On covering of products of $T$-generalized state machines

Masumeh Sadeghi$^1$, Hamid Alinejad-Rokny$^1$

$^1$Mathematical Department, University of Mazandaran, Babolsar, Iran
$^2$Department of Computer Engineering, Science and Research Branch, Islamic Azad University, Tehran, Iran

m.sadeghi@umz.ac.ir, alinejad@ualberta.ca

Abstract: We introduce the concepts of $T$-generalized state machines and coverings of products of them. Also some of algebraic properties of them are investigated. Finely some products such as direct sum and sum of $T$-generalized state machines are introduced. An interesting distributive property of cascade product over the sum of $T$-generalized state machines concerning to covering of $T$-generalized state machines is established.

Keywords: $T$-generalized state machine; Covering; Cascade product; Wreath product; Sum of two $T$-generalized state machines; Direct sum.

1. Introduction

Since Wee (1967) introduced the concept of fuzzy automata following Zadeh (1965), fuzzy automata theory has been developed by many researchers. Recently, Malik et al. (1994, 1997) introduced the concepts of fuzzy state machines and fuzzy transformation semigroups based on Wee's concept of fuzzy automata and related concepts and applied algebraic techniques. Kim et al. (1998) introduced the notion of $T$-generalized state machine that is extension of fuzzy state machine. Even if $T=$ our notion of generalized state machine is different from the notion of Malik (1994).

In this paper, we introduce On Covering of products of $T$-generalized state machines and investigate their algebraic structures. For the terminology in (crisp) algebraic automata theory, we refer to Holcombe (1982).

2. Preliminaries

Definition 2.1. (Kim et al. (1998)). A triple $M = (Q, X, \tau)$ where $Q$ and $X$ are finite nonempty sets and $\tau$ is a fuzzy subset of $Q \times X \times Q$, i.e., $\tau$ is a function from $Q \times X \times Q$ to $[0,1]$, is called a generalized state machine. If $\sum_{q \in Q} \tau(p, a, q) \leq 1$ for all $p \in Q$ and $a \in X$. If $\sum_{q \in Q} \tau(p, a, q) = 1$ for all $p \in Q$ and $a \in X$, then $M$ is said to be complete. Note that the concept of generalized state machines is different from the concept of fuzzy finite state machines of Malik et al. (1994) which is also a fuzzification of the concept of state machines. Their notion is based on the concept of fuzzy automata introduced by Wee (1967). While a generalized state machine $(Q, X, \tau)$ with $\tau( Q \times X \times Q) \subseteq {0,1}$ can always be regarded as a state machine, but a fuzzy finite state machine $(Q, X, \tau)$ with $\tau( Q \times X \times Q) \subseteq {0,1}$ cannot be regarded as a state machine generally. So the concept of generalized state machines is a generalization of the concept of state machines, whereas the concept of fuzzy finite state machines of Malik et al. may not be considered as a generalization of the concept of state machines in a certain sense. This means that the concept of generalized state machines is more adequate fuzzification of the concept of state machines than the concept of fuzzy finite state machines. Let $M = (Q, X, \tau)$ be a generalized state machine. Then $Q$ is called the set of states and $X$ is called the set of input symbols. Let $X^+$ denote the set of all words of elements of $X$ of finite length. Formally, every incomplete generalized state machine can be extended to a complete generalized state machine as follows:

$^1$ Department of Computer Engineering, Science and Research Branch, Islamic Azad University, Tehran, Iran
Definition 2.2. (Kim et al. (1998)). Let \( M = (Q, X, \rightarrow) \) be an incomplete generalized state machine. Let \( z \) be a state not in \( Q \). The completion \( M^c \) of \( M \) is the complete generalized state machine \( (\hat{Q}, X, \hat{\tau}) \) given by \( \hat{Q} = Q \cup \{ z \} \) and

\[
\hat{\tau}(\hat{p}, a, \hat{q}) = \begin{cases} 
\tau(p, a, q) & \text{if } p, q \in Q, \\
1 - \sum_{q \in Q} \tau(p, a, q) & \text{if } p \not\in Q \text{ and } q = z \\
0 & \text{if } p' = z \text{ and } q' \in Q \\
1 & \text{if } p', q' = z
\end{cases}
\]

For all \( \alpha \in X \). The new state \( z \) is called the sink state of \( M^c \). If \( M \) is complete, then we take \( M \) itself as \( M^c \).

