

A comparison of Semi-analytical Methods for Solving Partial Integro-Differential Equations

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Abstract: In this paper, we apply the Variational iteration method and homotopy perturbation method for solving linear and nonlinear partial integro-differential equation (PIDE). The efficiency and accuracy of the methods is validated by its application to several distinct test problems which have exact solutions. The results of applying these methods show the simplicity and efficiency of these methods.

Keywords: PIDE; Partial integro-differential equations; Variational iteration method; homotopy perturbation method.

1 Introduction

Mathematical modeling of real-life problems usually results in functional equations, such as partial differential equations, integral and integro-differential equation, and stochastic equations. Many mathematical formulations of physical phenomena contain integro-differential equations. Several techniques including finite difference, B-spline, homotopy perturbation, decomposition and variational iteration have been developed for solving partial integro-differential equation.

He [1, 2] developed the variational iteration and homotopy perturbation methods for solving linear, nonlinear partial integro-differential equation. It is worth mentioning that the origin of variational iteration method can be traced to Inokuti, Sekine and Mura [3], but the real potential of this technique was explored by He [4-10]. Moreover, He realized the physical significance of the variational iteration method, its compatibility with the physical problems and applied this promising technique to a wide class of linear and nonlinear, ordinary, partial integro-differential equation [1, 2]. He develops the homotopy perturbation method by merging two techniques, the standard homotopy and the perturbation [2, 11-13]. The homotopy perturbation method was formulated by taking the full advantage of the standard homotopy and perturbation methods.

Our contribution in this paper is to solve linear and nonlinear partial integro-differential equations in one dimensional space with non-homogeneous Dirichlet boundary conditions by Variational iteration method and homotopy perturbation method. The proposed techniques are programmed using Matlab ver. 7.8.0.347 (R2009a).

The paper is organized as follows: In Section 2, we give Analysis of the Variational iteration method for partial integro-differential equations with varying boundary conditions. In Section 3, we use Analysis of the homotopy perturbation method for solving partial integro-differential equations. In Section 4, the proposed schemes are directly applied to solve several numerical examples which have the exact solutions. Conclusions are drawn in Section 5.

2 Analysis of the Variational iteration method

To illustrate the basic concept of the Variational iteration method [7], we consider the following differential equation:

$$L(u) + N(u) = g(x, t) \quad (2.1)$$

where L is a linear operator, N is a nonlinear operator and $g(x, t)$ is an inhomogeneous term. Then we can construct a correct function as follows:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\xi) [L u_n(x, \xi) + N \tilde{u}_n(x, \xi) - g(x, \xi)] d\xi \quad (2.2)$$

where λ is a general Lagrange multiplier [3,8,10], which can be identified optimally via variational theory. The second term on the right is called the correction and \tilde{u}_n is considered as a restricted variation, i.e. $\delta \tilde{u}_n = 0$. With the determination of λ , the approximations $u_n(x, t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) \quad (2.3)$$

2.1. Linear partial integro-differential equation

We need to solve the following equation form:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_0^t k(t, s) u(x, s) ds = f(x, t), \quad x \in [0, 1], \quad t \in I = [0, T], \quad (2.1.1)$$

with the boundary conditions are defined as follows:

$$u(a, t) = 0, \quad u(b, t) = 0 \quad t \geq 0 \quad (2.1.2)$$

and the initial condition:

$$u(x, 0) = g(x) \quad (2.1.3)$$

We consider the equation (2.1.1) subject to the initial condition (2.1.3). According to the variational iteration method, we can construct the following correct functional:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\xi) \left\{ \frac{\partial}{\partial \xi} u_n(x, \xi) - \frac{\partial^2}{\partial x^2} u_n(x, \xi) - \int_0^\xi k(\xi, s) u_n(x, s) ds - f(x, \xi) \right\} d\xi \quad (2.1.4)$$

where $\delta \tilde{u}_n$ is considered as a restricted variation, i.e. $\delta \tilde{u}_n = 0$ and λ is the general Lagrange multiplier. Making the above correct functional stationary and noticing that $\delta \tilde{u}_n = 0$.

