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Some Recent Results on Hardy-Type Inequalities

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We present some recent results on Hardy type inequalities in $\mathbb{R}^n$, on open subset and for magnetic Dirichlet forms.

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1 Introduction

The inequalities of Hardy and Sobolev have a pivotal role in analysis and continue to be topics of intensive study. In its familiar basic form in $L^p(\mathbb{R}^n)$, the Hardy inequality takes the form

$$\int_{\mathbb{R}^n} |\nabla f|^p d\mathbf{x} \geq C_H(n,p) \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^p}{|\mathbf{x}|^p} d\mathbf{x}, \quad f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}), \tag{1.1}$$

with best possible constant $C_H(n,p) = \{(n-p)/p\}^p$; while the Sobolev inequality is, for $1 \leq p < n$ and $p^* := np/(n-p)$,

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C_S(n,p) \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{1.2}$$

with best possible constant

$$C_S(n,p) = \pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{(p-1)/p} \left\{\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right\}^{1/n},$$

for $1 < p < n$, and

$$C_S(n,1) = \pi^{-1/2} n^{-1} \left(\Gamma(1+n/2)\right)^{1/n}.$$

In the case $p = 2$, both inequalities are especially important in the spectral analysis of differential operators.

The two inequalities combine to give the following inequality: for $0 < \delta < C_H(n,p), 1 \le p < n$,

$$\|\nabla f\|^p_{L^p(\mathbb{R}^n)} - \delta \|f/|\cdot|\|^p_{L^p(\mathbb{R}^n)} \ge \{1-\delta/C_H(n,p)\}\|\nabla f\|^p_{L^p(\mathbb{R}^n)} \\ \ge [\{1-\delta/C_H(n,p)\}/C^p_S(n,p)]\,\|f\|^p_{L^{p^*}(\mathbb{R}^n)},$$

and so

$$\|f\|^p_{L^{p*}(\mathbb{R}^n)} \le C\left\{\|\nabla f\|^p_{L^p(\mathbb{R}^n)} - \delta\|f/|\cdot|\|^p_{L^p(\mathbb{R}^n)}\right\}, \tag{1.3}$$

where $C \le C^p_S(n,p)\{1-\delta/C_H(n,p)\}^{-1}$. In the case $p=2$, Stubbe [32] shows that the optimal value of the constant C is

$$C^2_S(n,2)[1-\delta/C_H(n,2)]^{-(n-1)/n}.$$

Similar inequalities, based on an affine invariant form of the Hardy inequality in which ∇f is replaced by $\mathbf{x}\cdot\nabla\mathbf{f}$, and a generalisation of Sobolev's inequality obtained by Ledoux in [26], were established in [4], and form the basis of the discussion in section 2. The aforementioned inequality of Ledoux is that, for every $1 \le p < q < \infty$ and every function f in the Sobolev space $W^{1,p}(\mathbb{R}^n)$,

$$\|f\|_{L^q(\mathbb{R}^n)} \le C\|\nabla f\|^\theta_{L^p(\mathbb{R}^n)}\|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}}, \tag{1.4}$$

where $\theta = p/q, C$ is a positive constant which depends only on p, q and n, and $B^\alpha_{\infty,\infty}$ is the homogenous Besov space of indices (α,∞,∞); see [33]. The latter is the space of tempered distributions for which the norm

$$\|f\|_{B^\alpha_{\infty,\infty}} := \sup_{t>0}\{t^{-\alpha/2}\|P_t f\|_{L^\infty(\mathbb{R}^n)}\}$$

is finite, where $P_t = e^{t\Delta}, t \ge 0$, is the heat semigroup on $\mathbb{R}^n$: recall that $\{P_t\}_{t\ge 0}$ is defined by $P_0 f = f$ and

$$P_t f(\mathbf{x}) = \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} f(\mathbf{y})\mathbf{e}^{-|\mathbf{x}-\mathbf{y}|^2/4\mathbf{t}}\mathbf{dy}$$

for $t > 0, \mathbf{x} \in \mathbb{R}^n$. Cases of (1.4) were earlier established in [9–11]. The inequality (1.4) is easily seen to include the classical Sobolev inequality (1.2). Ledoux's technique requires specific information on the heat semi-group $e^{t\Delta}$ in $L^2(\mathbb{R}^n)$. For the application in [4] discussed in section 2, there is a need to determine the operator semi-group e^{-tL^*L}, where $L = \mathbf{x}\cdot\nabla$.

In recent years there has been much interest in analogues of (1.1) on bounded domains, in particular the following for a bounded domain Ω :

$$\int_\Omega |\nabla f(\mathbf{x})|^p d\mathbf{x} \ge C\int_\Omega \frac{|f(\mathbf{x})|^p}{\delta(\mathbf{x})^p}d\mathbf{x}, \quad f \in C^\infty_0(\Omega), \tag{1.5}$$

where the positive constant C depends on p, n and Ω, and $\delta(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \partial\Omega)$ denotes the distance from $\mathbf{x}$ to the boundary $\partial\Omega$ of Ω. It is well-known that (1.5) requires some restrictions on Ω. For a convex Ω it is valid, with best constant $c_p := [(p-1)/p]^p$, although the convexity is not necessary for this result (see [13]). The sharp constant in (1.5) for general non-convex domains is unknown, although, for an arbitrary simply-connected domain Ω in $\mathbb{R}^2$ and $p = 2$, A. Ancona [1] proved the inequality (1.5) with $C = 1/16$. His proof was based on the Koebe one-quarter Theorem. In [24] a stronger version of the Koebe Theorem for some class of planar domains has been established. This yields better estimates for the constant appearing in the Hardy inequality (1.5).

For a general domain Ω in $\mathbb{R}^n$, what can be said is that there is such an inequality when δ is replaced by the *mean distance* δ_M introduced by Davies (see [16]) and defined by

$$\frac{1}{\delta_M^2(\mathbf{x})} := \int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu^2(\mathbf{x})} d\omega(\nu), \tag{1.6}$$

where $d\omega(\nu)$ is the normalised measure on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ and $\delta_\nu(\mathbf{x})$ is the distance from $\mathbf{x}$ to $\partial\Omega$ in the direction ν. A fairly comprehensive treatment of (1.5) in a general setting may be found in [17]. The inequality (1.5) and its various extensions are the subject of section 3. We shall be particularly concerned with the cases when Ω is either convex or the complement of a convex set.