Definition 2.3. (Schweizer and Sklar (1960)). A binary operation \( T \) on \([0,1]\) is called a \( \varepsilon \)-norm if for all \( a, b, c \in [0,1] \):

1. \( T(a,1) = a \),
2. \( T(a,b) \leq T(a,c) \) whenever \( b \leq c \),
3. \( T(a,b) = T(b,a) \),
4. \( T(a,T(b,c)) = T(T(a,b),c) \).

The maximum and minimum will be written as \( V \) and \( \Lambda \), respectively. \( T \) is clearly \( V \)-distributive, i.e., \( T(a \vee b,c) = T(a,c) \vee T(b,c) \) for all \( a, b, c \in [0,1] \). Define \( T_0 \) on \([0,1]\) by \( T_0(a,1) = a = T_0(1,a) \) and \( T_0(a,b) = 0 \) if \( a \neq 1 \) for all \( a, b \in [0,1] \). Then \( \Lambda \) is the greatest \( \varepsilon \)-norm on \([0,1] \) and \( T_0 \) is the least \( \varepsilon \)-norm on \([0,1] \), i.e., for any \( \varepsilon \)-norm \( T \), \( \Lambda (a, b) \geq T(a,b) \geq T_0(a,b) \) for all \( a, b \in [0,1] \). \( T \) will always mean a \( \varepsilon \)-norm on \([0,1]\). By an abuse of notation we will denote \( T(a_1, T(a_2, T(\ldots, T(a_{n-1}, a_n) \ldots))) \) by \( T(a_1, \ldots, a_n) \) where \( a_1, \ldots, a_n \in [0,1] \). The legitimacy of this abuse is ensured by the associativity of \( T \) (Definition 2.3[4]).

Definition 2.4. (Kim et al. (1998)). Let \( M = (Q, X, \rightarrow) \) be a generalized state machine. Define \( \tau^+ : Q \times X^+ \times Q \rightarrow [0,1] \) by

\[
\tau^+(p, x, q) = V \left\{ \tau(p, x, r), \tau^+(r, y, q) \mid r \in Q \right\}
\]

where \( p, q \in Q \) and \( a_1, \ldots, a_n \in X \). When \( T \) is applied to \( M \) as above, \( M \) is called a \( T \)-generalized state machine. Hereafter a generalized state machine will always be written as a \( T \)-generalized state machine because a generalized state machine always induces a \( T \)-generalized state machine as in Definition 2.4.

Definition 2.5. (Kim et al. (1998)). Let \( (Q, X, \rightarrow) \) be a \( T \)-generalized state machine. Then

\[
\tau^+(p, x, y, q) = V \{ \tau(p, x, r), \tau^+(r, y, q) \mid r \in Q \}
\]

For all \( p, q \in Q \) and \( x, y \in X^+ \).

Example 2.6. (Kim et al. (1998)).

\[
T = \Lambda, Q = \{ p_1, p_2, \ldots, p_3 \} \text{ and } X = \{ a \} \]

Let Then \( (Q, X, \rightarrow) \) is a \( T \)-generalized state machine. However,

\[
\sum_{q \in Q} \tau(p_1, a, q) = \tau(p_1, a, p_1) + \tau(p_2, a, p_2) + \tau(p_3, a, p_3)
\]

and

\[
\tau(p_1, a, p_7) = 0.6 + 0.4 + 0.1 + 0.3 > 1.
\]

3. Covering

Definition 3.1. (Kim et al. (1998)). Let \( M_1 = (Q_1, X_1, \rightarrow) \) and \( M_2 = (Q_2, X_2, \rightarrow) \) be \( T \)-generalized state machines. If \( \varepsilon : X_1 \rightarrow X_2 \) is a function and \( \eta : Q_2 \rightarrow Q_1 \) is a subjective partial function such that \( \tau_1^+(\eta(p), x, \eta(q)) \leq \tau_2^+ (p, \varepsilon(x), q) \) for all \( p, q \) in the domain of \( \eta \) and \( x \in X_1^+ \), then we say that \( (\eta, \varepsilon) \) is a covering of \( M_1 \) by \( M_2 \) and \( M_2 \) covers \( M_1 \) that denote by \( M_1 \leq M_2 \). Moreover, if the inequality turns out equality whenever the left-hand side of the inequality is not
zero (resp. the inequality always turns out equality), then we say that \((\eta, \varepsilon)\) is a strong covering (resp. a complete covering) of \(M_1\) by \(M_2\) and that \(M_2\) strongly covers \(M_1\) (resp. completely covers \(M_1\)) and denote by \(M_1 \leq_s M_2\) (resp. \(M_1 \leq_c M_2\)). In Definition 3.1 we abused the function \(\varepsilon\). We will write the natural semigroup homomorphism from \(X_1^+\) to \(X_2^+\) induced by \(\varepsilon\). Also for convenience sake, we give an example that is elementary and important.