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \delta \int_0^t \lambda(\xi) \left\{ \frac{\partial}{\partial \xi} u_n(x, \xi) - \frac{\partial^2}{\partial x^2} \tilde{u}_n(x, \xi) - \int_0^\xi k(\xi, s) \tilde{u}_n(x, s) ds - f(x, \xi) \right\} d\xi \quad (2.1.5)$$

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \lambda(\xi) \delta u_n(x, \xi) \Big|_{\xi=t} - \int_0^t \lambda'(\xi) \delta u_n(x, \xi) d\xi \quad (2.1.6)$$

which yields the following stationary conditions

$$1 + \lambda(t) = 0, \quad \lambda'(\xi) = 0 \quad (2.1.7)$$

Therefore, the general Lagrange multiplier can be readily identified as:

$$\lambda(\xi) = -1 \quad (2.1.8)$$

Substituting this value of the Lagrange multiplier into functional (3.4) gives the iteration formula

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left\{ \frac{\partial}{\partial \xi} u_n(x, \xi) - \frac{\partial^2}{\partial x^2} u_n(x, \xi) - \int_0^\xi k(\xi, s) u_n(x, s) ds - f(x, \xi) \right\} d\xi \quad (2.1.9)$$

2.2. Nonlinear partial integro-differential equation

We need to solve nonlinear problem given by the following integro-differential equations:

$$\frac{\partial}{\partial t} u(x, t) = \int_0^t k(t-s) \frac{\partial}{\partial x} \sigma(u_x(x, s)) ds - f(x, t), \quad t \geq 0 \quad (2.2.1)$$

with the initial condition:

$$u(x,0) = g(x) \quad (2.2.2)$$

We consider the equation (2.2.1) subject to the initial condition (2.2.2). According to the variational iteration method, we can construct the following correct functional:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left\{ \frac{\partial}{\partial \xi} u_n(x,\xi) - \int_0^\xi k(\xi-s) \frac{\partial}{\partial x} \sigma \left(\frac{\partial}{\partial x} \tilde{u}_n(x,s) \right) ds - f(x,\xi) \right\} d\xi \quad (2.2.3)$$

where $\delta \tilde{u}_n$ is considered as a restricted variation, i.e. $\delta \tilde{u}_n = 0$ and λ is the general Lagrange multiplier. Making the above correct functional stationary and noticing that $\delta \tilde{u}_n = 0$.

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\xi) \left\{ \frac{\partial}{\partial \xi} u_n(x,\xi) - \int_0^\xi k(\xi-s) \frac{\partial}{\partial x} \sigma \left(\frac{\partial}{\partial x} \tilde{u}_n(x,s) \right) ds - f(x,\xi) \right\} d\xi \quad (2.2.4)$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(\xi) \delta u_n(x,\xi) \Big|_{\xi=t} - \int_0^t \lambda'(\xi) \delta u_n(x,\xi) d\xi \quad (2.2.5)$$

which yields the following stationary conditions

$$1 + \lambda(t) = 0, \quad \lambda'(\xi) = 0 \quad (2.2.6)$$

Therefore, the general Lagrange multiplier can be readily identified as:

$$\lambda(\xi) = -1 \quad (2.2.7)$$

Substituting this value of the Lagrange multiplier into functional (4.3) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \frac{\partial}{\partial \xi} u_n(x,\xi) - \int_0^\xi k(\xi-s) \frac{\partial}{\partial x} \sigma \left(\frac{\partial}{\partial x} u_n(x,s) \right) ds - f(x,\xi) \right\} d\xi \quad (2.2.8)$$

