Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/amisl/030205

Characterizations of Hemirings by $(\in,\in\vee q_k)^*$-Intuitionistic Fuzzy h-Ideals

*Asim Hussain** *and Muhammad Shabir*

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

Received: 23 Oct. 2014, Revised: 15 Mar. 2015, Accepted: 17 Mar. 2015
Published online: 1 May 2015

Abstract: In this paper we define the concept of $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-ideals, $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideals, $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideals of hemiring. And also characterized h-hemiregular and h-*intra*-hemiregular hemiring by the properties of these $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-ideals, $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideals, and $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideals.

Keywords: $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-ideal, $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideals, and $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideas.

1 Introduction

A semiring is an algebraic structure consisting of a non-empty set R together with two associative binary operations, addition "$+$" and multiplication ". " such that ". "distributes over "$+$" from both sides. Semirings which are regarded as a generalization of rings, was first introduced by Vandiver. By a hemiring, we mean a semiring with a zero and with a commutative addition.

Ideals of hemirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals. Many results in rings apparently have no analogues in hemirings using only ideals. Henriksen defined in [7] a more restricted class of ideals in semirings, which is called the class of k-ideals, with the property that if the semiring R is a ring then a complex in R is a k-ideal if and only if it is a ring ideal. Another more restricted, but very important, class of ideals, called now h-ideals, has been given and investigated by Izuka in [10] and La Torre in [13].

The fundamental concept of fuzzy set, introduced by Zadeh in 1965 [19], was applied to generalize some of the basic concepts of algebra. In [15] Azirel Rosenfeld used the idea of fuzzy set to introduce the notions of fuzzy subgroups. In [1] Ahsan et al. initiated the study fuzzy semirings. The fuzzy algebraic structures play important role in mathematics with wide applications in theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [6,17].

The general properties of fuzzy k-ideals of semirings were described in [2,3,11,23], Jun [12] considered the fuzzy setting of h-ideals of hemirings. Moreover, Zhan et al. in [22] discussed the h-hemiregular hemirings by using the fuzzy h-ideals and they discussed the properties of L-fuzzy h-ideals with operators in hemirings [21]. As a continuation of this investigation, Yin et al. in [18] introduced the concepts of fuzzy h-bi-ideals and fuzzy h-quasi-ideals of hemirings. By using these fuzzy ideals, some characterization theorems of h-hemiregular and h-intra-hemiregular hemirings are obtained. Other important results related with fuzzy h-ideals of a hemiring were given in [4,5,8,20].

In [16] M. Shabir and T. Mahmood gave a characterization of h-hemiregular hemirings and h-intra-hemiregular hemirings in terms of $(\in,\in\vee q_k)$-fuzzy right and fuzzy left h-ideals. As a continuation of this paper, we characterize h-hemiregular hemirings and h-intra-hemiregular hemirings in terms of $(\in,\in\vee q_k)^*$-Intuitionistic fuzzy right h-ideal. In this paper we define the concept of $(\in,\in\vee q_k)^*$-Intuitionistic fuzzy right h-ideal, $(\in,\in\vee q_k)^*$-intuitionistic fuzzy left h-ideal and $(\in,\in\vee q_k)^*$-Intuitionistic fuzzy h-bi-ideal and $(\in,\in\vee q_k)^*$-Intuitionistic fuzzy h-quasi-ideal of R.

* Corresponding author e-mail: sh.asim.hussain@gmail.com

2 Preliminaries

Recall that a semiring is an algebraic system $(R, +, .)$ consisting of a non-empty set together with two binary operations "+" and ". " on R which are called addition and multiplication, respectively such that $(R, +)$ and $(R, .)$ are semi groups, linked by the following distributive laws:

$a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for all $a,b,c \in R$.

By a zero of a semiring $(R, +, .)$, we mean an element $0 \in R$ such that $0.\ x = x.\ 0 = 0$ and $0+x = x+0 = x$ for all $x \in R$. A semiring with a zero such that $(R,+)$ is a commutative semigroup is called a hemiring. A non-empty subset A of a hemiring R is called a sub hemiring of R if it contains 0 and closed with respect to addition and multiplication of R. A non-empty subset I of a hemiring R is called a left (right) ideal of R if I is closed under addition and $RI \subseteq I(IR \subseteq I)$. Furthermore I is called an ideal if it is both a left ideal and right ideal of R. A non-empty subset Q of a hemiring R is called a quasi-ideal of R if Q is closed under addition and $RQ \cap QR \subseteq Q$. A sub hemiring B of a hemiring R is called a bi-ideal of R if $BSB \subseteq B$. Every one sided ideal of a hemiring R is a quasi-ideal and every quasi-ideal is a bi-ideal but the converse is not true.

A left (right) ideal I of a hemiring R is called a left (right) h-ideal if for all $x,z \in R$ and for any $a,b \in I$, from $x+a+z = b+z$ it follows $x \in I$. A bi-ideal B of a hemiring R is called an h-bi-ideal of R, if for all $x,z \in R$ and $a,b \in B$, from $x+a+z = b+z$, it follows $x \in B$.

The h-closure $\overline{A}$ of a subset $A \neq \phi$ of a hemiring R is defined as

$\overline{A} = \{x \in R | x+a+z = b+z \text{ for some } a,b \in A, z \in R\}$.

A quasi-ideal Q of a hemiring R is called an h-quasi-ideal of R if $\overline{RQ} \cap \overline{QR} \subseteq Q$ and $x+a+z = b+z$ implies $x \in Q$, for all $x,z \in R$ and $a,b \in Q$. Every left (right) h-ideal of a hemiring R is an h-quasi-ideal of R and every h-quasi-ideal is an h-bi-ideal of R. However, the converse is not true in general.

2.1 Lemma [22]

Let A, B be subsets of a hemiring R containing 0. Then

(i) $A \subseteq \overline{A}$.

(ii) If $A \subseteq B \subseteq R$, then $\overline{A} \subseteq \overline{B}$.

(iii) $\overline{\overline{A}} = \overline{A}$, for all $A \subseteq R$.

(iv) $\overline{AB} = \overline{\overline{A}\,\overline{B}}$ and $\overline{ABC} = \overline{\overline{A}\,\overline{B}\,\overline{C}}$, for all $A,B,C \subseteq R$.

(v) For any left (right) h-ideal, h-bi-ideal or h-quasi-ideal A of R, we have $A = \overline{A}$.

Let X be a non-empty fixed set. An intuitionistic fuzzy subset A of X is an object having the form

$A = \{\langle x, \mu_A(x),\ \lambda_A(x)\rangle : x \in X\}$

where the functions $\mu_A : X \longrightarrow [0,1]$ and $\lambda_A : X \longrightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element of $x \in X$ to A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy subset (briefly, IFS) $A = \{\langle x, \mu_A(x),\ \lambda_A(x)\ : x \in X\rangle\}$. If $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ are intuitionistic fuzzy subsets of X, then

(1) $A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$.

(2) $A = B \Longleftrightarrow A \subseteq B$ and $B \subseteq A$.

(3) Complement of A is $A' = (\lambda_A, \mu_A)$.

If $\{A_i : i \in I\}$ is a family of intuitionistic fuzzy subset of X, then by the union and intersection of this family we mean an intuitionistic fuzzy subsets

(4) $\cup_{i \in I} A_i = (\vee_{i \in I} \mu_{A_i}, \wedge_{i \in I} \lambda_{A_i})$

(5) $\cap_{i \in I} A_i = (\wedge_{i \in I} \mu_{A_i}, \vee_{i \in I} \lambda_{A_i})$.

Let a be a point in a non-empty set X. If $\alpha \in (0,1]$ and $\beta \in [0,1)$ are two real numbers such that $0 \leq \alpha + \beta \leq 1$ then IFS

$a(\alpha, \beta) = \langle x, a_\alpha, 1 - a_{1-\beta}\rangle$

is called an intuitionistic fuzzy point(IFP) in X, where α and β are the degree of membership and nonmembership of $a(\alpha,\ \beta)$ respectively and $a \in X$ is the support of $a(\alpha,\ \beta)$.

Let $a(\alpha,\ \beta)$ be an IFP in X, and $A = (\mu_A, \lambda_A)$ is an IFS in X. Then $a(\alpha,\ \beta)$is said to belong to A, written $a(\alpha,\ \beta) \in A$, if $\mu_A(a) \geq \alpha$ and $\lambda_A(a) \leq \beta$ and quasi-coincident with A, written $a(\alpha,\ \beta)qA$, if $\mu_A(a) + \alpha > 1$, and $\lambda_A + \beta < 1$. $a(\alpha,\ \beta) \in \vee qA$, means that $a(\alpha,\ \beta) \in A$ or $a(\alpha,\ \beta)qA$ and $a(\alpha,\ \beta) \in \wedge qA$, means that $a(\alpha,\ \beta) \in A$ and $a(\alpha,\ \beta)qA$ and $a(\alpha,\ \beta)\overline{\in \vee q}A$, means that $a(\alpha,\ \beta) \in \vee qA$ doesn't hold.

Let $x(t,\ s)$ be an IFP in X, and $A = (\mu_A, \lambda_A)$ be an IFS in R. Then for all $x,y \in R$ and $t \in (0,1]$, $s \in [0,1)$, we define the following:

(*i*) $x(t,s)q_kA$ if $\mu_A(x) + t + k > 1$ and $\lambda_A(x) + s + k < 1$.

(*ii*) $x(t,s) \in \vee q_kA$ if $x(t,s) \in A$ or $x(t,s)q_kA$.

(*iii*) $x(t,s) \in \wedge q_kA$ if $x(t,s) \in A$ and $x(t,s)q_kA$.

(*iv*) $x(t,s)\overline{\in \vee q_k}A$ means that $x(t,s) \in \vee q_kA$ doesn't hold.

where $k \in [0,1)$.

2.2 Definition [9]

An IFS A of a hemiring R is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy sub hemiring of R, if $\forall\, x,y \in R$ and $t_1,t_2 \in (0,1]$, $s_1,s_2 \in [0,1)$

(1b) $x(t_1,s_1),\ y(t_2,s_2) \in A$

$\Rightarrow (x+y)(\min(t_1,t_2), \max(s_1,s_2)) \in \vee q_kA$

(2b) $x(t_1,s_1),\ y(t_2,s_2) \in A$

$\Rightarrow (xy)(\min(t_1,t_2), \max(s_1,s_2)) \in \vee q_kA$

2.3 Definition [9]

An IFS A of a hemiring R is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy left ideal of R, if $\forall\, x,y \in R$ and $t_1,t_2 \in (0,1]$, $s_1,s_2 \in [0,1)$

(1b) $x(t_1,s_1), y(t_2,s_2) \in A$
$\Rightarrow (x+y)(\min(t_1,t_2),\max(s_1,s_2)) \in \vee q_k A$
(3b) $x(t_1,s_1) \in A, y \in R$
$\Rightarrow (y.\,x)(\min(t_1,t_2),\max(s_1,s_2)) \in \vee q_k A$.