**Example 3.2.** (Kim et al. (1998)). Let \(M = (Q, X, \tau)\) be a \(T\)-generalized state machine. Define an equivalence relation \(\sim\) on \(X\) by \(a \sim b\) if and only if \(\tau(p, a, q) = \tau(p, b, q)\) for all \(p, q \in Q\). Construct a \(T\)-generalized state machine \(M_1 = (Q, X / \sim, \tau / \sim)\) by defining \(\tau / \sim(p, [a], q) = \tau(p, a, q)\). Now define \(\varepsilon : X \rightarrow X / \sim\) by \(\varepsilon(a) = [a]\) and \(\eta = 1_Q\). Then \((\eta, \varepsilon)\) is a complete covering of \(M\) by \(M_1\) clearly.

**Proposition 3.3.** (Cho et al. (2001)). Let \(M_1, M_2\) and \(M_3\) be \(T\)-generalized state machines. If \(M_1 \leq M_2\) (resp. \(M_1 \leq_s M_2\), \(M_1 \leq_c M_2\)) and \(M_2 \leq M_3\) (resp. \(M_2 \leq_s M_3\), \(M_2 \leq_c M_3\)), then \(M_1 \leq M_3\) (resp. \(M_1 \leq_s M_3\), \(M_1 \leq_c M_3\)).

### 4. Products

In this section, we consider cascade products and wreath products of \(T\)-generalized state machines, where \(T\) is less than or equal to the ordinary product. We will always assume that \(T\) is less than or equal to the ordinary product.

**Definition 4.1.** (Cho et al. (2001)). Let \(M_1 = (Q_1, X_1, \tau_1)\) and \(M_2 = (Q_2, X_2, \tau_2)\) be \(T\)-generalized state machines. The cascade product \(M_1 \omega_T M_2\) of \(M_1\) and \(M_2\) with respect to \(\omega : Q_2 \times X_2 \rightarrow X_1\) is the \(T\)-generalized state machine \((Q_1 \times Q_2, X_1 \times X_2, \tau_1 \times \omega_T \tau_2)\) with 

\[
(-1)_{Q_1 \times Q_2} \omega_T \tau_2((p_1, p_2), b, (q_1, q_2)) = \sum_{(a, \alpha) \in \varepsilon} \omega(p_1, b, q_1) \tau_2(p_2, b, q_2)
\]

In Definition 4.1, \((Q_1 \times Q_2, X_1 \times X_2, \tau_1 \times \omega_T \tau_2)\) is clearly a \(T\)-generalized state machine. In fact, we have 

\[
\sum_{(a, \alpha) \in \varepsilon} \omega(p_1, b, q_1) \tau_2(p_2, b, q_2) \leq 1
\]

Because for any \(\varepsilon\)-norm \(T\), \(\sum_{a \in Q_1} \omega(p_1, b, q_1) \tau_2(p_2, b, q_2) \leq 1\) for all \(p_1 \in Q_1, p_2 \in Q_2\) and \(b \in X_2\).

Let \(M_1 = (Q_1, X_1, \tau_1)\) and \(M_2 = (Q_2, X_2, \tau_2)\) be \(T\)-generalized state machines and \(\omega : Q_2 \times X_2 \rightarrow X_1\). Define \(\omega^* : Q_1 \times Q_2 \times X_2 \rightarrow X_1\) by 

\[
\omega^*(p_2, b_1, b_2, \ldots, b_n) = \omega(p_2, b_1) \omega(u_1, b_2) \ldots
\]

\[\omega(u_{n-1}, b_n)\), where \(p_2, u_1, u_2, \ldots, u_{n-1} \in Q_2\) and \(b_1, \ldots, b_n \in X_2\) such that \(\tau_1(p_1, \omega(p_2, b_1), \omega(u_1, b_2), \ldots, \omega(u_{n-1}, b_n), q_1) = \tau_1(p_1, \omega^*(p_2, x_1, q_1), \tau_2(p_2, x_2, q_2))\), where \(p_1, q_1 \in Q_1\) and \(b_2, b_2 \in X_2\).

**Lemma 4.2.** (Cho et al. (2001)). Let \(M_1 = (Q_1, X_1, \tau_1)\) and \(M_2 = (Q_2, X_2, \tau_2)\) be \(T\)-generalized state machines. Then 

\[
(\tau_1 \omega_T \tau_2)''((p_1, p_2), x, (q_1, q_2)) = \tau_1'(p_1, \omega^*(p_2, x, q_1), \tau_2''(p_2, x, q_2))
\]

where \(p_1, q_1 \in Q_1\) and \(b_2, b_2 \in X_2\).