When $n = 2$ in (1.1), the inequality is trivial. In [25], Laptev and Weidl showed, *inter alia*, that

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{a}} f(\mathbf{x})|^2 d\mathbf{x}, \geq \left(\mathrm{dist}(\widetilde{\Psi}, \mathbb{Z})\right)^2 \int_{\mathbb{R}^2} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^2} d\mathbf{x}, \quad f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \tag{1.7}$$

where $\nabla_{\mathbf{a}} := \nabla - i\mathbf{a}$ is the magnetic gradient associated with the magnetic potential $\mathbf{a}$ which, in polar co-ordinates, is of the form

$$\mathbf{a}(r, \theta) = \frac{\Psi(\theta)}{r} \left(-\sin\theta, \cos\theta\right), \quad \Psi \in L^\infty(0, 2\pi), \tag{1.8}$$

with magnetic flux $\widetilde{\Psi} = (1/2\pi) \int_0^{2\pi} \Psi(\theta) d\theta \notin \mathbb{Z}$. In (1.8), the magnetic field $\mathrm{curl}\,\mathbf{a} = 0$ in $\mathbb{R}^2 \setminus \{0\}$ and is of so-called *Aharonov-Bohm* type. If $\widetilde{\Psi} \in \mathbb{Z}$, the problem is equivalent to that with no magnetic field (by a gauge transformation) when there is no non-trivial inequality. Analogous inequalities for Aharonov-Bohm magnetic fields with multiple singularities are obtained in [2], and for general magnetic fields in [6]. These results are the subject of section 4.

2 Hardy and Hardy-Sobolev-Type Inequalities in $\mathbb{R}^n$

The Hardy inequality (1.1) and the Sobolev inequality (1.2) are both invariant under orthogonal transformations and scaling. But they are not invariant under general linear transformations. In [29] a new remarkable sharp affine L^p Sobolev inequality for functions on Euclidean n-space was established. This new inequality is significantly stronger than (and directly implies) the classical sharp L^p Sobolev inequality, even though it uses only the vector space structure and standard Lebesgue measure on $\mathbb{R}^n$. For this inequality, no inner product, norm, or conformal structure is needed; the inequality is invariant under all affine transformations of $\mathbb{R}^n$. Such affine invariant inequalities are important in many areas of image processing [3].

The next theorem is an affine invariant version of the Hardy inequality and is also stronger than the classical inequality (1.1).

Theorem 2.1. *Let $n \geq 1$ and $1 \leq p < \infty$. Then for all $f \in C_0^\infty(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} |(\mathbf{x}\cdot\nabla)f|^p d\mathbf{x} \geq \left(\frac{n}{p}\right)^p \int_{\mathbb{R}^n} |f|^p d\mathbf{x}. \tag{2.1}$$

Proof. For any differentiable function $V : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{div} V |f|^p d\mathbf{x} &= -p\, Re \int_{\mathbb{R}^n} (V\cdot\nabla f)|f|^{p-2}\overline{f} d\mathbf{x} \\ &\leq p\left(\int_{\mathbb{R}^n} |V\cdot\nabla f|^p d\mathbf{x}\right)^{1/p}\left(\int_{\mathbb{R}^n} |f|^p d\mathbf{x}\right)^{(p-1)/p} \\ &\leq \varepsilon^p \int_{\mathbb{R}^n} |V\cdot\nabla f|^p d\mathbf{x} + (p-1)\varepsilon^{-p/(p-1)}\int_{\mathbb{R}^n}|f|^p d\mathbf{x} \end{aligned} \tag{2.2}$$

for any $\varepsilon > 0$. Now choose $V(\mathbf{x}) = \mathbf{x}$ to get

$$\int_{\mathbb{R}^n} |(\mathbf{x}\cdot\nabla)f|^p d\mathbf{x} \geq K(n,\varepsilon)\int_{\mathbb{R}^n} |f|^p d\mathbf{x},$$

where

$$K(n,\varepsilon) = \varepsilon^{-p}\{n-(p-1)\varepsilon^{-p/(p-1)}\}.$$

This takes its maximum value $(n/p)^p$ when $\varepsilon^{p/(p-1)} = p/n$. This proves the theorem. □

Remark 2.1. The inequality (2.1) implies (1.1) for $1 \leq p \leq n$. For we have from

$$\nabla(|\mathbf{x}|f) = \frac{\mathbf{x}}{|\mathbf{x}|}f + |\mathbf{x}|\nabla f$$

that

$$\begin{aligned}\|\nabla(|\mathbf{x}|f)\|_{L^p(\mathbb{R}^n)} &\geq \||\mathbf{x}||\nabla f|\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)} \\ &\geq \|(\mathbf{x}\cdot\nabla)f\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)} \\ &\geq \left(\frac{n-p}{p}\right)\|f\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

whence (1.1) on replacing $f(\mathbf{x})$ by $f(\mathbf{x})/|\mathbf{x}|$.

Ledoux's inequality (1.5) is applied in [4] to an inequality involving $L = \mathbf{x}\cdot\nabla$, after first analysing the operator semi-group e^{-L^*L} in $L^2(\mathbb{R}^n)$. The operator L is readily seen to satisfy

$$L = iA - \frac{n}{2},$$

where A is the self-adjoint generator of the group of dilations $\{U(t) : t \in \mathbb{R}\}$ in $L^2(\mathbb{R}^n)$, namely

$$U(t)\psi(\mathbf{x}) := e^{tn/2}\psi(e^t\mathbf{x}),$$

and this gives

$$L^*L = (-iA - \frac{n}{2})(iA - \frac{n}{2}) = A^2 + \frac{n^2}{4}.$$

Hence, $L^*L \geq n^2/4$, in accordance with Theorem 2.1.