2.4 Theorem [9]

Let A be an intuitionistic fuzzy subset of a hemiring R. Then $(1b) \implies (1c)$, $(2b) \implies (2c)$, $(3b) \implies (3c)$, where $x,y \in R$ *and* $k \in [0,1)$,

(1c) $\mu_A(x+y) \geq \min\left\{\mu_A(x),\mu_A(y),\frac{1-k}{2}\right\}$ and $\lambda_A(x+y) \leq \max\left\{\lambda_A(x),\lambda_A(y),\frac{1-k}{2}\right\}$
(2c) $\mu_A(x.y) \geq \min\left\{\mu_A(x),\mu_A(y),\frac{1-k}{2}\right\}$ and $\lambda_A(x.y) \leq \max\left\{\lambda_A(x),\lambda_A(y),\frac{1-k}{2}\right\}$
(3c) $\mu_A(y.x) \geq \min\left\{\mu_A(x),\frac{1-k}{2}\right\}$ and $\lambda_A(y.x) \leq \max\left\{\lambda_A(x),\frac{1-k}{2}\right\}$

Converse of the above result may not be true in general.(See example 3.7.)

2.5 Definition [14]

Let A and B be intuitionistic fuzzy subsets of a hemiring R. Then the h-intrinsic product of A and B is defined by $A \odot B = \langle \mu_A \odot \mu_B,\ \lambda_A \odot \lambda_B \rangle$, where

$$(\mu_A \odot \mu_B)(x) = \begin{cases} \bigvee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(a_i)\right) \\ \wedge \left(\wedge_{i=1}^m \mu_B(b_i)\right) \\ \wedge \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \\ \wedge \left(\wedge_{j=1}^n \mu_B(b'_j)\right) \end{array} \right\} \\ 0 \text{ if } x \text{ cannot be expressed as} \\ x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z \end{cases}$$

$$(\lambda_A \odot \lambda_B)(x) = \begin{cases} \bigwedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left(\vee_{i=1}^m \lambda_A(a_i)\right) \\ \vee \left(\vee_{i=1}^m \lambda_B(b_i)\right) \\ \vee \left(\vee_{j=1}^m \lambda_A(a'_j)\right) \\ \vee \left(\vee_{j=1}^m \lambda_B(b'_j)\right) \end{array} \right\} \\ 1 \text{ if } x \text{ cannot be expressed as} \\ x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z \end{cases}$$

2.6 Definition [14]

If $S \subseteq R$, then intuitionistic characteristic function of S is denoted by $C_S = (\ \chi_S,\ \chi_S^c\ )$ and is defined by

$$\chi_S(x) = \begin{cases} 1 & if\ x \in S \\ 0 & if\ x \notin S \end{cases} \quad \text{and}$$

$$\chi_S^c(x) = \begin{cases} 0 & if\ x \in S \\ 1 & if\ x \notin S \end{cases}$$

In particular, we let $\overline{1} = (\ \chi_R,\ \chi_R^c\ )$ be the intuitionistic fuzzy set in R.

2.7 Lemma [14]

Let R be a hemiring and $P,Q \subseteq R$. Then we have
(1) $P \subseteq Q \Leftrightarrow C_P = (\ \chi_P,\ \chi_P^c\ ) \subseteq (\ \chi_Q,\ \chi_Q^c\ ) = C_Q$.
(2) $C_P \cap C_Q = C_{P\cap Q}$.
(3) $C_P \odot C_Q = C_{\overline{PQ}}$.

3 $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-ideals in hemiring

Throughout in this paper R will denote a hemiring.

3.1 Definition

An IFS A of a hemiring R is called an $(\in,\in\vee q_k)$-intuitionistic fuzzy left h-ideal of R, if $\forall$ $x,y,z,a,b \in R$ and $t_1,t_2 \in (0,1]$, $s_1,s_2 \in [0,1)$

(1b) $x(t_1,s_1), y(t_2,s_2) \in A$
$\Rightarrow (x+y)(\min(t_1,t_2),\max(s_1,s_2)) \in \vee q_k A$
(3b) $x(t_1,s_1) \in A, y \in R$
$\Rightarrow (yx)(\min(t_1,t_2),\max(s_1,s_2)) \in \vee q_k A$
(4b)$x+a+y = b+z$, $a(t_1,s_1), b(t_2,s_2) \in A$
$\implies (x)(\min(t_1,t_2),\max(s_1,s_2)) \in \vee q_k A$

$(\in,\in\vee q_k)$-intuitionistic fuzzy right h-ideal are defined similarly. An intuitionistic fuzzy set is called an $(\in,\in\vee q_k)$-intuitionistic fuzzy h-ideal of R if it is both $(\in,\in\vee q_k)$-intuitionistic fuzzy left h-ideal and $(\in,\in\vee q_k)$-intuitionistic fuzzy right h-ideal of R.

3.2 Definition

An IFS A of a hemiring R is called an $(\in,\in\vee q_k)$-intuitionistic fuzzy h-bi-ideal of R, if $\forall$ $x,y,z,a,b \in R$ and $t_1,t_2 \in (0,1]$, $s_1,s_2 \in [0,1)$

(1b) $x(t_1,s_1), y(t_2,s_2) \in A$
$\Rightarrow (x+y)(\min(t_1,t_2),\max(s_1,s_2)) \in \vee q_k A$
(2b) $x(t_1,s_1), y(t_2,s_2) \in A$
$\Rightarrow (xy)(\min(t_1,t_2),\max(s_1,s_2)) \in \vee q_k A$
(5b) $x(t_1,s_1), z(t_2,s_2) \in A$
$\implies (xyz)(\min(t_1,t_2),\max(s_1,s_2)) \in \vee q_k A$
(4b) $x+a+y = b+z$, $a(t_1,s_1), b(t_2,s_2) \in A$
$\implies (x)(\min(t_1,t_2),\max(s_1,s_2)) \in \vee q_k A$.

3.3 Theorem

Let $A = (\mu_A,\lambda_A)$ be an intuitionistic fuzzy subset of a hemiring R. Then the following holds, if $\forall\ x,y,z,a,b \in R$, $(4b) \implies (4c)$ and $(5b) \implies (5c)$, where

(4c) $x+a+y = b+z$
$\mu_A(x) \geq \min\{\mu_A(a),\mu_A(b),\frac{1-k}{2}\}$ and $\lambda_A(x) \leq \max\{\lambda_A(a),\lambda_A(b),\frac{1-k}{2}\}$

(5c) $\mu_A(xyz) \geq \min\{\mu_A(x),\mu_A(z),\frac{1-k}{2}\}$ and $\lambda_A(xyz) \leq \max\{\lambda_A(x),\lambda_A(z),\frac{1-k}{2}\}$

Proof.$(4b) \Rightarrow (4c)$

Let A be an intuitionistic fuzzy subset of a hemiring R and $(4b)$ holds. Suppose that $(4c)$ doesn't hold. Then there exist $x,a,b \in R$ such that

$\mu_A(x) < \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$ or $\lambda_A(x) > \max\left\{\lambda_A(a), \lambda_A(b), \frac{1-k}{2}\right\}$. So there exits three possible cases.

(i) $\mu_A(x) < \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$ and $\lambda_A(x) \leq \max\left\{\lambda_A(a), \lambda_A(b), \frac{1-k}{2}\right\}$

(ii) $\mu_A(x) \geq \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$ and $\lambda_A(x) > \max\left\{\lambda_A(a), \lambda_A(b), \frac{1-k}{2}\right\}$

(iii) $\mu_A(x) < \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$ and $\lambda_A(x) > \max\left\{\lambda_A(a), \lambda_A(b), \frac{1-k}{2}\right\}$.

For the first case, there exist $t \in (0,1]$ such that

$\mu_A(x) < t < \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$. Now choose $s = 1-t$, then clearly $a(t,s) \in A$ and $b(t,s) \in A$ but $(x)(t,s)\overline{\in \vee q_k}A$. Which is a contradiction. Second case is similar to this case.

Now consider case (iii), *i.e* $\mu_A(x) < \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$ and $\lambda_A(x) > \max\left\{\lambda_A(a), \lambda_A(b), \frac{1-k}{2}\right\}$. Then there exist $t \in (0,1]$ and $s \in [0,1)$, such that

$\mu_A(x) < t \leq \min\left\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\right\}$ and

$\lambda_A(x) > s \geq \max\left\{\lambda_A(a), \lambda_A(b), \frac{1-k}{2}\right\}$

$\Longrightarrow a(t,s) \in A$ and $b(t,s) \in A$ but $(x)(t,s)\overline{\in \vee q_k}A$. Which is again a contradiction. So our supposition is wrong. Hence $(4c)$ holds.

Similarly we can prove $(5b) \Longrightarrow (5c)$.

3.4 Definition

Let $A = (\mu_A, \lambda_A)$ be an IFS of a hemiring R. Then A is called an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of R if it satisfies the conditions $(1c)$, $(3c)$ and $(4c)$.

3.5 Definition

Let $A = (\mu_A, \lambda_A)$ be an IFS of a hemiring R. Then A is called an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-bi-ideal of R if it satisfies the conditions $(1c)$, $(2c)$, $(4c)$ and $(5c)$.

3.6 Remark

Every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal (h-*bi-ideal*) $A = (\mu_A, \lambda_A)$ of R need not be an $(\in, \in \vee q_k)$-intuitionistic fuzzy left h-ideal (h-*bi-ideal*) of R.

3.7 Example

Let $\mathbb{N}$ be the set of all non negative integers and $A = \langle \mu_A,\ \lambda_A \rangle$ be an IFS of $\mathbb{N}$ defined as follows

$$\mu_A(x) = \begin{cases} 0.4 & \text{if } x \in \langle 4 \rangle \\ 0.3 & \text{if } x \in \langle 2 \rangle - \langle 4 \rangle \\ 0 & \text{otherwise} \end{cases} \quad \lambda_A(x) = \begin{cases} 0.4 & \text{if } x \in \langle 4 \rangle \\ 0.3 & \text{if } x \in \langle 2 \rangle - \langle 4 \rangle \\ 0 & \text{otherwise} \end{cases}$$

For all $x,y,a,b \in \mathbb{N}$

$(1c)$ $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y), 0.4\}$ and $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y), 0.4\}$

$(3c)$ $\mu_A(xy) \geq \min\{\mu_A(y), 0.4\}$ and $\lambda_A(xy) \leq \max\{\lambda_A(y), 0.4\}$

$(4c)$ $x+a+y = b+y$

$\Longrightarrow \mu_A(x) \geq \min\{\mu_A(a), \mu_A(b), 0.4\}$ and $\lambda_A(x) \leq \max\{\lambda_A(a), \lambda_A(b), 0.4\}$

Thus $A = (\mu_A, \lambda_A)$ is an $(\in, \in \vee q_{0.2})^*$-intuitionistic fuzzy h-ideal of $\mathbb{N}$. But

$2(0.25,\ 0.35),\ 2(0.25,\ 0.35) \in A$

$\Longrightarrow (2.2)(0.25,\ 0.35)\overline{\in \vee q_{0.2}}A$. Thus $A = (\mu_A, \lambda_A)$ is not an $(\in, \in \vee q_{0.2})$-intuitionistic fuzzy h-ideal of $\mathbb{N}$.