**Definition 4.3.** (Cho et al. (2001)). Let \(M_1 = (Q_1, X_1, \tau_1)\) and \(M_2 = (Q_2, X_2, \tau_2)\) be \(T\)-generalized state machines. The wreath product \(M_1 \circ_T M_2\) of \(M_1\) and \(M_2\) is the \(T\)-generalized state machine 

\[
(Q_1 \times Q_2, X_1 \times X_2, \tau_1 \circ_T \tau_2)
\]

with 

\[
T(\tau_1(p_1, p_2, q_1), \tau_2(p_2, b, q_2)) \subseteq \{|f| Q_2 \rightarrow X_1\}.
\]

In Definition 4.3, 

\[
(Q_1 \times Q_2, X_1 \times X_2, \tau_1 \circ_T \tau_2)
\]

is clearly a \(T\)-generalized state machine.

In fact, we have
\[ \sum_{(q_1,q_2)\in Q_1\times Q_2} \left( \tau_1 \circ \tau_2 \right) \left( (p_1,p_2),(f,b),(q_1,q_2) \right) \]

\[ \sum_{q_2\in Q_2} \left( \tau_2(p_2,b),(q_1) \right) \sum_{q_1\in Q_1} \tau_1(p_1,(f,p_2,q_1)) \leq \sum_{q_2\in Q_2} \left( \tau_2(p_2,b),(q_2) \right) \sum_{q_1\in Q_1} \tau_1(p_1,(f,p_2,q_1)) \]

\[ \tau_1(p_1,(f,p_2,q_1)) \]

Theorem 4.4. (Cho et al. (2001)). Let \( M_1=(Q_1,X_1,\tau_1) \) and \( M_2=(Q_2,X_2,\tau_2) \) be T-generalized state machines. Then \( M_1 \omega_2 M_2 \preceq M_1 \circ \tau_2 M_2 \).

Theorem 4.5. (Cho et al. (2001)). Let \( M_1=(Q_1,X_1,\tau_1) \) and \( M_2=(Q_2,X_2,\tau_2) \) and \( M=(Q,X,\tau) \) are T-generalized state machines. Let \( M \preceq M_1 \omega_3 M_2 \). Then \( M \preceq M_1 \omega_3 M_2 \).

We now introduce two more ways of connecting T-generalized state machines.

Definition 4.6. (Kim et al. (1998)). Let \( M=(Q,X,\tau) \) and \( M'=((Q',X',\tau')) \) be two T-generalized state machines such that \( Q \cap Q' = \emptyset \) and \( X \cap X' = \emptyset \). Then the direct sum \( M \oplus M' \) of \( M \) and \( M' \) is T-generalized state machine \((Q \cup Q',X \cup X',\tau \oplus \tau')\) with

\[ \tau \oplus \tau'(p,x,q) = \begin{cases} \tau(p,x,q) & \text{if } p \in Q, x \in X \text{ and } q \in Q' \text{ or } (p,x) \in X \times X' \text{ and } q \in Q' \\ 0 & \text{otherwise} \end{cases} \]

Definition 4.7. (Kim et al. (1998)). Let \( M=(Q,X,\tau) \) and \( M'=(Q',X',\tau') \) be two T-generalized state machines such that \( Q \cap Q' = \emptyset \) and \( X \cap X' = \emptyset \). Then the sum \( M+M' \) of \( M \) and \( M' \) is T-generalized state machine \((Q \cup Q',X \cup X',\tau + \tau')\) with

\[ \tau + \tau'(p,x,q) = \begin{cases} \tau(p,x,q) & \text{if } p \in Q, x \in X \text{ and } q \in Q' \\ \tau'(p,x,q) & \text{if } p \in Q', x \in X' \text{ and } q \in Q \\ 0 & \text{otherwise} \end{cases} \]

5. Main Results

The following we proved that wreath product, sum and cascade products of T-generalized state machines are associative. However, it can easily be proved that the direct sum of T-generalized state machines is not associative.

Theorem 5.1. If \( M,M',M'' \) and \( M'' \) are T-generalized state machines, then

(i) \( (M \circ T) \circ T M'' = M \circ T (M \circ T M'') \);
(ii) \( (M+M') \circ M'' = M+M' \circ M'' \);
(iii) \( (M \omega_3 M') \omega_3 M'' = M \omega_3 (M \omega_3 M'') \).

where \( \omega_3 \) and \( \omega_4 \) are determined by \( \omega_1 \) and \( \omega_2 \) in a natural way.

