)$ some fundamental matrix of the linear differential system (12), and by $\xi : \mathbb{R}^k\times\mathbb{R}^{n-k}\to\mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1,\dots,x_n)=(x_1,\dots,x_k)$.

We assume that there exists a k-dimensional submanifold $\mathscr{Z}$ of Ω filled with $T-$periodic solutions of (11). Then an answer to the problem of bifurcation of $T-$periodic solutions from the periodic solutions contained in $\mathscr{Z}$ for system (10) is given in the following result.

Theorem 4.1. Let V be an open and bounded subset of $\mathbb{R}^k$, and let $\beta : \mathrm{Cl}(V)\to\mathbb{R}^{n-k}$ be a $\mathscr{C}^2$ function. We assume that

(i) $\mathscr{Z} = \{\mathrm{z}_\alpha = (\alpha,\beta(\alpha)), \alpha\in\mathrm{Cl}(V)\}\subset\Omega$ and that for each $\mathrm{z}_\alpha\in\mathscr{Z}$ the solution $\mathrm{x}(t,\mathrm{z}_\alpha)$ of (11) is T-periodic;

(ii) for each $\mathrm{z}_\alpha\in\mathscr{Z}$ *there is a fundamental matrix* $M_{\mathrm{z}_\alpha}(t)$ of (12) such that the matrix $M_{\mathrm{z}_\alpha}^{-1}(0)-M_{\mathrm{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k\times(n-k)$ zero matrix, and in the lower right corner a $(n-k)\times(n-k)$ matrix Δ_α with $\det(\Delta_\alpha)\neq 0$.

We consider the function $\mathscr{G} : \mathrm{Cl}(V)\to\mathbb{R}^k$

$$\mathscr{G}(\alpha) = \xi\left(\frac{1}{T}\int_0^T M_{\mathrm{z}_\alpha}^{-1}(t)G_1(t,\mathrm{x}(t,\mathrm{z}_\alpha,0))dt\right).$$

If there exists $a\in V$ with $\mathscr{G}(a)=0$ and $\det((d\mathscr{G}/d\alpha)(a))\neq 0$, then there is a $T-$periodic solution $\mathrm{x}(t,\varepsilon)$ of system (10) such that $\mathrm{x}(0,\varepsilon)\to\mathrm{z}_a$ as $\varepsilon\to 0$.

For a proof of Theorem 4.1 see Malkin [11] and Roseau [12], or [1] for shorter proof.

Acknowledgements

This work has been partially supported by MCYT/FEDER grant number MTM2011-22587.

The authors want to express their acknowledgment to Prof. Juan Luis García Guirao and Prof. Juan Antonio Vera for their encouragement and guidance during the present research.

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