3.8 Definition

Let A and B be intuitionistic fuzzy subsets of a hemiring R. Then the intuitionistic fuzzy subsets A_k, $A \cap_k B$, and $A \odot_k B$ are defined as following:

$$A \cap k = \left\langle \mu_A \wedge \tfrac{1-k}{2},\ \lambda_A \vee \tfrac{1-k}{2} \right\rangle = A_k$$

$$A \cap_k B = \left\langle \begin{array}{c} (\mu_A \wedge \mu_B) \wedge \frac{1-k}{2}, \\ (\lambda_A \vee \lambda_B) \vee \frac{1-k}{2} \end{array} \right\rangle = (A \cap B)_k$$

$$A \odot_k B = \left\langle \begin{array}{c} (\mu_A \odot \mu_B) \wedge \frac{1-k}{2}, \\ (\lambda_A \odot \lambda_B) \vee \frac{1-k}{2} \end{array} \right\rangle = (A \odot B)_k$$

3.9 Lemma

An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ of a hemiring R is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of R if and only if it satisfies $(1c)$, $(4c)$ and $\overline{1} \odot_k A \subseteq A_k$

*Proof.*Assume that $A = (\mu_A, \lambda_A)$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of R. It is sufficient to show that $\overline{1} \odot_k A \subseteq A_k$. Let $x \in R$ if $(\chi_R \odot_k \mu_A)(x) = 0$ and $(\chi_R^c \odot_k \lambda_A)(x) = 1$. Then $(\chi_R \odot_k \mu_A)(x) \subseteq (\mu_A)(x)$ and $(\lambda_A)(x) \subseteq (\chi_R^c \odot_k \lambda_A)(x)$. Otherwise there exist $a_i, b_i, a'_j, b'_j, z \in R$ such that

$x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z$ then we have

$(\chi_R \odot_k \mu_A)(x)$

$$= \bigvee_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$= \bigvee_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(b_i) \wedge \frac{1-k}{2}\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(b'_j) \wedge \frac{1-k}{2}\right) \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$$\leq \vee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\wedge_{i=1}^m \mu_A(a_ib_i)) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_jb'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$\leq \vee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left\{ \begin{array}{c} (\mu_A(\sum_{i=1}^m a_ib_i)) \wedge \\ \left(\mu_A(\sum_{j=1}^n a'_jb'_j)\right) \end{array} \right\}. \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$$\leq \left\{ \begin{array}{c} \left[\vee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \mu_A(x)\right] \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$= \mu_A(x) \wedge \frac{1-k}{2} = \left(\mu_A \wedge \frac{1-k}{2}\right)(x)$
$\Longrightarrow (\chi_R \odot_k \mu_A)(x) \subseteq \left(\mu_A \wedge \frac{1-k}{2}\right)(x)$
and $(\chi_R^c \odot_k \lambda_A)(x)$

$$= \wedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\vee_{i=1}^m \lambda_A(b_i)) \vee \\ \left(\vee_{j=1}^m \lambda_A(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$= \wedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left\{ \begin{array}{c} (\vee_{i=1}^m \lambda_A(b_i) \vee \frac{1-k}{2}) \vee \\ \left(\vee_{j=1}^m \lambda_A(b'_j) \vee \frac{1-k}{2}\right) \end{array} \right\} \\ \vee \frac{1-k}{2} \end{array} \right.$$

$$\geq \wedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\vee_{i=1}^m \lambda_A(a_ib_i)) \vee \\ \left(\vee_{j=1}^m \lambda_A(a'_jb'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$\geq \wedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left\{ \begin{array}{c} (\lambda_A(\sum_{i=1}^m a_ib_i)) \vee \\ \left(\lambda_A(\sum_{j=1}^n a'_jb'_j)\right) \end{array} \right\} \\ \vee \frac{1-k}{2} \end{array} \right.$$

$$\geq \left\{ \begin{array}{c} \left[\wedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \lambda_A(x)\right] \\ \vee \frac{1-k}{2} \end{array} \right.$$

$$= \lambda_A(x) \vee \frac{1-k}{2} = (\lambda_A \vee \frac{1-k}{2})(x)$$

$\Longrightarrow (\chi_R^c \odot_k \lambda_A)(x) \supseteq \left(\lambda_A \vee \frac{1-k}{2}\right)(x)$, thus $\overline{1} \odot_k A \subseteq A_k$.

Conversely, assume that given condition hold. It is sufficient to show that the condition (3c) of Theorem 2.4 is valid. Let $x, y \in R$. Then we have

$\mu_A(xy) \geq \mu_A(xy) \wedge \frac{1-k}{2} \geq (\chi_R \odot_k \mu_A)(xy) =$

$$\vee_{xy+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\wedge_{i=1}^m \chi_R(a_i)) \wedge \\ (\wedge_{i=1}^m \mu_A(b_i)) \wedge \\ \left(\wedge_{j=1}^n \chi_R(a'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$= \vee_{xy+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left\{ \begin{array}{c} (\wedge_{i=1}^m \mu_A(b_i)) \wedge \\ \left(\wedge_{j=1}^n \mu_A(b'_j)\right) \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$\geq \mu_A(y) \wedge \frac{1-k}{2}$
because $xy + 0y + z = xy + z$, and
$\lambda_A(xy) \leq \lambda_A(xy) \vee \frac{1-k}{2} \leq (\chi_R^c \odot_k \lambda_A)(xy) =$

$$= \wedge_{xy+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\vee_{i=1}^m \chi_R^c(a_i)) \vee \\ (\vee_{i=1}^m \lambda_A(b_i)) \\ \left(\vee_{j=1}^m \chi_R^c(a'_j)\right) \vee \\ \left(\vee_{j=1}^m \lambda_A(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$= \wedge_{xy+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left\{ \begin{array}{c} (\vee_{i=1}^m \lambda_A(b_i)) \vee \\ \left(\vee_{j=1}^m \lambda_A(b'_j)\right) \end{array} \right\} \\ \vee \frac{1-k}{2} \end{array} \right.$$

$\leq \lambda_A(y) \vee \frac{1-k}{2}$
because $xy + 0y + z = xy + z$.

This shows that A satisfies condition $(3c)$. So A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of R. Similarly we can prove the case of an$(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal of R.

3.10 Theorem

If A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal, and B is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of R, then $A \odot_k B \subseteq A \cap_k B$.

*Proof.*Let A and B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right and left h-ideals of R respectively. For any $x \in R$,

$(\mu_A \odot_k \mu_B)(x) =$

$$\vee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\wedge_{i=1}^m \mu_A(a_i)) \wedge \\ (\wedge_{i=1}^m \mu_B(b_i)) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_B(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$= \vee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\wedge_{i=1}^m \mu_A(a_i) \wedge \frac{1-k}{2}) \wedge \\ (\wedge_{i=1}^m \mu_B(b_i) \wedge \frac{1-k}{2}) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j) \wedge \frac{1-k}{2}\right) \wedge \\ \left(\wedge_{j=1}^n \mu_B(b'_j) \wedge \frac{1-k}{2}\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$\leq \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left\{ \begin{array}{c} (\wedge_{i=1}^{m} \mu_A(a_i b_i)) \wedge \\ (\wedge_{i=1}^{m} \mu_B(a_i b_i)) \wedge \\ \left(\wedge_{j=1}^{n} \mu_A(a'_j b'_j)\right) \wedge \\ \left(\wedge_{j=1}^{n} \mu_B(a'_j b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$\leq \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left\{ \begin{array}{c} (\mu_A(\sum_{i=1}^{m} a_i b_i)) \wedge \\ (\mu_B(\sum_{i=1}^{m} a_i b_i)) \wedge \\ \left(\mu_A(\sum_{j=1}^{n} a'_j b'_j)\right) \wedge \\ \left(\mu_B(\sum_{j=1}^{n} a'_j b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \\ \left\{ \begin{array}{c} (\mu_A(\sum_{i=1}^{m} a_i b_i)) \wedge \\ \left(\mu_A(\sum_{j=1}^{n} a'_j b'_j)\right) \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right\} \wedge \\ \left\{ \begin{array}{c} \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \\ \left\{ \begin{array}{c} (\mu_B(\sum_{i=1}^{m} a_i b_i)) \wedge \\ \left(\mu_B(\sum_{j=1}^{n} a'_j b'_j)\right) \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$\leq \min\left\{\mu_A(x),\ \mu_B(x),\ \frac{1-k}{2}\right\}$
$= (\mu_A \wedge_k \mu_B)(x)$.
Thus $(\mu_A \odot_k \mu_B)(x) \leq (\mu_A \wedge_k \mu_B)(x)$.
Now, $(\lambda_A \odot \lambda_B)(x) =$

$$\bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left\{ \begin{array}{c} (\vee_{i=1}^{m} \lambda_A(a_i)) \vee \\ (\vee_{i=1}^{m} \lambda_B(b_i)) \vee \\ \left(\vee_{j=1}^{m} \lambda_A(a'_j)\right) \vee \\ \left(\vee_{j=1}^{m} \lambda_B(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$= \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left\{ \begin{array}{c} \left\{ \begin{array}{c} (\vee_{i=1}^{m} \lambda_A(a_i) \vee \frac{1-k}{2}) \vee \\ (\vee_{i=1}^{m} \lambda_B(b_i) \vee \frac{1-k}{2}) \vee \\ \left(\vee_{j=1}^{m} \lambda_A(a'_j) \vee \frac{1-k}{2}\right) \vee \\ \left(\vee_{j=1}^{m} \lambda_B(b'_j) \vee \frac{1-k}{2}\right) \end{array} \right\} \\ \vee \frac{1-k}{2} \end{array} \right.$$

$$\geq \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left\{ \begin{array}{c} (\vee_{i=1}^{m} \lambda_A(a_i b_i)) \vee \\ (\vee_{i=1}^{m} \lambda_B(a_i b_i)) \vee \\ \left(\vee_{j=1}^{m} \lambda_A(a'_j b'_j)\right) \vee \\ \left(\vee_{j=1}^{m} \lambda_B(a'_j b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$\geq \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left\{ \begin{array}{c} (\lambda_A(\sum_{i=1}^{m} a_i b_i)) \vee \\ (\lambda_B(\sum_{i=1}^{m} a_i b_i)) \vee \\ \left(\lambda_A(\sum_{j=1}^{n} a'_j b'_j)\right) \vee \\ \left(\lambda_B(\sum_{j=1}^{n} a'_j b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \\ \left\{ \begin{array}{c} (\lambda_A(\sum_{i=1}^{m} a_i b_i)) \vee \\ \left(\lambda_A(\sum_{j=1}^{n} a'_j b'_j)\right) \end{array} \right\} \\ \vee \frac{1-k}{2} \end{array} \right\} \vee \\ \left\{ \begin{array}{c} \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \\ \left\{ \begin{array}{c} (\lambda_B(\sum_{i=1}^{m} a_i b_i)) \vee \\ \left(\lambda_B(\sum_{j=1}^{n} a'_j b'_j)\right) \end{array} \right\} \\ \vee \frac{1-k}{2} \end{array} \right\} \\ \vee \frac{1-k}{2} \end{array} \right.$$

$\geq \max\left\{\lambda_A(x),\ \lambda_B(x),\ \frac{1-k}{2}\right\}$
$= (\lambda_A \vee_k \lambda_B)(x)$.
Thus, $(\lambda_A \odot \lambda_B)(x) \geq (\lambda_A \vee_k \lambda_B)(x)$.
Hence $A \odot_k B \subseteq A \cap_k B$.