Proof. Let \( M=(Q,X,\tau) \) and \( M'=(Q',X',\tau') \) and \( M''=(Q'',X'',\tau'') \).

Recall that \( M \circ T M'=((Q \times Q') \times X',(X \times X') \times X',(\tau \circ T \tau') \circ T \tau'') \) and \( M \circ T M''=((Q \times Q') \times X'',(X \times X') \times X'',(\tau \circ T \tau') \circ T \tau'') \).

Let \( \alpha: (Q \times Q') \times X' \rightarrow Q \times ((Q' \times Q') \times X',(\tau \circ T \tau') \circ T \tau'') \).

Given a function \( f: Q' \rightarrow X' \) denote \( f_1=p_1 \circ f \) and \( f_2=p_2 \circ f \).

Define \( f_1:Q \times X',f_2:Q \times X' \) by \( f_1([q',q''])=f_1(q') \in (q''\times q''') \) and \( f_2(q')=q'' \times q''' \).

Then \( f_1(q')=f_1(q') \in (q''\times q''') \) and \( f_2(q')=q'' \times q''' \).

But

\[ \beta((f'x'))=\beta(f_{w',x'}) = f_{x'}(f_{w',x'}) = f_{x'}(f_{w',x'}) = f_{x'} = f_{x'}(f_{w',x'}) = f_{x'} \text{ and } x' = x'' \]

Then

\[ f_1(\alpha(q')=f_{x'}(q') \quad f_2 = f_{x'} \text{ and } x' = x'' \]

\[ f_1 = f_{x'}, f_2 = f_{x'} \text{ and } x' = x'' \]
\[ p_1 \circ f = p_1 \circ f \circ p_2 = p_2 \circ f \quad \text{and} \quad x'' = x_\alpha'' \]
\[ \Rightarrow f = f_\alpha \quad \text{and} \quad x'' = x_\alpha'' \]
\[ \Rightarrow (f, x') = (f_\alpha, x_\alpha') \]

Therefore \( \beta \) is injective.

Let \((g, (h, x'')) \in X^Q \times X^Q' \times (X^Q'')^X \times (X^Q'')^X\)

Define \( f : Q' \to X^Q' \times X' \) by \( f(q') = (g_{q'} \circ h(q'')) \) where \( g_{q'}(q'') = g(q', q'') \). Then \( \beta(f, x'') = (g, (h, x'')) \).

Thus, \( \beta \) is onto.

It can be easily seen that, \((\alpha, \beta)\) is a required isomorphism.

(i) Let, \( M + M'' = (QUQ', XuX', \tau + \tau') \), where,

\[
\tau + \tau'(p, x, q) = \begin{cases} 
(\tau(p, x, q), 1) & \text{if } p, q \in Q, x \in X \\
(\tau'(p, x, q), 1) & \text{if } p, q \in Q', x \in X' \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
(M + M') + M'' = (QUO'UQ', XuX') (XuX', \tau + \tau') + \tau''
\]

and

\[
\tau + \tau'(p, x, q) = \begin{cases} 
(\tau(p, x, q), 1) & \text{if } p, q \in Q \cup Q', x \in X \cup X' \\
(\tau'(p, x, q), 1) & \text{if } p, q \in Q', x \in X' \\
0 & \text{otherwise}
\end{cases}
\]

It immediately follows that, \( M' + M'' = (QUO'Q', XuX', \tau + \tau') \), and

\[
\tau + \tau'(p, x, q) = \begin{cases} 
(\tau(p, x, q), 1) & \text{if } p, q \in Q \cup Q', x \in X \\
(\tau'(p, x, q), 1) & \text{if } p, q \in Q', x \in X' \\
0 & \text{otherwise}
\end{cases}
\]

Moreover \( M + (M' + M'') = (QUUQ', XuX') \),

\[
\tau + \tau'(p, x, q) = \begin{cases} 
(\tau(p, x, q), 1) & \text{if } p, q \in Q, x \in X \\
(\tau'(p, x, q), 1) & \text{if } p, q \in Q', x \in X' \\
0 & \text{otherwise}
\end{cases}
\]

Set both \( \alpha \) and \( \beta \) as identity mappings on \( Q \cup Q' \cup Q'' \) and \( X \cup X' \cup X'' \), respectively.