2.3. Modified variational iteration method (MVIM)

To illustrate the basic concept of the variational homotopy perturbation method, we consider the following general differential equation

$$L(u) + N(u) = g(x) \quad (2.3.1)$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method, we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi) (Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)) d\xi, \quad (2.3.2)$$

where λ is a Lagrange multiplier, which can be identified optimally via variational iteration method. The subscripts n denote the n th approximation, \tilde{u}_n is considered as a restricted variation. i.e. $\delta \tilde{u}_n = 0$; eq (2.3.2) is called as a correction functional. Now, we apply the homotopy perturbation method

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p \int_0^x \lambda(\xi) \left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n) \right) d\xi - \int_0^x \lambda(\xi) g(\xi) d\xi, \quad (2.3.3)$$

which is the modified variational iteration method (MVIM) and is formulated by the coupling of variational iteration method and He's polynomials. The comparison of like power of P gives solutions of various orders.

3. Analysis of the homotopy perturbation method

To clarify the basic ideas of the homotopy perturbation method [14-16], let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega, \quad (3.1)$$

with boundary conditions

$$B(u, \frac{du}{dn}) = 0, \quad r \in \Gamma, \quad (3.2)$$

where A is a general differential operator, B is a boundary operator, u is a known analytical function, and Γ is the boundary of the domain Ω .

The operator A can be divided into two parts L and N , where L is linear, while N is nonlinear. Therefore eq. (5.2.1) can be rewritten as follows

$$L(u) + N(u) - f(r) = 0. \quad (3.3)$$

By the homotopy technique proposed by Liao [17], we can construct a homotopy $\mathfrak{v}(r, p) : \Omega \times [0, 1] \rightarrow R$ which satisfies

$$H(\mathfrak{v}, p) = (1-p)[L(\mathfrak{v}) - L(u_0)] + p[A(\mathfrak{v}) - f(r)] = 0, \quad (3.4)$$

or

$$H(\mathfrak{v}, p) = L(\mathfrak{v}) - L(u_0) + pL(u_0) + p[N(\mathfrak{v}) - f(r)] \quad (3.5)$$

where $r \in \Gamma$ and $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of (3.1), which satisfies the boundary conditions. By (3.4), it easily follows that

$$H(\mathfrak{v}, 0) = L(\mathfrak{v}) - L(u_0) = 0, \quad (3.6)$$

$$H(\mathfrak{v}, 1) = A(\mathfrak{v}) - f(r) = 0, \quad (3.7)$$

and the changing process of p from zero to unity is just that of $H(\mathfrak{v}, p)$ from $L(\mathfrak{v}) - L(u_0)$ to $A(\mathfrak{v}) - f(r)$. In topology, this is called deformation, $L(\mathfrak{v}) - L(u_0)$ and $A(\mathfrak{v}) - f(r)$ are called homotopic.

The embedding parameter p is introduced much more naturally, unaffected by artificial factors. Furthermore, it can be considered as a small parameter for $0 < p \leq 1$. By applying the perturbation technique used in [18, 19], we assume that the solution of eq. (3.4) can be expressed as

$$\mathfrak{v} = \mathfrak{v}_0 + p\mathfrak{v}_1 + p^2\mathfrak{v}_2 + \dots \quad (3.8)$$

Therefore, the approximate solution of eq. (3.1) can be readily obtained as follows:

$$u = \lim_{p \rightarrow 1} \mathfrak{v} = \mathfrak{v}_0 + \mathfrak{v}_1 + \mathfrak{v}_2 + \dots \quad (3.9)$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantage. The series (3.9) is convergent for most cases. However, the convergent rate depends on the nonlinear $A(\mathfrak{v})$. Moreover, the following suggestions were made by [1-3]:

The second derivative of $N(\mathfrak{v})$ with respect to \mathfrak{v} must be small because the parameter may be relatively large, i.e. $p \rightarrow 1$.

The norm of $L^{-1} \frac{\partial N}{\partial \mathfrak{v}}$ must be smaller than one so that the series converges.

