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Solving Deutsch's Problem using Entanglement Measurement Algorithm

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Abstract: Given a black-box representing an unknown Boolean function f, determining whether the unknown function $f(x)$ has $f(0) = f(1)$ or not using a single query to f is known as Deutsch's algorithm. Following [28], in this paper, we propose Deutsch's algorithm based on entanglement measurement. The proposed algorithm creates entanglement between the independent variable x represented by the test qubit and an ancilla qubit called detection qubit, where the entanglement is measured using concurrence measure to decide if the considered function $f(x)$ has $f(0) = f(1)$ or not.

Keywords: Entanglement measurement, quantum algorithms, Deutsch's problem

1 Introduction

Nowadays, the quest to develop quantum computers has owing back to the discovery that some algorithms work dramatically faster on a quantum computer rather than on a classical one. Harnessing the nature of quantum systems captured in superposition and interference, the scientists proposed using the quantum system to compute in a parallel way by proposing a set of amazing quantum algorithms [1,2,3,4,5]. First of these algorithms is called Deutsch's algorithm proposed by David Deutsch [1] in 1985, followed by a set of algorithms all have speed-up performance compared with classical algorithms that solve the same problems [2,3,5]. The attention to quantum computation incredibly increased when Shor's algorithm and Grover's algorithm are discovered [3,4,5]. In a fast consequence, the researchers have extended the scope to include other branches of computer science such as quantum teleportation [6,7,8], quantum machine learning [9,10,11], and quantum cryptography[12]. Quantum mechanics gains its influences and power from two astonishing phenomena which are superposition and entanglement. Quantum entanglement is a unique microscopic physical phenomenon that occurs when two qubits or more are interact in a way such that the quantum state of each qubit cannot be described independently of the other [13], even when the qubits are separated by a vast distance. Entanglement is a pivotal issue in quantum information and quantum computation theory and it is under continuous research [14,15,16,17,18,19,20]. This amazing phenomenon is proved experimentally via a set of experiments [21,22,23]. Entanglement is an area of extremely hot research by the communities of atomic physics and quantum information processing [24], with crucial utilization in many applications, for instance quantum teleportation [7,8], satellite-based quantum key distribution [25,26], and quantum Internet [27].

In order to solve more problems using quantum algorithms, using entanglement measurements as a crucial ingredient step in quantum algorithms have been proposed by M. Zidan et al. [28]. Following this proposal, some researchers applied it to increase the computational speed for testing junta variables[29]. In this paper, we follow [28] to propose an algorithm based on entanglement measurement to solve Deutsch's.

The paper is organized as follows: In Section 2, we briefly review some basic concepts of entangled two qubits and entanglement measurement. Section 3 presents a full

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description of the proposed algorithm to solve Deutsch's problem based on entanglement measurement. Section 4 demonstrates the analysis of the proposed algorithm. Finally, section 5 concludes the paper.

2 Entangled Two qubits and Entanglement Measurement

If the state of a 2-qubit composite system $|\chi\rangle$ cannot always be written in the product form as

$$|\chi\rangle \neq |\chi_1\rangle \otimes |\chi_1\rangle,$$

then $|\chi\rangle$ is called entangled state. In other words, If the 2 qubits are prepared independently, and maintained isolated, then each qubit forms a closed system, so the state can be written in the product form ; then, we say that the qubits are separable qubits. On the other hand, if the qubits are allowed to interact, then the closed system includes both qubits together, and it may not be possible to write the state in the product form; then, we say that the qubits are entangled. The maximally entangled states of two qubits are called Bell states [24,30] and are defined as:

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$|B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Bell states are used in plenty of applications such as an unknown qubit teleportation and quantum key distribution [6]. Entanglement measurements [31,32] are used to reveal if there is an entanglement into quantum systems which are governed by n-dimension Hilbert space such that $n > 1$. There are plenty of entanglement measurements defined based on different considerations such as concurrence, negativity, quantum discord, witness and so on [33,34]. Concurrence measurement [32,33] is considered one of the most popular measurements of entanglement quantification of bipartite system, and can be defined as follows:

$$C = |\langle\phi|\sigma_y \otimes \sigma_y|\phi^\dagger\rangle|,$$

where $\sigma_y = -i|1\rangle\langle 0| + i|0\rangle\langle 1|$, and $i = \sqrt{-1}$. The concurrence of a normalized state in the form $\alpha|00\rangle + \beta|11\rangle$ or $\alpha|01\rangle + \beta|10\rangle$ is calculated theoretically as follows:

$$C = 2|\alpha\beta|, \tag{1}$$

where $0 \leq C \leq 1$. There are a continues quest to implement entanglement measurement experimentally[35,36,37].

3 The proposed Algorithm

3.1 The Proposed Operator

Although, M_z operator [28], Fig. 1, can be used directly in this paper but in order to optimize the number of gates, a modified version is proposed according to the following definitions.