\[
(\tau + \tau') + \tau''(p, x, q) = \begin{cases} 
(\tau(p, x, q), 1) & \text{if } p, q \in Q, x \in X \\
(\tau'(p, x, q), 1) & \text{if } p, q \in Q', x \in X' \\
0 & \text{otherwise}
\end{cases}
\]

\[
\tau + \tau' = \begin{cases} 
(\tau(p, x, q), 1) & \text{if } p, q \in Q, x \in X \\
(\tau'(p, x, q), 1) & \text{if } p, q \in Q', x \in X' \\
0 & \text{otherwise}
\end{cases}
\]

\[
\tau + \tau''(p, x, q) = \begin{cases} 
(\tau(p, x, q), 1) & \text{if } p, q \in Q, x \in X \\
(\tau'(p, x, q), 1) & \text{if } p, q \in Q', x \in X' \\
0 & \text{otherwise}
\end{cases}
\]

\[
\tau''(p, x, q) = \begin{cases} 
(\tau(p, x, q), 1) & \text{if } p, q \in Q, x \in X \\
(\tau'(p, x, q), 1) & \text{if } p, q \in Q', x \in X' \\
0 & \text{otherwise}
\end{cases}
\]

\[
\tau + \tau' + \tau''(p, x, q) = \begin{cases} 
(\tau(p, x, q), 1) & \text{if } p, q \in Q, x \in X \\
(\tau'(p, x, q), 1) & \text{if } p, q \in Q', x \in X' \\
0 & \text{otherwise}
\end{cases}
\]
=τ+τ′(α[p,β[x,α[q]].
Hence,
(M+M′)+M″≡M+(M′+M″).
(iii) Consider, MωM′=(Q×Q′,X′,τωx,q), where ω2:Q×X→X and
τωx,q′=((p,p′),x′,q)→T(τωx,q′)=T(τ(p,x,q),1).
Hence, (MωM′)+M″≅M+(M′+M″).

(ii) Consider, MωM′=(Q×Q′,X′,τωx,q), where ω2:Q×X→X and
τωx,q′=((p,p′),x′,q)→T(τωx,q′)=T(τ(p,x,q),1).
Hence, (MωM′)+M″≅M+(M′+M″).

M=(Q,X,τ), M′=(Q′,X,τ′) be two T-generalized state machines.

(i) M≤M+M′;
(ii) M′≤M+M′;
(iii) M≤M⊕M′;
(iv) M′≤M⊕M′.

Proof. Let (η,ξ) be identity maps on Q∪Q′ and X∪X′ respectively.

(i) Case (a). If p,q∈Q and x∈X, then T(η(p),x,η(q),1)=τ+τ′(p,ξ(x),q) = T(τ(p,x,q),1).
Case (b). If p,q∈Q and x∈X, then T(η(p),x,η(q),1)=τ+τ′(p,ξ(x),q) = T(τ(p,x,q),1).
Case (c). If (p,x)∈Q×X and q∈Q then T(η(p),x,η(q),1)=0<τ+τ′(p,ξ(x),q) = 1.
In all other cases T(τ,1)=τ+τ′=0.

(ii) Clearly.

Theorem 6.2. Let M=(Q,X,τ), M′=(Q′,X,τ′) be two T-generalized state machines. Then
M⊕M≤M⊕M′.

Proof. Set both (η,ξ) as identity maps on Q∪Q′ and X∪X′ respectively.
Case (a). If \( p,q \in Q \) and \( x \in X \), then
\[
\tau + \tau'(\eta(p), x, \eta(q)) = T(\tau(p, x, q), 1) = \tau \oplus \tau'(p, \xi(x), q).
\]

Case (b). If \( p,q \in Q' \) and \( x \in X' \), then
\[
\tau + \tau'(\eta(p), x, \eta(q)) = T(\tau'(\eta(p), x, \eta(q), 1) = \tau \oplus \tau'(p, \xi(x), q).
\]

In all other cases \( \tau + \tau' = \tau + \tau' = 0 \).

**Theorem 6.3.** Let \( M = (Q, X, \tau) \), \( M' = (Q', X, \tau') \) and \( M'' = (Q'', X, \tau'') \) be three \( T \)-generalized state machines such that \( M \leq M' \). Then

(i) Given \( \omega_1 : Q'' \times X'' \to X \) there exists \( \omega_2 : Q' \times X' \to X' \) such that
\[
M \omega_1 \leq M' \omega_2.
\]

(ii) If \((\eta, \xi)\) is a covering of \( M \) by \( M' \) and \( \xi \) is onto, then for each \( \omega_1 : Q \times X \to X' \) there exists \( \omega_2 : Q' \times X' \to X' \), such that
\[
M'' \omega_1 \leq M' \omega_2.
\]

(iii) \( M \leq M'' \leq M' \).