3.11 Definition

An IFS A of a hemiring R is called an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal of R, if it satisfies the condition $(1c)$, $(4c)$ and
$(A \odot_k \overline{1}) \cap_k (\overline{1} \odot_k A) \subseteq A$.

3.12 Theorem

A non empty subset A of R is an h-ideal (h-bi-ideal,h-quasi-ideal) of R if and only if its intuitionistic characteristic function C_A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-ideal (h-bi-ideal,h-quasi-ideal) of R.

*Proof.*Proof is straightforward.

3.13 Lemma

Let A and B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal and $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of a hemiring R, respectively. Then $A \cap_k B$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-quasi ideal of R.

*Proof.*Let $x, y \in R$. Then
$(\mu_A \wedge_k \mu_B)(x+y)$
$= \min\left\{\mu_A(x+y), \mu_B(x+y), \frac{1-k}{2}\right\}$

$$\geq \min \left\{ \begin{array}{c} \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}, \\ \min\left\{\mu_B(x), \mu_B(y), \frac{1-k}{2}\right\}, \\ \frac{1-k}{2} \end{array} \right.$$

$$= \min\left\{\begin{array}{c}\min\left\{\mu_A(x),\mu_B(x),\frac{1-k}{2}\right\},\\ \min\left\{\mu_A(y),\mu_B(y),\frac{1-k}{2}\right\},\\ \frac{1-k}{2}\end{array}\right.$$

$$= \min\left\{\begin{array}{c}(\mu_A\wedge_k\mu_B)(x),\\ (\mu_A\wedge_k\mu_B)(y),\frac{1-k}{2}\end{array}\right\}$$

Thus $(\mu_A\wedge_k\mu_B)(x+y)\geq$

$$\min\left\{\begin{array}{c}(\mu_A\wedge_k\mu_B)(x),\\ (\mu_A\wedge_k\mu_B)(y),\frac{1-k}{2}\end{array}\right\}$$

Now, $(\lambda_A\vee_k\lambda_B)(x+y)$

$$= \max\left\{\lambda_A(x+y),\ \lambda_B(x+y),\frac{1-k}{2}\right\}$$

$$\leq \max\left\{\begin{array}{c}\max\left\{\lambda_A(x),\ \lambda_A(y),\frac{1-k}{2}\right\},\\ \max\left\{\lambda_B(x),\ \lambda_B(y),\frac{1-k}{2}\right\},\\ \frac{1-k}{2}\end{array}\right.$$

$$= \max\left\{\begin{array}{c}\max\left\{\lambda_A(x),\ \lambda_B(x),\frac{1-k}{2}\right\},\\ \max\left\{\lambda_A(y),\ \lambda_B(y),\frac{1-k}{2}\right\},\\ \frac{1-k}{2}\end{array}\right.$$

$$= \max\left\{\begin{array}{c}(\lambda_A\vee_k\lambda_B)(x),\\ (\lambda_A\vee_k\lambda_B)(y),\frac{1-k}{2}\end{array}\right\}.$$

Thus $(\lambda_A\vee_k\lambda_B)(x+y)$

$$\leq \max\left\{\begin{array}{c}(\lambda_A\vee_k\lambda_B)(x),\\ (\lambda_A\vee_k\lambda_B)(y),\frac{1-k}{2}\end{array}\right\}.$$

Now let $a,b,x,z\in R$ such that $x+a+z=b+z$. Then

$(\mu_A\wedge_k\mu_B)(x)$

$$= \min\left\{\mu_A(x),\mu_B(x),\frac{1-k}{2}\right\}$$

$$\geq \min\left\{\begin{array}{c}\min\left\{\mu_A(a),\ \mu_A(b),\frac{1-k}{2}\right\},\\ \min\left\{\mu_B(a),\ \mu_B(b),\frac{1-k}{2}\right\},\\ \frac{1-k}{2}\end{array}\right.$$

$$= \min\left\{\begin{array}{c}\min\left\{\mu_A(a),\ \mu_B(a),\frac{1-k}{2}\right\},\\ \min\left\{\mu_A(b),\ \mu_B(b),\frac{1-k}{2}\right\},\\ \frac{1-k}{2}\end{array}\right.$$

$$= \min\left\{\begin{array}{c}(\mu_A\wedge_k\mu_B)(a),\\ (\mu_A\wedge_k\mu_B)(b),\frac{1-k}{2}\end{array}\right\}$$

Thus $(\mu_A\wedge_k\mu_B)(x)$

$$\geq \min\left\{\begin{array}{c}(\mu_A\wedge_k\mu_B)(a),\\ (\mu_A\wedge_k\mu_B)(b),\frac{1-k}{2}\end{array}\right\}.$$

Now, $(\lambda_A\vee_k\lambda_B)(x)$

$$= \max\left\{\lambda_A(x),\ \lambda_B(x),\frac{1-k}{2}\right\}$$

$$\leq \max\left\{\begin{array}{c}\max\left\{\lambda_A(a),\ \lambda_A(b),\frac{1-k}{2}\right\},\\ \max\left\{\lambda_B(a),\ \lambda_B(b),\frac{1-k}{2}\right\},\\ \frac{1-k}{2}\end{array}\right\}$$

$$= \max\left\{\begin{array}{c}\max\left\{\lambda_A(a),\ \lambda_B(a),\frac{1-k}{2}\right\},\\ \max\left\{\lambda_A(b),\ \lambda_B(b),\frac{1-k}{2}\right\},\\ \frac{1-k}{2}\end{array}\right\}$$

$$= \max\left\{\begin{array}{c}(\lambda_A\vee_k\lambda_B)(a),\\ (\lambda_A\vee_k\lambda_B)(b),\frac{1-k}{2}\end{array}\right\}.$$

Thus $(\lambda_A\vee_k\lambda_B)(x)$

$$\leq \max\left\{\begin{array}{c}(\lambda_A\vee_k\lambda_B)(a),\\ (\lambda_A\vee_k\lambda_B)(b),\frac{1-k}{2}\end{array}\right\}.$$

On the other hand we have

$((A\cap_k B)\odot_k\overline{1})\cap_k(\overline{1}\odot_k(A\cap_k B))$
$\subseteq(A\odot_k\overline{1})\cap_k(\overline{1}\odot B)$
$\subseteq A_k\cap_k B_k\subseteq A\cap_k B.$

This completes the proof.

3.14 Lemma

Any $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi ideal of a hemiring R is an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideal of R

*Proof.*Let A be an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi ideal of a hemiring R. It is sufficient to show that conditions (2c) and (5c) hold.

Let $x,y,z\in R$, by assumption

$\mu_A(xyz)\geq((\mu_A\odot_k\chi_R)\cap_k(\chi_R\odot_k\mu_A))(xyz)$

$$= \min\left\{\begin{array}{c}(\mu_A\odot_k\chi_R)(xyz),\\ (\chi_R\odot_k\mu_A)(xyz),\frac{1-k}{2}\end{array}\right\}$$

$$= \min\left\{\begin{array}{c}\left\{\begin{array}{c}\bigvee_{xyz+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\\ \left\{\begin{array}{c}\mu_A(a_i)\wedge\\ \mu_A(a'_j)\end{array}\right\}\wedge\frac{1-k}{2}\end{array}\right\},\\ \left\{\begin{array}{c}\bigvee_{xyz+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\\ \left\{\begin{array}{c}\mu_A(b_i)\wedge\\ \mu_A(b'_j)\end{array}\right\}\wedge\frac{1-k}{2}\end{array}\right\},\\ \frac{1-k}{2}\end{array}\right\}$$

$$\geq \min\left\{\begin{array}{c}\min\left\{\mu_A(x),\mu_A(x),\frac{1-k}{2}\right\},\\ \left\{\mu_A(z),\mu_A(z),\frac{1-k}{2}\right\},\frac{1-k}{2}\end{array}\right\}$$

since $xyz+x0+0=x(yz)+0$ and $xyz+0z+0=(xy)z+0$

$= \min\left\{\mu_A(x),\mu_A(z),\frac{1-k}{2}\right\}$, and

$\lambda_A(xyz)$

$\leq((\lambda_A\odot_k\chi_R^c)\cup_k(\chi_R^c\odot_k\lambda_A))(xyz)$

$$= \max\left\{\begin{array}{c}(\lambda_A\odot_k\chi_R^c)(xyz),\\ (\chi_R^c\odot_k\lambda_A)(xyz),\frac{1-k}{2}\end{array}\right\}$$

$$= \max\left\{\begin{array}{c}\left\{\begin{array}{c}\bigwedge_{xyz+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\\ \left\{\begin{array}{c}\lambda_A(a_i)\vee\\ \lambda_A(a'_j)\end{array}\right\}\vee\frac{1-k}{2}\end{array}\right\},\\ \left\{\begin{array}{c}\bigwedge_{xyz+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\\ \left\{\begin{array}{c}\lambda_A(b_i)\vee\\ \lambda_A(b'_j)\end{array}\right\}\vee\frac{1-k}{2}\end{array}\right\},\\ \frac{1-k}{2}\end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c}\max\left\{\lambda_A(x),\lambda_A(x),\frac{1-k}{2}\right\},\\ \max\left\{\lambda_A(z),\lambda_A(z),\frac{1-k}{2}\right\},\\ \frac{1-k}{2}\end{array}\right\}$$

$= \max\left\{\lambda_A(x),\lambda_A(z),\frac{1-k}{2}\right\}$

Similarly we can show that condition (2c) holds. This completes the proof.

4 h-hemiregular hemirings

In this section we characterize h-hemiregular hemirings by the properties of their $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-ideals, $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi ideals and $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi ideals.

4.1 Definition [18]

A hemiring R is said to be h-hemiregular if for each $x \in R$, there exist $a, a', z \in R$ such that $x + xax + z = xa'x + z$

4.2 Lemma [18]

A hemiring R is h-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $\overline{IL} = I \cap L$.

4.3 Lemma [18]

Let R be a hemiring. Then the following conditions are equivalent.

(i) R is h-hemiregular.
(ii) $B = \overline{BRB}$ for every h-bi-ideal B of R.
(iii) $Q = \overline{QRQ}$ for every h-quasi-ideal Q of R.

4.4 Theorem

For a hemiring R the following conditions are equivalent:

(i) R is h-hemiregular.
(ii) $A \cap_k B = A \odot_k B$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal A and every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal B of R.