Consider the co-ordinate change determined by the map $\Phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}\times\mathbb{S}^{n-1})$ defined by

$$(\Phi\psi)(s,\omega) := e^{sn/2}\psi(e^s\omega) \tag{2.3}$$

for $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$. Note that we equip $\mathbb{R}\times\mathbb{S}^{n-1}$ with the usual one dimensional Lebesgue measure on $\mathbb{R}$ and the usual surface measure on $\mathbb{S}^{n-1}$. Thus Φ preserves the L^2 norm. The inverse of Φ satisfies $\Phi^{-1} : L^2(\mathbb{R}\times\mathbb{S}^{n-1}) \to L^2(\mathbb{R}^n)$ and is given by

$$(\Phi^{-1}\varphi)(\mathbf{x}) = r^{-n/2}\varphi\big(\ln r, \omega\big).$$

Let P_t denote e^{-tA^2}. Then, from [4, Theorem],

$$\Phi P_t \Phi^{-1}\varphi(r,\omega) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} \exp\{-\frac{1}{4t}(r-s)^2\}\varphi(s\omega)ds. \tag{2.4}$$

The fact that $\Phi e^{-tA^2}\Phi^{-1}$ in (2.4) is essentially radial means that the analogue of (1.4) derived by Ledoux's technique is a consequence of the one-dimensional case of (1.4). Defining B^α to be the space of all tempered distributions g on $\mathbb{R}\times\mathbb{S}^{n-1}$ for which the norm

$$\|g\|_{B^\alpha} := \sup_{t>0}\{t^{-\alpha/2}\|\Phi e^{-tA^2}\Phi^{-1}g\|_{L^\infty(\mathbb{R}\times\mathbb{S}^{n-1})}\} < \infty, \tag{2.5}$$

one obtains from the $n = 1$ case of (1.4), that for any $\omega \in \mathbb{S}^{n-1}$,

$$\begin{aligned}
&\int_{\mathbb{R}} |g(r,\omega)|^q dr \leq C^q \int_{\mathbb{R}} \left|\frac{\partial g(r,\omega)}{\partial r}\right|^p dr \\
&\times \left(\sup_{t>0, r\in\mathbb{R}} t^{\theta/2(1-\theta)}\left|\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-(r-s)^2/4t} g(s,\omega)ds\right|\right)^{q(1-\theta)} \\
&= C^q \int_{\mathbb{R}} \left|\frac{\partial g(r,\omega)}{\partial r}\right|^p dr \left(\sup_{t>0, r\in\mathbb{R}} t^{\theta/2(1-\theta)}\left|\Phi e^{-tA^2}\Phi^{-1}g(r,\omega)\right|\right)^{q(1-\theta)} \\
&\leq C^q \int_{\mathbb{R}} \left|\frac{\partial g(r,\omega)}{\partial r}\right|^p dr \left(\sup_{t>0} t^{\theta/2(1-\theta)}\left\|\Phi e^{-tA^2}\Phi^{-1}g\right\|_{L^\infty(\mathbb{R}\times\mathbb{S}^{n-1})}\right)^{q(1-\theta)} \\
&\leq C^q \int_{\mathbb{R}} \left|\frac{\partial g(r,\omega)}{\partial r}\right|^p dr \|g\|^{q(1-\theta)}_{B^{\theta/(\theta-1)}}.
\end{aligned}$$

Integrating with respect to ω over $\mathbb{S}^{n-1}$ yields

Theorem 2.2. *Let $1 \leq p < q < \infty$ and suppose that g is such that $\Phi A \Phi^{-1} g \equiv -i(\partial/\partial r) g \in L^p(\mathbb{R} \times \mathbb{S}^{n-1})$ and $g \in B^{\theta/(\theta-1)}, \theta = p/q$. Then there exists a positive constant C, depending on p and q, such that*

$$\|g\|_{L^q(\mathbb{R}\times\mathbb{S}^{n-1})} \leq C \|(\partial/\partial r) g\|^{\theta}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} \|g\|^{1-\theta}_{B^{\theta/(\theta-1)}}. \tag{2.6}$$

The following corollary is obtained in [4, Corollary 2].

Corollary 2.1. (i) *Let $p^* := np/(n-p), 1 \leq p \leq n-1$, and suppose $(\partial/\partial r) g \in L^p(\mathbb{R} \times \mathbb{S}^{n-1})$ and $\sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot,\omega)\|_{L^p(\mathbb{R})} < \infty$. Then*

$$\|g\|_{L^{p^*}(\mathbb{R}\times\mathbb{S}^{n-1})} \leq C \|(\partial/\partial r) g\|^{1/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} \sup_{\omega\in\mathbb{S}^{n-1}} \|g(\cdot,\omega)\|^{(n-1)/n}_{L^p(\mathbb{R})}. \tag{2.7}$$

(ii) *If $G = \mathcal{M}(g)$ denotes the integral mean of g, namely,*

$$G(r) = \mathcal{M}(g)(r) := \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} g(r,\omega) d\omega,$$

then if $g, (\partial/\partial r) g \in L^p(\mathbb{R} \times \mathbb{S}^{n-1})$,

$$\|G\|_{L^{p^*}(\mathbb{R})} \leq C \|(\partial/\partial r) g\|^{1/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} \|g\|^{(n-1)/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{2.8}$$

If g is supported in $[-\Lambda, \Lambda] \times \mathbb{S}^{n-1}$, then

$$\|g\|_{L^{p^*}(\mathbb{R}\times\mathbb{S}^{n-1})} \leq C \Lambda^{(n-1)/n^2} \|(\partial/\partial r) g\|^{1/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} \sup_{\omega\in\mathbb{S}^{n-1}} \|g(\cdot,\omega)\|^{(n-1)/n}_{L^{p^*}(\mathbb{R})}; \tag{2.9}$$

also

$$\|G\|_{L^{p^*}(\mathbb{R})} \leq C \Lambda^{(n-1)/n} \|(\partial/\partial r) g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{2.10}$$

The case $p = 2$ of Corollary 2.1 is of special interest and gives analogues of Stubbe's Hardy-Sobolev inequality (1.3).