Proof. Since \( M \leq M' \) there exist a partial onto mapping \( \eta : Q' \to Q \) and a mapping \( \xi : X \to X' \) such that
\[
\tau(\eta(p'), x, \eta(q)) \leq \tau'(p', \xi(x), q).
\]

(i) Given \( \omega_1 : Q'' \times X'' \to X \), set \( \omega_2 = \xi \circ \omega_1 \) and \( \eta' : Q' \times Q'' \to Q' \times Q'' \) and \( \xi' = \xi \).

(ii) Given \( \omega_1 : Q \times X \to X'' \), construct \( \omega_2 : Q' \times X' \to X'' \) such that
\[
\tau'( \eta(p'), x, \eta(q')) \leq \tau(\eta(p), x, \eta(q)).
\]

Since \( \xi \) is onto and \( \Sigma \) is finite, such \( \omega_2 \) exists. Clearly \( \omega_2 \) is not unique.

Define \( \eta' : Q' \times X'' \to X'' \) by
\[
\eta'(p', p'' \in Q' \times Q'' \to Q' \times Q'').
\]

Define \( \xi' : X' \times X'' \to X'' \) by \( \xi'(x', x'' \in X' \times X'' \to X'' \).

Since \( \xi \) is onto and \( \Sigma \) is finite, such \( \omega_2 \) exists. Clearly \( \omega_2 \) is not unique.

Define \( \eta' : Q' \times X'' \to X'' \) by \( \eta'(p', p'' \in Q' \times Q'' \to Q' \times Q'') \) and \( \xi' : X' \times X'' \to X'' \) by
\[
\xi'(x', x'' \in X' \times X'' \to X' \times X'').
\]

Define \( \eta' : Q' \times X'' \to X'' \) by \( \eta'(p', p'' \in Q' \times Q'' \to Q' \times Q'' \) and \( \xi' : X' \times X'' \to X'' \) by
\[
\xi'(x', x'' \in X' \times X'' \to X' \times X'')\).

By \( T(\tau'(p', \xi(x'), q'), \tau(p', x, q)) \leq T(\tau'(p', \xi(x'), q'), \tau(p', x, q)) \)

(iv) Recall that \( M + M'' = (Q \cup Q'', X \cup X'', \tau + \tau'') \), where,
\[
\tau + \tau'(p, x) = \begin{cases} \tau(p, x) & \text{if } p \in Q, x \in X \\ \tau'(p, x) & \text{otherwise} \end{cases}
\]

and \( M' + M'' = (Q' \cup Q'', X' \cup X'', \tau + \tau'') \), where,
\[
\tau + \tau'(p, x) = \begin{cases} \tau(p, x) & \text{if } p \in Q', x \in X' \\ \tau'(p, x) & \text{otherwise} \end{cases}
\]
Define $\eta':Q'\cup Q''\rightarrow Q\cup Q''$ by $\eta'(p') = \{\begin{array}{ll}
\eta(p) & \text{if } p' \in Q'

p' & \text{otherwise}
\end{array}$

And $\xi':X'\cup X''\rightarrow X'\cup X''$ by $\xi'(x) = \{\begin{array}{ll}
\xi(x) & \text{if } x \in X

x & \text{otherwise}
\end{array}$

since $\eta$ is a partial onto mapping so is $\eta'$. We claim that $\tau+\tau'( (\eta'(p),x,\eta'(q')) \leq \tau'+\tau''(p',\xi'(x),q')$.

If $p',q',q''\in Q'$ and $x\in X$ or $p',q',q''\in Q''$ and $x\in X''$, then obviously $\tau+\tau'( (\eta'(p),x,\eta'(q')) \leq \tau'+\tau''(p',\xi'(x),q')$.

in all other cases $\tau+\tau'( (\eta'(p),x,\eta'(q')) \leq \tau'+\tau''(p',\xi'(x),q')$.

The proofs of (v),(vi) and (vii) are now obvious. It can be easily seen that $M''\circ M'$, in general, does not cover $M''\circ M$, even though $M\leq M'$. This fact makes it imperative to introduce a weaker notion of covering. Hence, the following definition:

**Definition 6.4.** Let $M=(Q,X,\tau),M'=(Q',X',\tau')$ be two T-generalized state machines. Let $\eta:Q'\rightarrow Q$ be a partial onto mapping and $\xi:X\rightarrow X'$ be a partial mapping. The ordered pair $(\eta,\xi)$ is called weak covering, if $\tau(\eta(p'),x,\eta(q')) \leq \tau'(p',\xi(x),q')$, for all $p',q'$ in the domain of $\eta$ and $x$ in the domain of $\xi$. Symbolically we denote this fact by $M\leq_w M'$. 