Proof. Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal and B an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of R and $x \in R$. Then there exist $a, a', z \in R$, such that $x + xax + z = xa'x + z$. Now

$$(\mu_A \odot_k \mu_B)(x) = \bigvee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(a_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_B(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_B(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$\geq \left[\begin{array}{c} \mu_A(xa) \wedge \mu_A(xa') \\ \wedge \mu_B(x) \wedge \frac{1-k}{2} \end{array} \right]$$
$$\geq \left[\mu_A(x) \wedge \mu_B(x) \wedge \tfrac{1-k}{2}\right]$$
$$= (\mu_A \wedge_k \mu_B)(x)$$ and

$$(\lambda_A \odot_k \lambda_B)(x) = \bigwedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left(\vee_{i=1}^m \lambda_A(a_i)\right) \vee \\ \left(\vee_{j=1}^m \lambda_A(a'_j)\right) \vee \\ \left(\vee_{i=1}^m \lambda_B(b_i)\right) \vee \\ \left(\vee_{j=1}^m \lambda_B(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$\leq \left[\begin{array}{c} \lambda_A(xa) \vee \lambda_A(xa') \\ \vee \lambda_B(x) \vee \frac{1-k}{2} \end{array} \right]$$
$$\leq \left[\lambda_A(x) \vee \lambda_B(x) \vee \tfrac{1-k}{2}\right]$$
$$= (\lambda_A \vee_k \lambda_B)(x).$$

Thus $A \cap_k B \subseteq A \odot_k B$.

By Theorem 3.10, we know that $A \odot_k B \subseteq A \cap_k B$. Hence $A \odot_k B = A \cap_k B$

$(ii) \Longrightarrow (i)$ Let A and B be right h-ideal and left h-ideal of R, respectively. Then C_A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal and C_B is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of R. By assumption

$C_A \odot_k C_B = C_A \cap_k C_B \Longrightarrow (C_A \odot C_B)_k = (C_A \cap C_B)_k \Longrightarrow (C_{\overline{AB}})_k = (C_{A\cap B})_k \Longrightarrow \overline{AB} = A \cap B$.
Thus by lemma 4.2 R is h-hemiregular.

4.5 Theorem

The following conditions are equivalent for a hemiring R:

(i) R is h-hemiregular.
(ii) $A_k \subseteq (A \odot_k \overline{1} \odot_k A)$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-bi-ideal A of R.
(iii) $A_k \subseteq (A \odot_k \overline{1} \odot_k A)$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal A of R.

Proof. $(i) \Longrightarrow (ii)$ Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-bi-ideal A of R, and $x \in R$. Then there exist $a, a', z \in R$ such that $x + xax + z = xa'x + z$. Now

$$(\mu_A \odot_k \chi_R \odot_k \mu_A)(x)$$

$$= \bigvee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \left(\mu_A \odot_k \chi_R\right)(a_i)\right) \wedge \\ \left(\wedge_{j=1}^n \left(\mu_A \odot_k \chi_R\right)(a'_j)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_A(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(b'_j)\right) \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$$\geq \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left(\mu_A \odot_k \chi_R\right)(xa) \wedge \\ \left(\mu_A \odot_k \chi_R\right)(xa') \wedge \left(\mu_A(x)\right) \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \bigvee_{xa+\sum_{i=1}^m c_id_i+z'=\sum_{j=1}^n c'_jd'_j+z'} \\ \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(c_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(c'_i)\right) \end{array} \right\} \wedge \frac{1-k}{2} \end{array} \right) \wedge \\ \left(\begin{array}{c} \bigvee_{xa'+\sum_{i=1}^m c_id_i+z=\sum_{j=1}^n c'_jd'_j+z'} \\ \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(c_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(c'_i)\right) \end{array} \right\} \wedge \frac{1-k}{2} \end{array} \right) \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$$\geq \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left(\begin{array}{c} \min \left\{ \begin{array}{c} \mu_A(xax), \\ \mu_A(xa'x) \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right) \wedge \\ \left(\begin{array}{c} \min \left\{ \begin{array}{c} \mu_A(xax), \\ \mu_A(xa'x) \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right) \end{array} \right\} \\ \wedge \frac{1-k}{2} \end{array} \right.$$

because $xa+xaxa+za=xa'xa+za$ and $xa'+xaxa'+za'=xa'xa'+za'$ $\geq \mu_A(x)\wedge\frac{1-k}{2}$, and
$\left(\lambda_A\odot_k\chi_k^c\odot_k\lambda_A\right)(x)=$

$$\wedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\left\{\begin{array}{c}\left(\vee_{i=1}^m(\lambda_A\odot_k\chi_k^c)(a_i)\right)\vee\\ \left(\vee_{j=1}^m(\lambda_A\odot_k\chi_k^c)(a'_j)\right)\vee\\ \left(\vee_{i=1}^m\lambda_A(b_i)\right)\vee\\ \left(\vee_{j=1}^m\lambda_A(b'_j)\right)\end{array}\right\}\vee\frac{1-k}{2}$$

$$\leq\left\{\begin{array}{c}\left\{\begin{array}{c}\left(\lambda_A\odot_k\chi_k^c\right)(xa)\vee\\ \left(\lambda_A\odot_k\chi_k^c\right)(xa')\vee(\lambda_A(x))\end{array}\right\}\\ \vee\frac{1-k}{2}\end{array}\right.$$

$$=\left\{\begin{array}{c}\left\{\begin{array}{c}\left(\begin{array}{c}\wedge_{xa+\sum_{i=1}^m c_id_i+z'=\sum_{j=1}^n c'_jd'_j+z'}\\ \left\{\begin{array}{c}\left(\vee_{i=1}^m\lambda_A(c_i)\right)\vee\\ \left(\vee_{j=1}^m\lambda_A(c'_j)\right)\end{array}\right\}\\ \vee\frac{1-k}{2}\end{array}\right)\vee\\ \left(\begin{array}{c}\wedge_{xa'+\sum_{i=1}^m c_id_i+z'=\sum_{j=1}^n c'_jd'_j+z'}\\ \left\{\begin{array}{c}\left(\vee_{i=1}^m\lambda_A(c_i)\right)\vee\\ \left(\vee_{j=1}^m\lambda_A(c'_j)\right)\end{array}\right\}\\ \vee\frac{1-k}{2}\end{array}\right)\end{array}\right\}\\ \vee\frac{1-k}{2}\end{array}\right.$$

$$\leq\left\{\begin{array}{c}\left\{\begin{array}{c}\left(\begin{array}{c}\max\{\lambda_A(xax),\\ \lambda_A(xa'x)\}\vee\frac{1-k}{2}\end{array}\right)\\ \vee\left(\begin{array}{c}\max\{\lambda_A(xax),\\ \lambda_A(xa'x)\}\vee\frac{1-k}{2}\end{array}\right)\end{array}\right\}\\ \vee\frac{1-k}{2}\end{array}\right.$$

$\leq\lambda_A(x)\vee\frac{1-k}{2}$.
Thus $A_k\subseteq\left(A\odot_k\overline{1}\odot_kA\right)$.
(ii) $\Longrightarrow$ (iii) This is straightforward.
(iii) $\Longrightarrow$ (i) Let Q be any h-quasi-ideal of R. Then by Theorem 3.12 C_Q is an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal of R. Now by given condition $(C_Q)_k\subseteq(C_Q\odot_k\overline{1}\odot_kC_Q)\implies(C_Q)_k\subseteq(C_Q\odot\overline{1}\odot C_Q)_k=(C_{\overline{QRQ}})_k$ $\Longrightarrow Q\subseteq\overline{QRQ}$. Also $\overline{QRQ}\subseteq\overline{RQ}\cap\overline{QR}=Q$. Thus $Q=\overline{QRQ}$. Therefore by Lemma 4.3 R is h-hemiregular.

4.6 Theorem

The following conditions for a hemiring R are equivalent:
(i) R is h-hemiregular.
(ii) $A\cap_kB\subseteq A\odot_kB\odot_kA$ for every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideal A and every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-ideal B of R.
(iii) $A\cap_kB\subseteq A\odot_kB\odot_kA$ for every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal A and every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-ideal B of R.

Proof.(i) $\Longrightarrow$ (ii) Let A be an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideal A and B be an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-ideal of R, and $x\in R$. Then there exist $a,a',z\in R$ such that $x+xax+z=xa'x+z$. Now
$\left(\mu_A\odot_k\mu_B\odot_k\mu_A\right)(x)=$

$$\vee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\left\{\begin{array}{c}\left\{\begin{array}{c}\left(\wedge_{i=1}^m\left(\mu_A\odot_k\mu_B\right)(a_i)\right)\\ \wedge\left(\wedge_{j=1}^n\left(\mu_A\odot_k\mu_B\right)(a'_j)\right)\wedge\\ \left(\wedge_{i=1}^m\mu_A(b_i)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_A(b'_j)\right)\end{array}\right\}\\ \wedge\frac{1-k}{2}\end{array}\right.$$

$$\geq\left\{\begin{array}{c}\left\{\begin{array}{c}\left(\mu_A\odot_k\mu_B\right)(xa)\wedge\\ \left(\mu_A\odot_k\mu_B\right)(xa')\wedge(\mu_A(x))\end{array}\right\}\\ \wedge\frac{1-k}{2}\end{array}\right.$$

$$=\left\{\begin{array}{c}\left(\begin{array}{c}\vee_{xa+\sum_{i=1}^m c_id_i+z'=\sum_{j=1}^n c'_jd'_j+z'}\\ \left\{\begin{array}{c}\left(\wedge_{i=1}^m\mu_A(c_i)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_A(c'_j)\right)\wedge\\ \left(\wedge_{i=1}^m\mu_B(d_i)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_B(d'_j)\right)\end{array}\right\}\wedge\frac{1-k}{2}\end{array}\right)\wedge\\ \left(\begin{array}{c}\vee_{xa'+\sum_{i=1}^m c_id_i+z'=\sum_{j=1}^n c'_jd'_j+z'}\\ \left\{\begin{array}{c}\left(\wedge_{i=1}^m\mu_A(c_i)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_A(c'_j)\right)\wedge\\ \left(\wedge_{i=1}^m\mu_B(d_i)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_B(d'_j)\right)\end{array}\right\}\wedge\frac{1-k}{2}\end{array}\right)\end{array}\right\}\wedge\frac{1-k}{2}$$

$$\geq\left\{\begin{array}{c}\left(\min\left\{\begin{array}{c}\mu_A(x),\mu_B(axa),\\ \mu_B(a'xa)\end{array}\right\}\wedge\frac{1-k}{2}\right)\wedge\\ \left(\min\left\{\begin{array}{c}\mu_A(x),\mu_B(axa'),\\ \mu_B(a'xa')\end{array}\right\}\wedge\frac{1-k}{2}\right)\end{array}\right\}\wedge\frac{1-k}{2}$$