Corollary 2.2. (i) *Let f be such that $Lf \in L^2(\mathbb{R}^n), L = \mathbf{x} \cdot \nabla$, and*

$$\sup_{\omega\in\mathbb{S}^{n-1}} \|f(\cdot,\omega)\|_{L^2(\mathbb{R}^+;d\mu)} < \infty.$$

Then, for $n \geq 3$,

$$\|rf(r\omega)\|^2_{L^{2^*}(\mathbb{R}^n)} \leq C \left\{ \|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4} \|f\|^2_{L^2(\mathbb{R}^n)} \right\}^{1/n} \sup_{\omega\in\mathbb{S}^{n-1}} \|f(\cdot,\omega)\|^{2(1-1/n)}_{L^2(\mathbb{R}^+;d\mu))}, \tag{2.11}$$

where $2^* = 2n/(n-2)$ and $d\mu = r^{n-1} dr$.

(ii) *If* $f, Lf \in L^2(\mathbb{R}^n)$, *then, with* $F := \mathcal{M}(f)$,

$$\|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C \left\{ \|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)} \right\}^{1/n} \|f\|^{2(1-1/n)}_{L^2(\mathbb{R}^n)}. \qquad (2.12)$$

For $0 \leq \delta < n^2/4$, *we have*

$$\|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C \left(n^2/4 - \delta\right)^{-(n-1)/n} \left\{ \|Lf\|^2_{L^2(\mathbb{R}^n)} - \delta\|f\|^2_{L^2(\mathbb{R}^n)} \right\}. \qquad (2.13)$$

The following local Hardy-Sobolev type inequalities are also consequences.

Corollary 2.3. *Let* f *be supported in the annulus* $A_R := \{\mathbf{x} \in \mathbb{R}^n : 1/R \leq |\mathbf{x}| \leq R\}$. *Then*

$$\|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C(\ln R)^{2(n-1)/n} \left\{ \|Lf\|^2_{L^2(\mathbb{R}^n)} - (n^2/4)\|f\|^2_{L^2(\mathbb{R}^n)} \right\}; \qquad (2.14)$$

$$\|F\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C(\ln R)^{2(n-1)/n} \left\{ \|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \left[\frac{n-2}{2}\right]^2 \left\| \frac{f}{|\cdot|} \right\|^2_{L^2(\mathbb{R}^n)} \right\}. \qquad (2.15)$$

The inequality (2.15) is reminiscent of the case $s = 1$ of (2.6) in [22, Section 6.4]; this is also proved in [8]. To be specific, it is that if $f \in C_0^\infty(\Omega)$ and $2 \leq q < 2^*$,

$$\|f\|^2_{L^q(\mathbb{R}^n)} \leq C|\Omega|^{2(1/q-1/2^*)} \left\{ \|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \left[\frac{n-2}{2}\right]^2 \left\| \frac{f}{|\cdot|} \right\|^2_{L^2(\mathbb{R}^n)} \right\}, \qquad (2.16)$$

where $|\Omega|$ denotes the volume of Ω. It is noted in [22, Remark 2.4] that, in contrast to (2.15), the q in (2.15) must be strictly less than the critical Sobolev exponent $2^* = 2n/(n-2)$ if Ω includes the origin.

3 On Hardy-Type Inequalities on Open Subsets

It is known (see [30] and [31]) that if Ω is a convex domain in $\mathbb{R}^n$, the best constant in (1.5) is $C = c_p := [(p-1)/p]^p$, but there are smooth domains for which $C < c_p$. Numerous extensions of this result have been proved in recent years. In [7], Brézis and Markus proved that for any convex Ω, the largest possible constant $\lambda(\Omega)$ such that

$$\int_\Omega |\nabla f(\mathbf{x})|^2 d\mathbf{x} \geq (1/4)\int_\Omega \frac{|f(\mathbf{x})|^2}{\delta(\mathbf{x})^2} d\mathbf{x} + \lambda(\Omega)\int_\Omega |f(\mathbf{x})|^2 d\mathbf{x}, \quad f \in C_0^\infty(\Omega), \qquad (3.1)$$

satisfies

$$\lambda(\Omega) \geq (4\,\mathrm{diam}(\Omega)^2)^{-1}. \qquad (3.2)$$

Various improvements of this result are discussed in [18]. Of particular interest is the following analogue of the Hardy-Sobolev inequality (1.3) established in [21] in $L^p(\Omega)$, when Ω is convex with finite internal diameter: for $1 < p < n$ and $p \le q < p^*$,

$$\|f\|^p \le C(n,p,q)|\Omega|^{p(1/q-1/p^*)} \left\{ \|\nabla f\|^p_{L^p(\Omega)} - c_p \|f/\delta\|^p_{L^p(\Omega)} \right\}. \tag{3.3}$$

Our main concern will be with inequalities for an open set Ω which is either the complement of a convex set or is convex.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^n$ whose complement Ω^c is convex, and let $\delta(\mathbf{x}) := \mathrm{dist}(\mathbf{x}, \Omega^c)$. Then, for all $f \in C_0^\infty(\Omega)$,*

$$\int_\Omega \delta^{2(m-1)} |\nabla \delta^2 \cdot \nabla f|^p d\mathbf{x} \ge \left(\frac{2(2m-1)}{p} \right)^p \int_\Omega \delta^{2(m-1)} |f|^p d\mathbf{x}. \tag{3.4}$$

Proof. In (2.2), set $V(\mathbf{x}) = \nabla \delta^{2m}(\mathbf{x})$. Then, in any compact subset of Ω,

$$\begin{aligned} \mathrm{divV} &= \sum_{i=1}^n \partial_i [\partial_i \delta^{2m}] \\ &= m\delta^{2(m-1)} \Delta \delta^2 + 4m(m-1)\delta^{2(m-1)} |\nabla \delta|^2 \\ &= m\delta^{2(m-1)} \Delta \delta^2 + 4m(m-1)\delta^{2(m-1)}, \end{aligned}$$

since $|\nabla \delta| = 1, a.e.$ in Ω. On substituting in (2.2) and using the Hölder and Young inequalities, we get

$$\begin{aligned} &\int_\Omega \left\{ m\Delta\delta^2 + 4m(m-1) \right\} \delta^{2(m-1)} |f|^p d\mathbf{x} \le p \int_\Omega |\nabla \delta^{2m} \cdot \nabla f| |f|^{p-1} d\mathbf{x} \\ &\quad \le p \left(\int_\Omega |\nabla \delta^{2m} \cdot \nabla f|^p \delta^{-2(p-1)(m-1)} d\mathbf{x} \right)^{1/p} \left(\int_\Omega \delta^{2(m-1)} |f|^p d\mathbf{x} \right)^{1-1/p} \\ &\quad \le m^p \varepsilon^p \int_\Omega |\nabla \delta^2 \cdot \nabla f|^p \delta^{2(m-1)} d\mathbf{x} + (p-1)\varepsilon^{-p/(p-1)} \int_\Omega \delta^{2(m-1)} |f|^p d\mathbf{x}. \end{aligned} \tag{3.5}$$