3.1. Linear partial integro-differential equation

We consider the partial integro-differential equation (2.1.1-2.1.3). According to the homotopy perturbation method, we can construct the homotopy $\Omega \times [0, 1] \rightarrow R$ which satisfies

$$H(\mathfrak{v}, p) = (1-p) \left[\frac{\partial \mathfrak{v}(x, t)}{\partial t} - \frac{\partial u_0(x, t)}{\partial t} \right] + p \left[\frac{\partial \mathfrak{v}(x, t)}{\partial t} - \frac{\partial^2 \mathfrak{v}}{\partial x^2} - \int_0^t K(t, s) \mathfrak{v}(x, s) ds - f(x, t) \right] = 0 \quad (3.1.1)$$

Substituting eq. (3.8) in to eq. (3.1.1), we get

$$\begin{aligned} & (1-p) \left[\frac{\partial}{\partial t} (\mathfrak{v}_0 + p\mathfrak{v}_1 + p^2\mathfrak{v}_2 + \dots) - \frac{\partial u_0}{\partial t} \right] + \\ & + p \left[\frac{\partial}{\partial t} (\mathfrak{v}_0 + p\mathfrak{v}_1 + p^2\mathfrak{v}_2 + \dots) - \frac{\partial^2}{\partial x^2} (\mathfrak{v}_0 + p\mathfrak{v}_1 + p^2\mathfrak{v}_2 + \dots) \right] + \\ & + p \left[- \int_0^t K(t, s) (\mathfrak{v}_0 + p\mathfrak{v}_1 + p^2\mathfrak{v}_2 + \dots) ds - f(x, t) \right] = 0 \end{aligned} \quad (3.1.2)$$

and rearranging the resultant based on powers of P terms, one has:

$$p^0 : \frac{\partial \mathfrak{v}_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \quad u_0 = g(x), \quad (3.1.3)$$

$$p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial v_0}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} - \int_0^t K(t, s) v_0 ds - f(x, t) = 0, \quad (3.1.4)$$

$$p^2 : \frac{\partial v_2}{\partial t} - \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} - \int_0^t K(t, s) v_1 ds = 0, \quad (3.1.5)$$

$$p^i : \frac{\partial v_i}{\partial t} - \frac{\partial v_{i-1}}{\partial t} + \frac{\partial v_{i-1}}{\partial t} - \frac{\partial^2 v_{i-1}}{\partial x^2} - \int_0^t K(t, s) v_{i-1} ds = 0, \quad i = 2, 3, \dots \quad (3.1.6)$$

With the following conditions:

$$\begin{cases} v_0(x, 0) = g(x), & v_0(0, t) = 0, & v_0(1, t) = 0, \\ v_i(x, 0) = 0, & v_i(0, t) = 0, & v_i(1, t) = 0, \quad i = 1, 2, \dots \end{cases} \quad (3.1.7)$$

In order to obtain the unknown, we should solve eqs. (3.1.4) and (3.1.5), considering the initial conditions of eq. (3.1.7), and having the initial approximations of eq. (3.1.4). So we have:

$$v_0 = g(x), \quad (3.1.8)$$

and solving the above equations, we obtain from eq. (3.1.4)

$$\frac{\partial v_1}{\partial t} = \frac{\partial^2 g(x)}{\partial x^2} + \int_0^t K(t, s) v_0(x, s) ds + f(x, t), \quad (3.1.9)$$

$$v_1 = \int_0^t \left[\frac{\partial^2 g(x)}{\partial x^2} + \int_0^t K(t, s) v_0(x, s) ds + f(x, t) \right] dt, \quad (3.1.10)$$

so we can find v_1 , from v_1 in eq. (3.1.5), we can find v_2 , etc.