$\left(\begin{array}{c}\text{because } xa+xaxa+za=xa'xa+za\\ \text{and } xa'+xaxa'+za'=xa'xa'+za'\end{array}\right)$
$\geq\left(\mu_A\wedge\mu_B\right)(x)\wedge\frac{1-k}{2}=\left(\mu_A\wedge_k\mu_B\right)(x)$ and
$\left(\lambda_A\odot_k\lambda_B\odot_k\lambda_A\right)(x)$

$$=\wedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\left\{\begin{array}{c}\left(\vee_{i=1}^m(\lambda_A\odot_k\lambda_B)(a_i)\right)\vee\\ \left(\vee_{j=1}^m(\lambda_A\odot_k\lambda_B)(a'_j)\right)\vee\\ \left(\vee_{i=1}^m\lambda_A(b_i)\right)\vee\\ \left(\vee_{j=1}^m\lambda_A(b'_j)\right)\end{array}\right\}\vee\frac{1-k}{2}$$

$$\leq\left\{\begin{array}{c}\left\{\begin{array}{c}\left(\lambda_A\odot_k\lambda_B\right)(xa)\vee\\ \left(\lambda_A\odot_k\lambda_B\right)(xa')\vee(\lambda_A(x))\end{array}\right\}\\ \vee\frac{1-k}{2}\end{array}\right.$$

$$= \left\{\begin{array}{c}\left\{\begin{array}{c}\begin{pmatrix}\bigwedge_{xa+\sum_{i=1}^{m}c_id_i+z'=\sum_{j=1}^{n}c'_jd'_j+z'}\\ \left\{\begin{array}{c}\left(\vee_{i=1}^{m}\lambda_A(c_i)\right)\vee\\ \left(\vee_{j=1}^{m}\lambda_A(c'_j)\right)\vee\\ \left(\vee_{i=1}^{m}\lambda_B(d_i)\right)\vee\\ \left(\vee_{j=1}^{m}\lambda_B(d'_j)\right)\end{array}\right\}\\ \vee\frac{1-k}{2}\end{pmatrix}\vee\\ \begin{pmatrix}\bigwedge_{xa'+\sum_{i=1}^{m}c_id_i+z'=\sum_{j=1}^{n}c'_jd'_j+z'}\\ \left\{\begin{array}{c}\left(\vee_{i=1}^{m}\lambda_A(c_i)\right)\vee\\ \left(\vee_{j=1}^{m}\lambda_A(c'_j)\right)\vee\\ \left(\vee_{i=1}^{m}\lambda_B(d_i)\right)\vee\\ \left(\vee_{j=1}^{m}\lambda_B(d'_j)\right)\end{array}\right\}\\ \vee\frac{1-k}{2}\end{pmatrix}\end{array}\right\}\\ \vee\frac{1-k}{2}\end{array}\right.$$

$$\le \left\{\begin{array}{c}\left\{\begin{array}{c}\begin{pmatrix}\max\{\lambda_A(x),\lambda_B(axa),\\ \lambda_B(a'xa)\}\vee\frac{1-k}{2}\end{pmatrix}\vee\\ \begin{pmatrix}\max\{\lambda_A(x),\lambda_B(axa'),\\ \lambda_B(a'xa')\}\vee\frac{1-k}{2}\end{pmatrix}\end{array}\right\}\\ \vee\frac{1-k}{2}\end{array}\right.$$

$\le (\lambda_A\vee\lambda_B)(x)\vee\frac{1-k}{2}$
$= (\lambda_A\vee_k\lambda_B)(x)$.

Thus $A\cap_k B\subseteq A\odot_k B\odot_k A$.

(ii) $\Longrightarrow$ (iii) This is straightforward.

(iii) $\Longrightarrow$ (i) Let A be an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal of R. Since $\overline{1}$ is an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-ideal of R, so by given condition $\left(A\cap_k\overline{1}\right)\subseteq\left(A\odot_k\overline{1}\odot_k A\right)\Longrightarrow A_k\subseteq\left(A\odot_k\overline{1}\odot_k A\right)$. Therefore by Theorem 4.5, R is h-hemiregular.

4.7 Theorem

Let R be a hemiring. Then the following condition are equivalent.

(1) R is h-hemiregular.

(2) $A\cap_k B\subseteq A\odot_k B$ for every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideal A and every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy left h-ideal B of R.

(3) $A\cap_k B\subseteq A\odot_k B$ for every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal A and every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy left h-ideal B of R.

(4) $A\cap_k B\subseteq A\odot_k B$ for every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy right h-ideal A and every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideal B of R.

(5) $A\cap_k B\subseteq A\odot_k B$ for every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy right h-ideal A and every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal B of R.

(6) $A\cap_k B\cap_k C\subseteq(A\odot_k B\odot_k C)$ for every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy right h-ideal A, every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideal B, and every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy left h-ideal C of R.

(7) $A\cap_k B\cap_k C\subseteq(A\odot_k B\odot_k C)$ for every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy right h-ideal A, every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal B, and every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy left h-ideal C of R.

Proof.(1) $\Longrightarrow$ (2) let A be an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideal and B be an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy left h-ideal of R. Since R is h-hemiregular so for $x\in R$, there exist $a,a',z\in R$ such that $x+xax+z=xa'x+z$. Thus we have

$(\mu_A\odot_k\mu_B)(x)$

$$=\bigvee_{x+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z}\left\{\begin{array}{c}\left(\wedge_{i=1}^{m}\mu_A(a_i)\right)\wedge\\ \left(\wedge_{j=1}^{n}\mu_A(a'_j)\right)\wedge\\ \left(\wedge_{i=1}^{m}\mu_B(b_i)\right)\wedge\\ \left(\wedge_{j=1}^{n}\mu_B(b'_j)\right)\end{array}\right\}\wedge\frac{1-k}{2}$$

$$\ge\left\{\begin{array}{c}\left\{\begin{array}{c}\mu_A(x)\wedge\mu_B(ax)\\ \wedge\mu_B(a'x)\end{array}\right\}\\ \wedge\frac{1-k}{2}\end{array}\right.$$

$$\ge\left\{\begin{array}{c}\mu_A(x)\wedge\mu_B(x)\\ \wedge\frac{1-k}{2}\end{array}\right\}\wedge\frac{1-k}{2}$$

$$=\left\{\begin{array}{c}\mu_A(x)\wedge\mu_B(x)\\ \wedge\frac{1-k}{2}\end{array}\right\}$$

$=(\mu_A\wedge_k\mu_B)(x)$, and

$(\lambda_A\odot_k\lambda_B)(x)$

$$=\bigwedge_{x+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z}\left\{\begin{array}{c}\left(\vee_{i=1}^{m}\lambda_A(a_i)\right)\vee\\ \left(\vee_{j=1}^{m}\lambda_A(a'_j)\right)\vee\\ \left(\vee_{i=1}^{m}\lambda_B(b_i)\right)\vee\\ \left(\vee_{j=1}^{m}\lambda_B(b'_j)\right)\end{array}\right\}\vee\frac{1-k}{2}$$

$$\le\left\{\begin{array}{c}\lambda_A(x)\vee\lambda_B(ax)\vee\\ \lambda_B(a'x)\end{array}\right\}\vee\frac{1-k}{2}$$

$$\le\left\{\begin{array}{c}\lambda_A(x)\vee\lambda_B(x)\\ \vee\frac{1-k}{2}\end{array}\right\}\vee\frac{1-k}{2}$$

$$=\left\{\begin{array}{c}\lambda_A(x)\vee\lambda_B(x)\\ \vee\frac{1-k}{2}\end{array}\right\}=(\lambda_A\vee_k\lambda_B)(x).$$

Thus (2) holds.

(2) $\Longrightarrow$ (3) This is obvious because every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal is an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-bi-ideal.

(3) $\Longrightarrow$ (1) Let A be an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy right h-ideal and B be an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy left h-ideal of R. Since every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy right h-ideal is an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy h-quasi-ideal, so by (3), we have $A\cap_k B\subseteq A\odot_k B$. By Theorem 3.10, we know that $A\odot_k B\subseteq A\cap_k B$. Hence $A\cap_k B=A\odot_k B$ for every $(\in,\in\vee q_k)^*$-intuitionistic fuzzy right h-ideal A and every $(\in,$

$\in \vee q_k)^*$-intuitionistic fuzzy left h-ideal B of R. Thus by Theorem 4.4 R is h-hemiregular.

Similarly we can show that (1)⇔(4)⇔(5).

(1) $\Longrightarrow$ (6) Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal, B be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-bi ideal and C be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal. Since R is h-hemiregular so for $x \in R$ there exist $a, a', z \in R$ such that $x + xax + z = xa'x + z$. Thus we have

$$(\mu_A \odot_k \mu_B \odot_k \mu_C)(x)$$

$$= \bigvee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \left\{ \begin{array}{c} \left(\wedge_{i=1}^m (\mu_A \odot_k \mu_B)(a_i)\right) \wedge \\ \left(\wedge_{j=1}^n (\mu_A \odot_k \mu_B)(a'_j)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_C(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_C(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2} \right.$$

$$\geq \left\{ \begin{array}{c} (\mu_A \odot_k \mu_B)(x) \wedge \mu_C(ax) \\ \wedge \mu_C(a'x) \wedge \frac{1-k}{2} \end{array} \right\}$$

$$\geq \left\{ \begin{array}{c} \left[\begin{array}{c} \bigvee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \\ \left(\begin{array}{c} \left(\wedge_{i=1}^m \mu_A(a_i)\right) \wedge \\ \left(\wedge_{j=1}^n (\mu_A)(a'_j)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_B(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n (\mu_B)(b'_j)\right) \end{array} \right) \\ \wedge \frac{1-k}{2} \end{array} \right] \\ \wedge \mu_C(ax) \wedge \mu_C(a'x) \wedge \frac{1-k}{2} \end{array} \right.$$

$$\geq \left\{ \begin{array}{c} \mu_A(xa) \wedge \mu_A(xa') \wedge \mu_B(x) \\ \wedge \mu_C(ax) \wedge \mu_C(a'x) \wedge \frac{1-k}{2} \end{array} \right\}$$

$$\geq \left\{ \begin{array}{c} \mu_A(x) \wedge \mu_B(x) \wedge \\ \mu_C(x) \wedge \frac{1-k}{2} \end{array} \right\}$$

$= (\mu_A \wedge_k \mu_B \wedge_k \mu_C)(x)$.

Thus $(\mu_A \odot_k \mu_B \odot_k \mu_C) \geq (\mu_A \wedge_k \mu_B \wedge_k \mu_C)$. And $(\lambda_A \odot_k \lambda_B \odot_k \lambda c)(x) =$

$$\bigwedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \left\{ \begin{array}{c} \left(\vee_{i=1}^m (\lambda_A \odot_k \lambda_B)(a_i)\right) \vee \\ \left(\vee_{j=1}^m (\lambda_A \odot_k \lambda_B)(a'_j)\right) \vee \\ \left(\vee_{i=1}^m \lambda_C(b_i)\right) \vee \\ \left(\vee_{j=1}^m \lambda_C(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2} \right.$$

$$\leq \left\{ \begin{array}{c} (\lambda_A \odot_k \lambda_B)(x) \vee \lambda_C(ax) \\ \vee \lambda_C(a'x) \vee \frac{1-k}{2} \end{array} \right\}$$

$$\leq \left\{ \begin{array}{c} \left[\begin{array}{c} \bigwedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \\ \left(\begin{array}{c} \left(\vee_{i=1}^m \lambda_A(a_i)\right) \vee \\ \left(\vee_{j=1}^m \lambda_A(a'_j)\right) \vee \\ \left(\vee_{i=1}^m \lambda_B(b_i)\right) \vee \\ \left(\vee_{j=1}^m \lambda_B(b'_j)\right) \end{array} \right) \vee \frac{1-k}{2} \end{array} \right] \\ \vee \lambda_C(ax) \vee \lambda_C(a'x) \vee \frac{1-k}{2} \end{array} \right.$$

$$\leq \left\{ \begin{array}{c} \lambda_A(xa) \vee \lambda_A(xa') \vee \lambda_B(x) \\ \vee \lambda_C(ax) \vee \lambda_C(a'x) \vee \frac{1-k}{2} \end{array} \right\}$$

$$\leq \left\{ \lambda_A(x) \vee \lambda_B(x) \vee \lambda_C(x) \vee \frac{1-k}{2} \right\}$$

$= (\lambda_A \vee_k \lambda_B \vee_k \lambda_C)(x)$.