We claim that $\Delta\delta^2 \ge 2$. To see this, for any $\mathbf{x} \in \mathbf{\Omega}$, rotate the co-ordinate system so that $\mathbf{x} = (\xi_{\mathbf{1}}, \xi')$, where $\xi_1 = \delta(\mathbf{x})$ measured along the line L_1 from $\mathbf{x}$ to its nearest point on the boundary of Ω and $\xi' = (\xi_2, \ldots, \xi_n)$ lies in the $(n-1)$-dimensional orthogonal complement $L_{(n-1)}$ of L_1 in $\mathbb{R}^n$. Then, in view of the rotational invariance of the Laplacian, we have that

$$\Delta\delta^2(\mathbf{x}) = \partial_1^2 \xi_1^2 + \Delta' \delta^2(\mathbf{x}),$$

where Δ' denotes the Laplacian in $L_{(n-1)}$. Since Ω^c is convex, $\Delta'\delta^2(\mathbf{x}) \ge 0$ and so $\Delta\delta^2 \ge 2$, as asserted. It follows from (3.5) that

$$\int_\Omega |\nabla \delta^2 \cdot \nabla f|^p \delta^{2(m-1)} d\mathbf{x} \ge \int_\Omega \delta^{2(m-1)} K(\varepsilon) |f|^p d\mathbf{x}, \tag{3.6}$$

where

$$K(\varepsilon) = \left(\frac{2(2m-1)}{m^{(p-1)}}\right)\varepsilon^{-p} - \left(\frac{p-1}{m^p}\right)\varepsilon^{-p^2/(p-1)}.$$

It is readily shown that K attains its maximum value of $[2(2m-1)/p]^p$ at $\varepsilon = [p/2m(2m-1)]^{(p-1)/p}$. Hence, the theorem follows from (3.6), □

Corollary 3.1. *Let Ω be an open subset of $\mathbb{R}^n$ whose complement is convex, and let $\delta(\mathbf{x}) := \operatorname{dist}(\mathbf{x}, \Omega^c)$. Then, for all $g \in C_0^\infty(\Omega)$ and $\gamma > -1/p$,*

$$\int_\Omega \delta^{p(\gamma+1)}|\nabla g|^p d\mathbf{x} \geq (\gamma + 1/p)^p \int_\Omega \delta^{p\gamma}|g|^p d\mathbf{x}. \tag{3.7}$$

Proof. Put $f = g/\delta^\alpha$ in (3.4). Then we have

$$|\nabla\delta^2 \cdot \nabla f| \leq 2\{\delta^{-\alpha+1}|\nabla g| + \alpha\delta^{-\alpha}|g|\}$$

and

$$\left\|\delta^{[2(m-1)/p-\alpha+1]}\nabla g\right\| \geq \left[\frac{(2m-1)}{p} - \alpha\right]\left\|\delta^{[2(m-1)/p-\alpha]}g\right\|$$

where $\|\cdot\|$ denotes the L^p norm. The inequality (3.7) follows on setting $\gamma = 2(m-1)/p-\alpha$. □

The technique used in the proof of Theorem 3.1 can be used to give the following inequality for a convex Ω, which contains the Hardy inequality (1.5) with the optimal constant.

Theorem 3.2. *Let Ω be an open convex subset of $\mathbb{R}^n$. Then, for all $f \in C_0^\infty(\Omega)$,*

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p d\mathbf{x} \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \delta^{-p}|f|^p d\mathbf{x}, \tag{3.8}$$

and hence,

$$\int_\Omega |\nabla f|^p d\mathbf{x} \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \delta^{-p}|f|^p d\mathbf{x}. \tag{3.9}$$

Proof. In (2.2), we now choose $V = \nabla\delta^{-2m}$, so that

$$\mathrm{div\ V} = \mathrm{m}\delta^{-2(\mathrm{m}-1)}\Delta\delta^{-2} + 4\mathrm{m}(\mathrm{m}-1)\delta^{-2(\mathrm{m}+1)}|\nabla\delta|^2. \tag{3.10}$$

Since Ω is convex, we have in the notation used in the proof of Theorem 3.1, that $\Delta'\delta^{-2} \geq 0$, and

$$\Delta\delta^{-2} \geq \partial_1^2\xi_1^{-2} = 6\xi_1^{-4} = 6\delta^{-4}.$$

Therefore, in (3.10),

$$\mathrm{div\ V} \geq 2\mathrm{m}(2\mathrm{m}+1)\delta^{-2(\mathrm{m}+1)}.$$

On substituting this in (2.2) and applying the Hölder and Young inequalities, as in the proof of Theorem 3.1, we have

$$\int_\Omega |\nabla\delta^{-2}\cdot\nabla f|^p\delta^{4p-2(m+1)}d\mathbf{x} \geq \int_\Omega J(\varepsilon)\delta^{-2(m+1)}d\mathbf{x},$$

where

$$J(\varepsilon) = \frac{2(2m+1)}{m^{p-1}}\varepsilon^{-p} - \frac{p-1}{m^p}\varepsilon^{-p^2/(p-1)} \leq \left(\frac{2(2m+1)}{p}\right)^p,$$

the maximum being attained. It follows that

$$\int_\Omega |\nabla\delta^{-2}\cdot\nabla f|^p\delta^{4p-2(m+1)}d\mathbf{x} \geq \left(\frac{2(2m+1)}{p}\right)^p \int_\Omega J(\varepsilon)\delta^{-2(m+1)}d\mathbf{x}.$$