By continuing the calculation, we thus have the solution given by

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (3.1.11)$$

3.2. Nonlinear partial integro-differential equation with a modified homotopy perturbation method

We consider the nonlinear partial integro-differential equations (2.2.1, 2.2.2) Eq. (2.2.1) is an example of the general nonlinear equation:

$$u'(t) = -\int_0^t K(t-s)g(u(s))ds + f(t) \quad (3.2.1)$$

which is defined on a Hilbert space H with g a given nonlinear unbounded operator. To solve eqs (2.2.1) and (2.2.2), we construct the following homotopy with $L(u) = u_t$ and

$$N(u) = -\int_0^t K(t-s) \frac{\partial}{\partial x} \sigma(u_x(x, s)) ds \quad (3.2.2)$$

$$L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0. \quad (3.2.3)$$

To deal with the nonlinear term, we will employ He's polynomials which is given by

$$N(u) = N(v_0) + N(v_0, v_1) p + N(v_0, v_1, v_2) p^2 + \dots + N(v_0, v_1, \dots, v_n) p^n + \dots, \quad (3.2.4)$$

$$N(v_0, v_1, \dots, v_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^n p^k v_k\right)_{p=0}, \quad n = 1, 2, \dots, \quad (3.2.5)$$

Substituting (3.2.4) into (3.2.3), and equating coefficients of like powers of p , we obtain

$$p^0 : L(v_0) - L(u_0) = 0, \quad (3.2.6)$$

$$p^1 : L(v_1) + L(u_0) + N(v_0) = 0, \quad (3.2.7)$$

$$p^2 : L(v_2) + N(v_0, v_1) = 0, \quad (3.2.8)$$

$$p^3 : L(v_3) + N(v_0, v_1, v_2) = 0, \quad (3.2.9)$$

$$p^{n+1} : L(v_{n+1}) + N(v_0, v_1, \dots, v_n) = 0, \quad (3.2.10)$$

and so on, which forms the basis of a complete determination of the components v_0, v_1, v_2, \dots . We let $u_0(x, t) = 0$ for convenience. We therefore obtain the following equations for the components:

$$(v_0)_t = 0, \quad v_0(x, 0) = g(x), \quad (3.2.11)$$

$$(v_1)_t - \int_0^t K(t-s) \frac{\partial}{\partial x} \sigma(u_{0x}(x, s)) ds, \quad v_1(x, 0) = 0, \quad (3.2.12)$$

$$(v_2)_t + \frac{\partial}{\partial p} N \left(\sum_{k=0}^1 p^k v_k \right) = 0, \quad v_2(x, 0) = 0, \quad (3.2.13)$$

$$(v_3)_t + \frac{1}{2!} \frac{\partial^2}{\partial p^2} N \left(\sum_{k=0}^2 p^k v_k \right) = 0, \quad v_3(x, 0) = 0, \quad (3.2.14)$$

and so on

4 Numerical Example

The accuracy of our proposed numerical method is measured by computing the difference between numerical and exact solution.

Example 1:

$$u_t(x, t) = \int_0^t e^{-(t-s)} \frac{\partial}{\partial x} (u_x(x, s))^2 ds + f(x, t) \quad (4.1)$$

if we assume that,

$$f(x, t) = -x(1-x)e^{-x-t} + (8-34x+40x^2-16x^3+2x^4)(e^{-2x-t} - e^{-2x-2t}) \quad (4.2)$$

then the exact solution is given by

$$u(x, t) = x(1-x)e^{-x-t} \quad (4.3)$$

and the initial condition is:

$$u(x, 0) = x(1-x)e^{-x} \quad (4.4)$$

The numerical experiment is carried out for $t = 0.01313$ and $x = 0.1, 0.2, \dots, 1$. Table (1) exhibits the numerical results.

Example 2:

$$u_t(x, t) = \int_0^t e^{-2(t-s)} \frac{\partial}{\partial x} (u_x(x, s))^2 ds + f(x, t) \quad (4.5)$$

if we assume that,

$$f(x, t) = \cos(x+t) + \frac{1}{4} \left[\sin 2(x+t) - \cos 2(x+t) - e^{-2t} (\sin 2x - \cos 2x) \right] \quad (4.6)$$

then the exact solution is given by

$$u(x, t) = \sin(x+t) \quad (4.7)$$

and the initial condition is:

$$u(x, 0) = \sin(x). \quad (4.8)$$

The numerical experiment is carried out for $t = 0.02$ and $x = 0.1, 0.2, \dots, 1$. Table (2) exhibits the numerical results.