Thus $(\lambda_A \vee_k \lambda_B \vee_k \lambda_C)$ $\supseteq (\lambda_A \odot_k \lambda_B \odot_k \lambda c)$.

Hence $A \cap_k B \cap_k C \subseteq (A \odot_k B \odot_k C)$.

(6) $\Longrightarrow$ (7) This is straightforward.

(7) $\Longrightarrow$ (1) Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal and B be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of R. Then

$(A \cap_k B) = (A \cap_k \overline{1} \cap_k B) \subseteq (A \odot_k \overline{1} \odot_k B) \subseteq (A \odot_k B)$.

But $(A \odot_k B) \subseteq (A \cap_k B)$ always. Hence $(A \cap_k B) = (A \odot_k B)$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal A and for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal B of R. Thus by Theorem 4.4, R is h-hemiregular.

5 *h*-intra-hemiregular hemirings

5.1 Definition [18]

A hemiring R is said to be h-intra-hemiregular if for each $x \in R$ there exist $a_i, a'_i, b_j, b'_j, z \in R$ such that $x + \sum_{i=1}^m a_i x^2 a'_i + z = \sum_{j=1}^n b_j x^2 b'_j + z$

5.2 Lemma [18]

A hemiring R is an h-intra-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have$I \cap L \subseteq \overline{LI}$.

5.3 Lemma [18]

The following conditions are equivalent for a hemiring R.

(i) R is both h-hemiregular and h-intra-hemiregular.

(ii) $B = \overline{B^2}$ for every h-bi-ideal B of R.

(iii) $Q = \overline{Q^2}$ for every h-quasi-ideal Q of R.

5.4 Lemma

For a hemiring R, the following condition are equivalent:

(1) R is h-intra-hemiregular.

(2)$A \cap_k B \subseteq A \odot_k B$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal A and every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal B of R.

Proof.(1) $\Longrightarrow$ (2) Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal and B be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal of R. As R is h-intra-hemiregular so for each $x \in R$ there exist $a_i, a'_i, b_j, b'_j, z \in R$ such that

$x + \sum_{i=1}^m a_i x^2 a'_i + z = \sum_{j=1}^n b_j x^2 b'_j + z$. Thus we have

$$(\mu_A \odot_k \mu_B)(x)$$

$$= \vee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\wedge_{i=1}^m \mu_A(a_i)) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \wedge \\ (\wedge_{i=1}^m \mu_B(b_i)) \wedge \\ \left(\wedge_{j=1}^n \mu_B(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$\geq \begin{pmatrix} \mu_A(a_ix) \wedge \mu_A(b_jx) \wedge \\ \mu_B(xa'_i) \wedge \mu_B(xb'_j) \wedge \frac{1-k}{2} \end{pmatrix},$$

$$\begin{pmatrix} \text{because } x+\sum_{i=1}^m (a_ix)(xa'_i)+z \\ = \sum_{j=1}^n (b_jx)(xb'_j)+z \end{pmatrix}$$

$\geq (\mu_A \wedge_k \mu_B)(x)$, and

$$(\lambda_A \odot_k \lambda_B)(x)$$

$$= \wedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\vee_{i=1}^m \lambda_A(a_i)) \vee \\ \left(\vee_{j=1}^m \lambda_A(a'_j)\right) \vee \\ (\vee_{i=1}^m \lambda_B(b_i)) \vee \\ \left(\vee_{j=1}^m \lambda_B(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$\leq \begin{pmatrix} \lambda_A(a_ix) \vee \lambda_A(b_jx) \vee \\ \lambda_B(xa'_i) \vee \lambda_B(xb'_j) \vee \frac{1-k}{2} \end{pmatrix}$$

$\leq (\lambda_A \vee_k \lambda_B)(x)$

Thus $A \cap_k B \subseteq A \odot_k B \implies$ (2) holds.

(2) $\implies$ (1) Assume that A and B be left and right h-ideals of R, respectively. Thus by Theorem 3.12, the intuitionistic characteristic functions C_A and C_B are $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal and $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal of R respectively. By assumption $C_A \cap_k C_B \subseteq C_A \odot_k C_B \implies (C_A \cap C_B)_k \subseteq (C_A \odot C_B)_k \implies (C_{A\cap B})_k \subseteq (C_{\overline{AB}})_k \implies A \cap B \subseteq \overline{AB}$. Thus by lemma 5.2, R is h-intra-hemiregular.

5.5 Theorem

For a hemiring R, the following conditions are equivalent:

(1) R is both h-hemiregular and h-intra-hemiregular.

(2)$A_k = A \odot_k A$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-bi ideal A of R.

(3)$A_k = A \odot_k A$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-quasi ideal A of R.

Proof.(1) $\implies$ (2) Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-bi ideal of R, and $x \in R$. Since R is both h-hemiregular and h-intra-hemiregular, so there exist $a, a', a'_i, a'_i, b_j, b'_j, z \in R$ such that $x + xax + z = xa'x + z$ and $x + \sum_{i=1}^m a_ix^2a'_i + z = \sum_{j=1}^n b_jx^2b'_j + z$.

Then (as given in Lemma 5.6, [18]) there exist $a_1, a_2, p_i.\ p'_i, q_j, q'_j, z \in R$ such that

$$x + \sum_{j=1}^n (xa_2q_jx)(xq'_ja_1x) +$$
$$\sum_{j=1}^n (xa_1q_jx)(xq'_ja_2x) +$$
$$\sum_{i=1}^m (xa_1p_ix)(xp'_ia_1x) +$$
$$\sum_{i=1}^m (xa_2p_ix)(xp'_ia_2x) + z$$
$$= \sum_{i=1}^m (xa_2p_ix)(xp'_ia_1x) +$$
$$\sum_{i=1}^m (xa_1p_ix)(xp'_ia_2x)$$
$$+ \sum_{j=1}^n (xa_1q_jx)(xq'_ja_1x) +$$
$$\sum_{j=1}^n (xa_2q_jx)(xq'_ja_2x) + z.$$

$$(\mu_A \odot_k \mu_A)(x)$$

$$= \vee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z} \left\{ \begin{array}{c} (\wedge_{i=1}^m \mu_A(a_i)) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \wedge \\ (\wedge_{i=1}^m \mu_A(b_i)) \wedge \\ \left(\wedge_{j=1}^n \mu_A(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$\geq \left\{ \begin{array}{c} \left[\begin{array}{c} \begin{pmatrix} (\wedge_{i=1}^m \mu_A(xa_1p_ix)) \wedge \\ (\wedge_{i=1}^m \mu_A(xp'_ia_1x)) \wedge \\ (\wedge_{i=1}^m \mu_A(xa_2p_ix)) \wedge \\ (\wedge_{i=1}^m \mu_A(xp'_ia_2x)) \end{pmatrix} \wedge \\ \begin{pmatrix} \left(\wedge_{j=1}^n \mu_A(xa_2q_jx)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(xq'_ja_1x)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(xa_1q_jx)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(xq'_ja_2x)\right) \end{pmatrix} \end{array} \right] \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$\geq \min\left\{\mu_A(x), \frac{1-k}{2}\right\} \wedge \frac{1-k}{2}$

$= \mu_A(x) \wedge \frac{1-k}{2} = (\mu_A \wedge \frac{1-k}{2})(x)$.

Thus $(\mu_A \odot_k \mu_A) \geq (\mu_A \wedge \frac{1-k}{2})$ similarly we can prove $(\lambda_A \vee \frac{1-k}{2}) \geq (\lambda_A \odot_k \lambda_A)$. This implies that $A_k \subseteq A \odot_k A$.

On the other hand

if $x + \sum_{i=1}^m a_ib_i + z = \sum_{j=1}^n a'_jb'_j + z$, we have

$$(\mu_A \wedge \tfrac{1-k}{2})(x)$$

$$\geq \min \left\{ \begin{array}{c} \left\{ \begin{array}{c} \mu_A(\sum_{i=1}^m a_ib_i), \\ \mu_A(\sum_{j=1}^n a'_jb'_j), \frac{1-k}{2} \end{array} \right\}, \\ \frac{1-k}{2} \end{array} \right.$$

$$\geq \min\left\{\begin{array}{c}\left(^{m}_{i=1}\mu_A(a_ib_i)\right),\\ \left(^{m}_{j=1}\mu_A(a'_jb'_j)\right), \frac{1-k}{2}\end{array}\right\}$$

$$\geq \min\left\{\begin{array}{c}\left(\begin{array}{c}^{m}_{i=1}\mu_A(a_i)\\ \wedge\mu_A(b_i)\end{array}\right),\\ \left(\begin{array}{c}^{m}_{j=1}\mu_A(a'_j)\\ \wedge\mu_A(b'_j)\end{array}\right), \frac{1-k}{2}\end{array}\right\}.$$

Thus $(\mu_A \odot_k \mu_A)(x)$

$$= \vee_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z}\left\{\begin{array}{c}\left(\wedge_{i=1}^{m}\mu_A(a_i)\right)\wedge\\ \left(\wedge_{j=1}^{n}\mu_A(a'_j)\right)\wedge\\ \left(\wedge_{i=1}^{m}\mu_A(b_i)\right)\wedge\\ \left(\wedge_{j=1}^{n}\mu_A(b'_j)\right)\end{array}\right\}\wedge\frac{1-k}{2}$$

$\leq (\mu_A \wedge \frac{1-k}{2})(x)$,

So $(\mu_A \odot_k \mu_A) \leq (\mu_A \wedge \frac{1-k}{2})$.

Similarly we can prove

$(\lambda_A \vee \frac{1-k}{2}) \leq (\lambda_A \odot_k \lambda_A)$.

This implies $A_k \supseteq A \odot_k A$. Consequently $A_k = A \odot_k A$.

(2) $\Longrightarrow$ (3) This is straightforward.

(3) $\Longrightarrow$ (1) Let Q be an h-quasi ideal of R. Then by Theorem 3.12, C_Q is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-quasi ideal of R, thus by assumption $(C_Q)_k = (C_Q \odot_k C_Q) = (C_Q \odot C_Q)_k = (C_{\overline{Q^2}})_k \implies Q = \overline{Q^2}$. Hence by Lemma 5.3, R is both h-hemiregular and h-intra-hemiregular.