The theorem follows on choosing $m = (p/2) - 1$. □

4 Hardy-Type Inequalities with Magnetic Fields in 2 Dimensions

Consider the magnetic form

$$h_{\mathbf{a}}[u] = \int |(-\imath\nabla - \mathbf{a})u|^2\, d\mathbf{x} \tag{4.1}$$

on $u \in C_0^1(\mathbb{R}^n)$, $n \geq 2$, with an appropriate vector potential $\mathbf{a} \in L^2(\mathbb{R}^n)$. In view of the diamagnetic inequality [28, p. 179],

$$h_{\mathbf{a}}[u] \geq \int |\nabla|u||^2\, d\mathbf{x}\,, \qquad \forall u \in C_0^1(\mathbb{R}^n),$$

the Hardy inequality implies the same bound for the magnetic form (4.1). In dimension $n = 2$, however, the Hardy inequality (1.1) is trivial. Nevertheless the introduction of a magnetic field can improve this situation. In the present paper we consider the magnetic form (4.1) only in dimension $n = 2$.

For symmetric operators

$$L_1 = -\imath\,\frac{\partial}{\partial x} - a_1(x,y) \quad \text{and} \quad L_2 = -\imath\,\frac{\partial}{\partial y} - a_2(x,y)$$

with $\mathbf{a} = (a_1(x,y), a_2(x,y))$, we have $(L_1 \pm \imath L_2)(L_1 \pm \imath L_2)^* \geq 0$. This implies that $L_1^2 + L_2^2 \geq \pm\imath[L_1, L_2] = \pm B$, where $B := \operatorname{curl}\mathbf{a}$ is a magnetic field. The last inequality is the standard lower bound for the magnetic form (4.1) in dimension $d = 2$, namely,

$$h_{\mathbf{a}}[u] \geq \int \pm B|u|^2\, dxdy\,, \tag{4.2}$$

which holds with either of the signs $\pm$.

If B is positive (or negative) and big enough, then (4.2) gives a very good lower bound for (4.1). It is worth pointing out that very little is known about a lower bound for (4.1) in the case of a B with variable sign. In the case of $B = 0$ or in the case when the support of u is located outside of the support of B, the inequality (4.2) is useless and another approach is needed. In this section we explain in the case of regular B how to combine (4.2) and the Hardy inequality for domains with Lipschitz boundaries to get a bound of the form

$$h_{\mathbf{a}}[u] \geq c \int \widetilde{B}|u|^2 \, dxdy \,,$$

with an *effective* magnetic field $\widetilde{B}$, which coincides with $\pm B$ on its support and decays outside the support as $\operatorname{dist}\{\mathbf{x}, \operatorname{supp} B\}^{-2}$ (see [6] for more details).

We also show how to get a Hardy type inequality in the case of Aharonov-Bohm magnetic potentials, i.e., when $B = 0$ on a punctured plane $M = \mathbb{R}^2 \setminus \{P_1, \ldots, P_n\}$ (see [2] for details).

It was shown in [5] that for Aharonov-Bohm magnetic potentials we have Sobolev and Cwickel-Lieb-Rosenblum inequalities in dimension two. In [19] the authors show that introducing a magnetic field also improves the Relich inequality in dimension 4 and the results obtained were used in [20] to count eigenvalues of biharmonic operators with magnetic fields.

Let $P_1 = (x_1, y_1), \ldots, P_n = (x_n, y_n)$ be n different points in $\mathbb{R}^2$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and the points $P_1, \ldots, P_n$ then correspond to the complex numbers $z_1 = x_1 + \imath y_1, \ldots, z_n = x_n + \imath y_n$. Consider a smooth vector potential $\mathbf{a} = (a_1(x,y), a_2(x,y))$ in the punctured plane $M = \mathbb{R}^2 \setminus \{P_1, \ldots, P_n\}$ with magnetic field

$$B := \text{curl}\, \mathbf{a} = 0 \,. \tag{4.3}$$

Such a vector potential $\mathbf{a}$ is known as a magnetic vector potential of Aharonov-Bohm type.

Let us denote by $\omega_{\mathbf{a}}$ the differential 1-form $a_1(x,y)\, dx + a_2(x,y)\, dy$. Then (4.3) is equivalent to $d\,\omega_{\mathbf{a}} = 0$, i.e. $\omega_{\mathbf{a}}$ is a closed differential form. The condition (4.3) implies that in any simply connected open subset of M, there exists a gauge function f such that $\mathbf{a} = \nabla f$.

For each point P_k $(k = 1, \ldots, n)$ let us define a circulation of $\mathbf{a}$ around P_k as

$$\Phi_k = \frac{1}{2\pi} \int\limits_{\gamma_k} \omega_{\mathbf{a}} \,, \tag{4.4}$$

where γ_k is a small circle in M which winds once around P_k in an anticlockwise direction. Condition (4.3) implies that (4.4) is invariant under continuous deformations of γ_k inside M.

For $n = 1$ and $P_1 = (0,0)$ Laptev and Weidl in [25] proved that

$$h_{\mathbf{a}}[u] \geq \min_{n \in \mathbb{Z}} |\Phi_1 - n|^2 \int \frac{|u|^2}{|\mathbf{x}|^2} \, d\mathbf{x} \,, \qquad u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \,. \tag{4.5}$$

The proof of (4.5) is based on the observation that for a suitable choice of gauge, the angular part of the quadratic form $h_{\mathbf{a}}[u]$ is separated from zero if the flux Φ_1 stays away from the set of integers. Unfortunately when $n \geq 2$ there is no natural decomposition of $h_{\mathbf{a}}[u]$ into radial and spherical parts and a new approach is needed.

We are looking for a lower bound for (4.1) by a Hardy-type expression

$$h_{\mathbf{a}}[u] \geq \int\limits_M H(x,y)|u(x,y)|^2\,dxdy\,, \quad u \in C_0^\infty(M) \tag{4.6}$$

with a suitable non-negative function $H(x,y)$ on M.