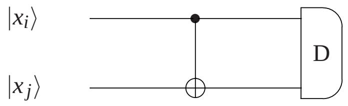

Fig. 1: Quantum circuit for the proposed M_z operator.

Definition 3.1. Consider two arbitrary qubits t and d, such that the t qubit is called the test qubit and d qubit is called the detection qubit.
Definition 3.2. A measuring device of entanglement $D_{i,j}$ measures the concurrence between the test and detection qubits.
Definition 3.3. For arbitrary two-qubit system, an operator E_z is the operator which is applying a black box U_f on the test qubit and the detection qubit, according to the following formula:

$$U_f : |t,d\rangle \rightarrow |t, d \oplus f(t)\rangle. \tag{2}$$

then the device $D_{i,j}$ measures the entanglement in between. The cuircuit model of the proposed E_z operator is shown in Fig. 2.

Remark:
If the oracle U_f is represented by $CNOT$ gate then E_z implies to M_z [28].

3.2 The Proposed algorithm

In this section, we propose a quantum algorithm that uses entanglement measure as an essential step for solving

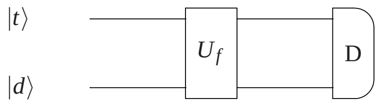

Fig. 2: Quantum circuit for the proposed E_z operator.

Deutsch's problem. In other words, the proposed algorithm shows how to use the proposed operator E_z to create an entanglement between two separable qubits, and then use concurrence measurement to solve the problem at hand. The abstract of Deutsch's problem is defined as follows:

Given: A black-box for computing an unknown function f:$\{0,1\} \to \{0,1\}$.
Goal: Determine whether $f(0) = f(1)$ or not using a single query to f.

The given orcale represents a function take one of the following cases:

$$f(x) = \begin{cases} f(0) = f(1), \\ \\ or \quad f(0) \neq f(1). \end{cases} \tag{3}$$

Classically, there are two queries for f must be conducted to determine if $f(0) = f(1)$?. Deutsch showed that his algorithm capable of answering this question by making only a single query to quantum orcle for f. In this paper, we re-propose another version of this algorithm from quantum measurement perspective. Although, the proposed algorithm has no more speed up comparing the original version but it is more easier to understand. The algorithm is proposed in the following steps:

1.Register Preparation: initialize the two qubit register $|td\rangle$ by the vacuum state as follows

$$|\psi_0\rangle = |td\rangle = |00\rangle.$$

2.Apply Hadamard-gate H on the first qubit as:

$$|\psi_1\rangle = (H \otimes I)|00\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}},$$

where I is the $2x2$ indentity operator.

3.Apply E_z on $|\psi_1\rangle$

$$|\psi_2\rangle = E_z|\psi_1\rangle.$$

(i) If $C = 0$ then
$f(0) = f(1)$.

(ii) If $C = 1$ then
$f(0) \neq f(1)$.

The cuircuit model of the proposed algorithm based on entanglement measurement, E_m, is shown in Fig. 3 .

4 Analysis of the Proposed Algorithm

In this section, we discuss the proposed algorithm with the suggested operator introduced in section 3. We analyze the proposed algorithm assuming that the given oracle is a black box U_f.

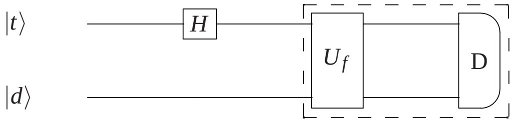

Fig. 3: Quantum circuit for the proposed algorithm via Entanglement measurement.

In step 2, the proposed algorithm applies the Hadamard gate to create equal superposition of state $|0\rangle$ and state $|1\rangle$ on the first qubit as

$$|t\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

where the state of qubit $|d\rangle$ keeps unchanged in state $|0\rangle$.
Step 3 of the proposed algorithm pertains applying E_z. The function under scrutiny, through the given black box U_f, take one of two cases depicted in Eq. (3). The effect of E_z on $|\psi_1\rangle$ can be viewed as, apply the U_f gate between the test qubit $|t\rangle$ and the detection qubit $|d\rangle$ according to Eq. (2) as follows:

(i) In this case, if $f(0) = f(1)$ then the state of $|td\rangle$ will not produce a measurable entanglement between the test and detection qubits because it is a separable state. So, the concurrence between those qubits is $C = 0$.

(ii) But on the other hand, if $f(0) \neq f(1)$ then the state of $|td\rangle$ will be maxmuim entanglement between the test and detection qubits. So, the concurrence between those qubits is $C = 1$.

5 Perspective

In this paper, we propose an algorithm to use entanglement measurement, concretely concurrence, as a crucial ingredient step to decide if a function f under investigation has $f(0) = f(1)$ or not.

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