5.6 Theorem

For a hemiring R, the following conditions are equivalent:

(1) R is both h-hemiregular and h-intra-hemiregular.

(2) $A \cap_k B \subseteq A \odot_k B$ for all $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-bi ideals A and B of R.

(3) $A \cap_k B \subseteq A \odot_k B$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-bi ideal A and every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-quasi ideal B of R.

(4) $A \cap_k B \subseteq A \odot_k B$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-quasi ideal A and every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-bi ideal B of R.

(5) $A \cap_k B \subseteq A \odot_k B$ for all $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-quasi ideals A and B of R.

Proof.(1) $\Longrightarrow$ (2) Let A and B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-bi ideals of R and $x \in R$. Since R is both h-hemiregular and h-intra-hemiregular, so there exist $a, a', a'_i, a'_i, b_j, b'_j, z \in R$ such that $x + xax + z = xa'x + z$ and $x + \sum_{i=1}^{m} a_ix^2a'_i + z = \sum_{j=1}^{n} b_jx^2b'_j + z$.

Then (as given in Lemma 5.6, [18]) there exist $a_1, a_2, p_i.\ p'_i, q_j, q'_j, z \in R$ such that

$$\begin{aligned}&x+\sum_{j=1}^{n}(xa_2q_jx)(xq'_ja_1x)+\\ &\sum_{j=1}^{n}(xa_1q_jx)(xq'_ja_2x)+\\ &\sum_{i=1}^{m}(xa_1p_ix)(xp'_ia_1x)+\\ &\sum_{i=1}^{m}(xa_2p_ix)(xp'_ia_2x)+z\\ =&\sum_{i=1}^{m}(xa_2p_ix)(xp'_ia_1x)+\\ &\sum_{i=1}^{m}(xa_1p_ix)(xp'_ia_2x)+\\ &\sum_{j=1}^{n}(xa_1q_jx)(xq'_ja_1x)+\\ &\sum_{j=1}^{n}(xa_2q_jx)(xq'_ja_2x)+z.\end{aligned}$$

$(\mu_A \odot_k \mu_B)(x)$

$$= \vee_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z}\left\{\begin{array}{c}\left(\wedge_{i=1}^{m}\mu_A(a_i)\right)\wedge\\ \left(\wedge_{j=1}^{n}\mu_A(a'_j)\right)\wedge\\ \left(\wedge_{i=1}^{m}\mu_B(b_i)\right)\wedge\\ \left(\wedge_{j=1}^{n}\mu_B(b'_j)\right)\end{array}\right\}\wedge\frac{1-k}{2}$$

$$\geq \left\{\begin{array}{c}\left[\begin{array}{c}\left(\begin{array}{c}\left(\wedge_{j=1}^{n}\mu_A(xa_2q_jx)\right)\wedge\\ \left(\wedge_{j=1}^{n}\mu_B(xq'_ja_1x)\right)\wedge\\ \left(\wedge_{j=1}^{n}\mu_A(xa_1q_jx)\right)\wedge\\ \left(\wedge_{j=1}^{n}\mu_B(xq'_ja_2x)\right)\end{array}\right)\wedge\\ \left(\begin{array}{c}\left(\wedge_{i=1}^{m}\mu_A(xa_1p_ix)\right)\wedge\\ \left(\wedge_{i=1}^{m}\mu_B(xp'_ia_1x)\right)\wedge\\ \left(\wedge_{i=1}^{m}\mu_A(xa_2p_ix)\right)\wedge\\ \left(\wedge_{i=1}^{m}\mu_B(xp'_ia_2x)\right)\end{array}\right)\end{array}\right]\\ \wedge\frac{1-k}{2}\end{array}\right.$$

$\geq \min\left\{\mu_A(x), \mu_B(x), \frac{1-k}{2}\right\} \wedge \frac{1-k}{2}$

$= \min\left\{\mu_A(x), \mu_B(x), \frac{1-k}{2}\right\}$

$= \mu_A(x) \wedge_k \mu_B(x) = (\mu_A \wedge_k \mu_B)(x)$.

Thus $(\mu_A \odot_k \mu_B) \geq (\mu_A \wedge_k \mu_B)$.

Now $(\lambda_A \odot_k \lambda_B)(x)$

$$= \wedge_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z}\left\{\begin{array}{c}\left(\vee_{i=1}^{m}\lambda_A(a_i)\right)\vee\\ \left(\vee_{j=1}^{m}\lambda_A(a'_j)\right)\vee\\ \left(\vee_{i=1}^{m}\lambda_B(b_i)\right)\vee\\ \left(\vee_{j=1}^{m}\lambda_B(b'_j)\right)\end{array}\right\}\vee\frac{1-k}{2}$$

$$\leq \left\{ \begin{array}{c} \left[\begin{array}{c} \left(\begin{array}{c} \left(\vee_{j=1}^{m} \lambda_A(xa_2q_jx)\right) \vee \\ \left(\vee_{j=1}^{m} \lambda_B(xq'_ja_1x)\right) \vee \\ \left(\vee_{j=1}^{m} \lambda_A(xa_1q_jx)\right) \vee \\ \left(\vee_{j=1}^{m} \lambda_B(xq'_ja_2x)\right) \end{array} \right) \vee \\ \left(\begin{array}{c} \left(\vee_{i=1}^{m} \lambda_A(xa_1p_ix)\right) \vee \\ \left(\vee_{i=1}^{m} \lambda_B(xp'_ia_1x)\right) \vee \\ \left(\vee_{i=1}^{m} \lambda_A(xa_2p_ix)\right) \vee \\ \left(\vee_{i=1}^{m} \lambda_B(xp'_ia_2x)\right) \end{array} \right) \end{array} \right] \\ \vee \frac{1-k}{2} \end{array} \right.$$

$\leq \max\left\{\lambda_A(x), \lambda_B(x), \frac{1-k}{2}\right\} \vee \frac{1-k}{2}$
$= \max\left\{\lambda_A(x), \lambda_B(x), \frac{1-k}{2}\right\}$
$= \lambda_A(x) \vee_k \lambda_B(x) = (\lambda_A \vee_k \lambda_B)(x).$
Thus $(\lambda_A \odot_k \lambda_B) \leq (\lambda_A \vee_k \lambda_B)$.
This implies that $A \cap_k B \subseteq A \odot_k B$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (5) and (2) $\Longrightarrow$ (4) $\Longrightarrow$ (5) are straightforward.

(5) $\Longrightarrow$ (1) Let A be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right h-ideal of R and B be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left h-ideal of R. Then A and B are $(\in, \in \vee q_k)^*$-intuitionistic fuzzy h-bi ideals of R. So by assumption $A \cap_k B \subseteq A \odot_k B$ and by using Theorem 3.10, $A \odot_k B \subseteq A \cap_k B$. Thus $A \odot_k B = A \cap_k B$. Hence by Theorem 4.4, R is h-hemiregular. On the other hand by hypothesis we also have $A \cap_k B \subseteq A \odot_k B$. By Lemma 5.4, R is h-intra-hemiregular.

References

[1] J. Ahsan, K. Saifullah, M. F. Khan, Fuzzy semirings, Fuzzy Sets and Systems 60 (1993) 309–320.
[2] T. K. Dutta, B. K. Biswas, Fuzzy prime ideals of a semiring, Bull. Malaysian Math. Soc. 17 (1994) 9–16.
[3] T. K. Dutta, B. K. Biswas, Fuzzy k-ideals of semirings, Bull Cal. Math. Soc. 87 (1995) 91–96
[4] W. A. Dudek, Intuitionistic fuzzy h-ideals of hemirings, WSEAS Trans. Math. 12 (2006) 1315–1331.
[5] W. A. Dudek, Special types of intuitionistic fuzzy left h-ideals of hemirings, Soft Comput. 12 (2008) 359–364.
[6] J. S. Golan, Semirings and their Applications, Kluwer Acad. Publ. , 1999
[7] M. Henriksen, Ideals in semirings with commutative addition, Amer. Math. Soc. Notices 6 (1958) 321.
[8] X. K. Huang, H. J. Li, Y. Q. Yin, The h-hemiregular fuzzy Duo hemirings, Int. J. Fuzzy Syst. 9 (2) (2007) 105–109.
[9] A. Hussain, M. Shair, Hemiring Characterized by $(\in, \in \vee q_k)^*$-intuitionistic fuzzy ideal. (Submitted).
[10] K. Iizuka, On the Jacobson radical of a semiring, Tohuku Math. J. 11 (2) (1959) 409–421.
[11] Y. B. Jun, H. S. Kim, M. A. Öztürk, Fuzzy k-ideals in semirings, J. Fuzzy Math. 13 (2005) 351–364.
[12] Y. B. Jun, M. A. Öztürk, S. Z. Song, On fuzzy h-ideals in hemirings, Inform. Sci. 162 (2004) 211–226.
[13] D. R. La Torre, On h-ideals and k-ideals in hemirings, Publ. Math. Debrecen 12 (1965) 219–226.
[14] X. Lin and J. Wang, The characterization of h-hemiregular hemirings and h-intra-hemiregular hemirings in terms of intuitionistic fuzzy h-ideals of hemiring, Southeast Asian Bulletin of Mathematics (2010) 34: 893–904
[15] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[16] M. Shabir, T. Mahmood, Characterizations of hemirings by $(\in, \in \vee q_k)$-fuzzy ideals, Computers and Mathematics with Applications 61 (2011) 1059–1078
[17] W. Wechler, The Concept of Fuzziness in Automata and Language Theory, Akademic Verlag, Berlin, 1978
[18] Y. Q. Yin, H. X. Li, The characterization of h-hemiregular hemirings and h-intra-hemiregular hemirings, Inform Sci. 178 (2008) 3451–3464.
[19] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–35
[20] J. Zhan, On properties of fuzzy left h-ideals in hemirings with t-norms, Int. J. Math. Sci. 19 (2005) 3127–3144.
[21] J. Zhan, B. Davvaz, L-fuzzy h-ideals with operators in hemirings, Northeast. Math. J. 23 (2007) 1–14.
[22] J. Zhan, W. A. Dudek, Fuzzy h-ideals of hemirings, Inform. Sci. 177 (2007) 876–886.
[23] J. Zhan, Z. Tan, T-fuzzy k-ideals of semirings, Sci. Math. Japan 58 (2003) 597–601.

Asim Hussain is a Ph. D. scholar in Mathematics Department, Quaid-i-Azam University, Islamabad, Pakistan. He did his B. Sc. in 2007 and M. Sc. Mathematics from Punjab University Lahore, and obtained 2nd position in 2009. He did his M. Phil in Mathematics from Quaid-i-Azam University, Islamabad, Pakistan. His research interest area is fuzzy and soft algebraic structures and algebraic techniques in Automata Theory.

Muhammad Shabir is a Professor in Mathematics Department of Quaid-iAzam University, Islamabad, Pakistan. He has published more than 100 research papers in the international journals. He is also a coauthor of 4 books including the book ”Fuzzy Semirings with Applications to Automata Theory” published by Springer Verlag. His current research interests include Fuzzy Algebraic structures, Soft Algebraic Structures and their applications.