Example 3:

$$u_t(x, t) = \int_0^t e^{-(t-s)} \frac{\partial}{\partial x} (u_x(x, s))^2 ds + f(x, t) \quad (4.9)$$

if we assume that,

$$f(x, t) = e^{-(x+t)} + 2e^{-2x}(e^{-t} - e^{-2t}) \quad (4.10)$$

then the exact solution is given by

$$u(x, t) = e^{-x-t} \quad (4.11)$$

and the initial condition is:

$$u(x, 0) = e^{-x} \quad (4.12)$$

The numerical experiment is carried out for $t = 0.001$ and $x = 0.1, 0.2, \dots, 1$. Table (3) exhibits the numerical results.

Table 1: The approximate and exact solutions at $t = 0.001$

x	Approximate Solution	Exact Solution	Error
0	7.994669E-006	0.000000E+000	7.994669E-006
0.1	8.151268E-002	8.135397E-002	1.587113E-004
0.2	1.311261E-001	1.308660E-001	2.600708E-004
0.3	1.557268E-001	1.554163E-001	3.104484E-004
0.4	1.610377E-001	1.607160E-001	3.216704E-004
0.5	1.517845E-001	1.514811E-001	3.034354E-004
0.6	1.318468E-001	1.315831E-001	2.636581E-004
0.7	1.043874E-001	1.041787E-001	2.087592E-004
0.8	7.196469E-002	7.182078E-002	1.439097E-004
0.9	3.662793E-002	3.655470E-002	7.323711E-005
1	0.000000E+000	0.000000E+000	0.000000E+000

Table 2: The approximate and exact solutions at $t = 0.01$

x	Approximate Solution	Exact Solution	Error
0	7.946932E-004	0.000000E+000	7.946932E-004
0.1	8.184030E-002	8.062507E-002	1.215226E-003
0.2	1.321222E-001	1.296935E-001	2.428748E-003
0.3	1.570661E-001	1.540239E-001	3.042275E-003
0.4	1.624853E-001	1.592761E-001	3.209215E-003
0.5	1.531734E-001	1.501239E-001	3.049516E-003
0.6	1.330612E-001	1.304042E-001	2.656976E-003
0.7	1.053501E-001	1.032453E-001	2.104851E-003
0.8	7.262749E-002	7.117729E-002	1.450198E-003
0.9	3.696442E-002	3.622718E-002	7.372391E-004
1	0.000000E+000	0.000000E+000	0.000000E+000

Table 3: The approximate and exact solutions at $t = 0.0001$

x	Approximate Solution	Exact Solution	Error
0	5.000500E-005	1.000000E-004	4.999500E-005
0.1	9.977298E-002	9.993292E-002	1.599359E-004
0.2	1.985979E-001	1.987673E-001	1.694357E-004
0.3	2.954377E-001	2.956157E-001	1.780366E-004
0.4	3.893252E-001	3.895104E-001	1.852478E-004
0.5	4.793227E-001	4.795133E-001	1.905771E-004
0.6	5.645315E-001	5.647250E-001	1.935520E-004
0.7	6.441004E-001	6.442942E-001	1.937419E-004
0.8	7.172350E-001	7.174258E-001	1.907777E-004
0.9	7.832047E-001	7.833891E-001	1.843704E-004
1	8.413507E-001	8.415250E-001	1.743267E-004

5 Conclusion

Analytically, partial integro-differential equations are usually difficult to solve. In many cases, it is required to obtain the approximate solutions. In this work, we proposed the homotopy perturbation method and variational iteration method for solving linear and nonlinear partial integro-differential equation.

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