For any real number Ψ denote by $p(\Psi)$ the distance from Ψ to the set of integers $\mathbb{Z}$, i.e.

$$p(\Psi) = \min_{k\in\mathbb{Z}} |k - \Psi|\,. \tag{4.7}$$

We are interested in those functions $H(x,y)$ which satisfy the following conditions.

1. $H(x,y)$ depends on $\mathbf{a}$ only throughout the circulations $\Phi_1, \ldots, \Phi_n$ and the coordinates of $P_j,\ \ j = 1, \ldots, n$.

2. $H(x,y)$ behaves like

$$\frac{(p(\Phi_j))^2}{(x-x_j)^2 + (y-y_j)^2}$$

near each point $P_j,\ \ j = 1, \ldots, n$, since around point P_j only circulation Φ_j is present. Near infinity, $H(x,y)$ behaves like

$$\frac{(p(\Phi_1 + \ldots + \Phi_n))^2}{x^2+y^2},$$

since $\Phi_1 + \ldots + \Phi_n$ is the circulation around infinity.

For the reader's convenience we finish this introduction by giving an example of $H(x,y)$ in the case of two points $P_1 = -1$ and $P_2 = 1$ in $\mathbb{C}$ with the circulations $c_1 \equiv \Phi_1$ and $c_2 \equiv \Phi_2$, respectively.

Example 4.1. Let $P_1 = (-1,0)$, $P_2 = (1,0)$ be two points in $\mathbb{R}^2$, $M = \mathbb{R}^2 \setminus \{P_1, P_2\}$ and $\mathbf{a}$ is a magnetic vector potential of Aharonov-Bohm type in M with the circulations c_j round P_j, $j = 1, 2$. Then the inequality (4.6) holds with

$$H(x,y) = C(x,y) \cdot \left|\frac{2z}{z^2-1}\right|^2, \quad z = x + iy,$$

where $C(x,y)$ is the piecewise constant function on $\mathbb{R}^2$ shown in Figure 4.1.

In the figure, C is the curve $(x^2 - y^2 - 1) + 4x^2y^2 = 1$ which divides the plane $\mathbb{R}^2$ into three regions Ω_1, Ω_2 and Ω_∞, where $P_1 \in \Omega_1$ and $P_2 \in \Omega_2$; $C(x,y)$ equals $(p(c_1))^2$ in Ω_1, $(p(c_2))^2$ in Ω_2 and $(p(c_1+c_2))^2/4$ in Ω_∞.

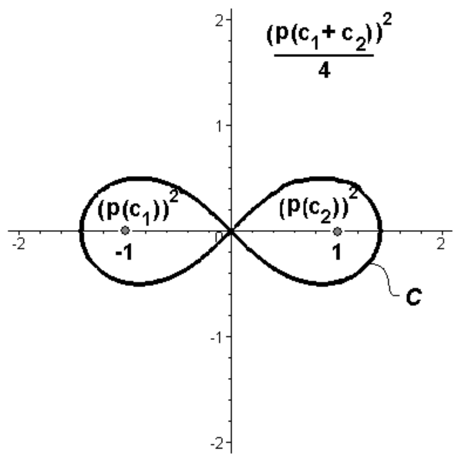

Figure 4.1: Function C(x,y)

Our approach is based on the conformal invariance of magnetic Dirichlet forms with Aharonov-Bohm potentials. The strategy is first to establish a Hardy-type inequality for doubly connected domains in $\mathbb{C}$ using uniformization, and second to use an analytic function F to decompose $\mathbb{C}$ into doubly connected domains with explicit uniformizations.

Let us show that any analytic function $F(z)$ on $\mathbb{C}$ with zero set $\{P_1, \dots, P_n\}$ and $F(\infty) = \infty$ generates a function $H(x, y)$ with properties **1** and **2** above.

Denote by $\mathrm{ord}_{P_j} F$ the order of zero of F at P_j. Let $\{Q_1, \dots, Q_l\}$ be a zero set of the complex derivative F'_z of the function F, i.e. $\{Q_1, \dots, Q_l\} = (F'_z)^{-1}(0)$, and denote by Crit_F the following subset of $\mathbb{R}_+ = \{x \in \mathbb{R} | \, x \geq 0\}$:

$$\mathrm{Crit}_F = \{0, |F(Q_1)|, \dots, |F(Q_l)|\} .$$

Under the map $|F| : \mathbb{C} \to \mathbb{R}_+$ the pre-image of Crit_F is a zero measure set $\mathcal{F}_c$.

Let us define a piecewise constant function C_F on $\mathbb{R}^2$. For any $(x, y) \in \mathbb{R}^2$, $x + \imath y \notin \mathcal{F}_c$, the set $|F|^{-1}(|F|(x+\imath y))$ is a disjoint union of smooth simple curves in $\mathbb{C}$; let $\gamma_{(x,y)}$ denote one of them that goes through the point (x, y). This $\gamma_{(x,y)}$ divides $\mathbb{C}$ into two domains, a bounded domain $\Omega_{\text{int}}(\gamma_{(x,y)})$ and an unbounded domain $\Omega_{\text{ext}}(\gamma_{(x,y)})$. Then

$$\mathrm{C}_F(x, y) := \frac{\left(p\left(\sum_{P_k \in \Omega_{\text{int}}(\gamma_{(x,y)})} \Phi_k\right)\right)^2}{(\mathrm{ord}_{\gamma_{(x,y)}} F)^2} \,, \tag{4.8}$$

where Φ_k is a circulation of **a** round P_k and $\mathrm{ord}_{\gamma_{(x,y)}} F = \sum_{P_k \in \Omega_{\text{int}}(\gamma_{(x,y)})} \mathrm{ord}_{P_k} F$.

Theorem 4.1. *Let C_F be defined in (4.8) for the analytic function F. For any $u \in C_0^\infty(M)$*

the following inequality holds

$$\int\limits_M |(-\imath\nabla - \mathbf{a})u|^2 \, dxdy \geq \int\limits_M C_F(x,y) \left| \frac{F_z'(x+\imath y)}{F(x+\imath y)} \right|^2 |u(x,y)|^2 \, dxdy \,. \tag{4.9}$$

The function

$$C_F(x,y) \left| \frac{F_z'(x+\imath y)}{F(x+\imath y)} \right|^2$$

is a function $H(x,y)$ with properties **1** and **2** above.