**Remark 6.5.** A weak covering differs from a covering only in one sense. $\xi$ in Definition 6.2 is a partial mapping, while $\xi$ in Definition 3.1 is a function. Thus, every covering is a weak covering.

Theorem 6.6. If $M,M',M''$ be Three T-generalized state machines and $M\leq M'$, then $M''\circ M \leq M''\circ M'$.

**Proof.** Since $M\leq M'$ there exist a partial onto mapping $\eta:Q'\rightarrow Q$ and a mapping $\xi:X\rightarrow X'$ such that $\tau(\eta(p'),x,\eta(q')) \leq \tau'(p',\xi(x),q')$.

we have, $M''\circ M = (Q''\times Q, (X')^\tau \times X', (\tau'' \circ \tau))$ and $M''\circ M' = (Q''\times Q', (X')', (\tau'' \circ \tau'))$.

Define $\eta':Q'\times X'\rightarrow Q''\times X''$ by $\eta'(p',p') = (\eta'(p'),\eta(p))$ and $\xi':X'\rightarrow X''$ by $\xi'(x) = (\eta(\xi(x))$). Obviously $\eta'$ is a partial onto mapping and $\xi'$ is a partial mapping.

Consider $\tau'' \circ \tau( (\eta'(p'),p'),(x,\eta'(q')),q'(q'))$

$= (\tau'' \circ \tau( (\eta'(p'),p'),(x,\eta'(q'))) \wedge (\eta'(p'),\eta(p))$.

However, since $\tau(\eta(p'),x,\eta(q')) \leq \tau'(p',\xi(x),q')$, it follows that $\tau'' \circ \tau( (\eta'(p'),p'),(x,\eta'(q')),q'(q')) \leq (\tau''(p',\eta(\xi(x))),q'(q'))$.

The following theorems are easy consequences of the transitive property of coverings of T-generalized state machines and Theorem 6.1.

**Theorem 6.7.** If $M,M',M''$ be three T-generalized state machines and $M\leq M'$, then

(i) $M \leq M'' \leq M'' \circ M'$;

(ii) $M'' \circ M \leq M'' \circ M''$;

(iii) $M + M'' \leq M'' + M'$;

(iv) $M'' + M \leq M'' + M''$.

**Proof.** Since $M\leq M'$ there exist a partial onto mapping $\eta:Q'\rightarrow Q$ and a mapping $\xi:X\rightarrow X'$ such that $\tau(\eta(p'),x,\eta(q')) \leq \tau'(p',\xi(x),q')$.

(i) Given $\omega_T:Q''\times X'' \rightarrow X$ and $\eta:Q'\times X'\rightarrow Q''\times X''$. $\xi_2$ as an identity mapping on $X''$. Let $\xi_2 = \{\begin{array}{ll}
\xi_1(x') & \text{if } x' \in X

x' & \text{otherwise}
\end{array}$

that $\xi_1(x):Q''\rightarrow X'$.

$\omega_T M'' = (\tau \circ \omega_T)( (p,p'),x'',\eta'(q'))$

$= (\tau(p,p'),\omega_T(p'',x'),q'),(\tau''(p'',x''),q''))$

$\leq (\tau'(p',\xi_1(x'),q'),(\tau''(p'',x''),q''))$

$= (\tau''(p',\xi_1(x'),q'),(\tau''(p'',x''),q''))$

$= (\tau''(p',\xi_1(x'),q'),(\tau''(p'',x''),q''))$

$= (\tau''(p',\xi_1(x'),q'),(\tau''(p'',x''),q''))$

$= (\tau''(p',\xi_1(x'),q'),(\tau''(p'',x''),q''))$

The following theorems are easy consequences of the transitive property of coverings of T-generalized state machines and Theorem 6.1.
\[=\tau'' \omega_T \tau((p'',\eta(p''),x,(q'',\eta(q')))
=\tau'(p'',\omega_T (\eta(p''),\xi(x)),q''),\tau(\eta(p''),x,\eta(q'))
\leq \tau'(p'',\omega_T (\eta(p''),\xi(x)),q''),\tau(\eta(p''),x,\eta(q'))
=\tau'(p'',\omega_T (\eta(p''),\xi(x)),q''),\tau(\eta(p''),x,\eta(q'))
\]
References