Let now $a = (a_1, a_2) \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ be a real-valued vector function. Assume that the magnetic field

$$B = \operatorname{curl} \mathbf{a}$$

exists in the sense of distribution and it is measurable on $\mathbb{R}^2$. As was explained at the beginning of this section, we want to combine the estimate (4.2) in the domain Ω where B is large and with the Hardy inequality for the domain $\Omega' = \mathbb{R}^2 \setminus \Omega$. Since a function u from the form-domain of $h_{\mathbf{a}}[u]$ has to be regular, we can't just restrict u to Ω and Ω'. Some sort of partition of unity is required. Before stating the main results, let us introduce an important constant depending on Ω.

Suppose that the boundary of Ω is Lipschitz. Let $\delta(\mathbf{x})$ be the distance from $\mathbf{x} \in \mathbb{R}^2$ to Ω. Then there exists a positive constant $\mu \leq 1/4$ such that for any $u \in H_0^1(\Omega')$, $\Omega' = \mathbb{R}^2 \setminus \Omega$, one has the following Hardy inequality (see [14, 15, 27]):

$$\int\limits_{\Omega'} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \geq \mu \int\limits_{\Omega'} \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^2} \, d\mathbf{x} \,. \tag{4.10}$$

If Ω' is a union of convex connected components, one has $\mu = 1/4$. In view of the diamagnetic inequality we have

$$\int\limits_{\Omega'} |(-\imath\nabla - \mathbf{a})u(\mathbf{x})|^2 \, d\mathbf{x} \geq \mu \int\limits_{\Omega'} \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^2} \, d\mathbf{x} \,, \qquad \forall u \in C_0^1(\Omega') \,. \tag{4.11}$$

For other results connected with the inequality (4.10) and further references see e.g. [7, 23].

We also need to introduce a positive continuous function ℓ which plays the role of a slowly varying spatial scale reflecting variation of the magnetic field. We associate with the function ℓ the open ball

$$\mathcal{K}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{y}| < \ell(\mathbf{x})\} \,.$$

The scale ℓ is assumed to satisfy the conditions

$$\ell \in C^1(\mathbb{R}^2); \qquad |\nabla \ell(\mathbf{x})| \leq 1, \quad \ell(\mathbf{x}) > 0, \quad \forall\, \mathbf{x} \in \mathbb{R}^2 \,. \tag{4.12}$$

To specify further conditions on B we need to divide $\mathbb{R}^2$ into sets relevant to the strength of the field. For a (measurable) set $\mathcal{C} \subset \mathbb{R}^2$ define

$$\mathcal{C}^{\uparrow} = \bigcup_{x \in \mathcal{C}} \mathcal{K}(\mathbf{x}) .$$

With the field B we associate two open sets $\Omega, \Lambda \subset \mathbb{R}^2$, such that $\Omega^{\uparrow} \subset \Lambda$ and $(\mathbb{R}^2 \setminus \Lambda)^{\uparrow} \cap \Omega^{\uparrow} = \emptyset$. The case $\Lambda = \mathbb{R}^2$ is not excluded. Let $\lambda_0 > 0$ be the lowest eigenvalue of the Laplace operator $-\Delta$ on the unit disk with Dirichlet boundary conditions. If Ω has Lipschitz boundary and $\mathbb{R}^2 \setminus \Lambda \neq \emptyset$, then there exists a convenient partition of unity (see [6, Lemma 3.2]): $\zeta, \eta \in C^1(\mathbb{R}^2)$ such that

(i) $0 \le \zeta \le 1$,

(ii) $\xi(\mathbf{x}) = 1$ for $\mathbf{x} \in \Omega$, $\quad \eta(\mathbf{x}) = 1$ for $\mathbf{x} \in \mathbb{R}^2 \setminus \Lambda$,

(iii) $\zeta^2 + \eta^2 = 1$,

(iv) $|\nabla \zeta| \le A\ell^{-1}$, $|\nabla \eta| \le A\ell^{-1}$ with any $A > A_0 = 5(2 + 4\sqrt{\lambda_0})/\sqrt{2}$.

Assume that

$$|B(\mathbf{x})|\, \ell(\mathbf{x})^2 \ge 2A_0^2 , \qquad a.a. \quad \mathbf{x} \in \Lambda . \tag{4.13}$$

The physical meaning of the sets Ω, Λ, is that, on Ω, the field B is large, on $\mathbb{R}^2 \setminus \Lambda$ the field B is negligibly small, and the set $\Lambda \setminus \Omega$ is a transition zone.

Since $\zeta^2 + \eta^2 = 1$, we have for any $u \in C_0^1(\mathbb{R}^2)$

$$\begin{aligned} h_{\mathbf{a}}[u] &= \int |\zeta(-\imath\nabla - \mathbf{a})u|^2 \, d\mathbf{x} + \int |\eta(-\imath\nabla - \mathbf{a})u|^2 \, d\mathbf{x} \\ &= h[\zeta u] + h[\eta u] - \int (|\nabla\zeta|^2 + |\nabla\eta|^2)\, |u|^2 \, d\mathbf{x} . \end{aligned}$$

We can estimate $h[\zeta u]$ using (4.2) and $h[\eta u]$ using (4.11) and $|\nabla\zeta|^2 + |\nabla\eta|^2$ by property (iv) of the partition of unity. We summarise all this in

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^2$ be an open set with Lipschitz boundary. Let the function ℓ be as specified in (4.12), and let the field B satisfy (4.13). Suppose also that the field B is either non-negative or non-positive a.a. $\mathbf{x} \in \mathbb{R}^2$. Then*

$$h_{\mathbf{a}}[u] \ge \frac{\mu}{2} \int \frac{|u(\mathbf{x})|^2}{\ell(\mathbf{x})^2 + \delta(\mathbf{x})^2} \, d\mathbf{x}$$

for all $u \in D[h_{\mathbf{a}}]$.

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