On Generalized Left Derivations in BCI-Algebras

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Abstract: In the present paper, we introduce the notion of generalized left derivation of a BCI-algebra \(X\), construct several examples, and investigate related properties. Also establish some results on regular generalized left derivation. Furthermore, for a generalized left derivation \(H\), the concept of a \(H\)-invariant generalized left derivation is introduced, and examples are discussed. Using this concept a condition for a generalized left derivation to be regular is provided. Finally, some results on p-semisimple BCI-algebra are obtained and it is shown that let \(H\) be a self map in a p-semisimple BCI-algebra \(X\). Then \(H\) is a generalized left derivation if and only if it is a derivation on \(X\).

Keywords: Derivations, BCI-Algebras

1 Introduction

The notion of BCK-algebras and BCI-algebras were introduced by Y. Imai and K. Iseki in 1966 [9, 10]. BCK-algebras and BCI-algebras are algebraic formulation of BCK-system and BCI-system in combinatory logic. Later, the notion of BCI algebras have been extensively investigated by many researchers (see [2, 3, 14] and references there in). BCI-algebra is a generalization of a BCK-algebra that is every BCK-algebra is a BCI-algebra but not vice versa (see [6]). Therefore, most of the algebras related to the t-norm based logic such as MTL [5], BL, hoop, MV [4] (i.e. lattice implication algebra) and Boolean algebras etc., are extensions of BCK-algebras which have a lot of applications in computer science (see [19]). Cosequently, BCK/BCI-algebras are considerably general structures.

Throughout the present paper \(X\) will denote a BCI-algebra. Jun and Xin [11] applied the notion of derivation in ring and near-ring theory to BCI-algebras in the year 2004 and introduced a new concept called a (regular) derivation in BCI-algebras, and investigated some of its properties. Using the notion of a regular derivation, they also established characterizations of a p-semisimple BCI-algebra. For a self map \(d\) of a BCI-algebra, they defined a \(d\)-invariant ideal, and gave conditions for an ideal to be \(d\)-invariant. During the last 10 years, a greater interest has been devoted to the study on derivations in BCI-algebras and a number of research articles have been published in this direction on various aspects (see [1, 8, 15, 16, 17, 18, 20]).

Motivated by notions of left derivations [1] and generalized derivations [18] in the theory of BCI-algebras, in this paper, we introduced the notion of generalized left derivations on BCI-algebras and investigate related properties. The concept of generalized left derivations covers the concept of left derivations on BCI-algebras. Further, we obtain some results on regular generalized left derivations. Also, for a generalized left derivation \(H\), we introduce the concept of a \(H\)-invariant generalized left derivations and give some examples. Using this concept we provide a condition for a generalized left derivation to be regular. Finally, we characterize the notion of p-semisimple BCI-algebra \(X\) by using the concept of generalized left derivation and show that let \(H\) be a self map in a p-semisimple BCI-algebra \(X\). Then \(H\) is a generalized left derivation if and only if it is a derivation on \(X\).
2 Preliminaries

In this section, we collect the following definitions and properties from the existing literature that will be needed in the sequel.

A nonempty set $X$ with a constant 0 and a binary operation $*$ is called a BCI-algebra if for all $x, y, z \in X$ the following conditions hold:

1. $(I)(x * y) * (x * z) = (z * y) * x$,
2. $(II)(x * (x * y)) * y = 0$,
3. $(III)x * x = 0$,
4. $(IV)x * y = 0$ and $y * x = 0$ imply $x = y$.

Define a binary relation $\leq$ on $X$ by letting $x \leq y = 0$ if and only if $x \leq y$. Then $(X, \leq)$ is a partially ordered set. A BCI-algebra $X$ satisfying $0 \leq x$ for all $x \in X$, is called BCK-algebra.

A BCI-algebra $X$ has the following properties: for all $x, y, z \in X$

(a1)$x * 0 = x$.
(a2)$x * y * z = (x * z) * y$.
(a3)$x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
(a4)$x * (z * y) \leq x * y$.
(a5)$x * (x * (x * y)) = x * y$.
(a6)$0 * (x * y) = (0 * x) * (0 * y)$.
(a7)$x * 0 = 0$ implies $x = 0$.

For a BCI-algebra $X$, denote by $X_+$ (resp. $G(X)$) the BCK-part (resp. the BCI-G part) of $X$, i.e., $X_+$ is the set of all $x \in X$ such that $0 \leq x$ (resp. $G(X) := \{ x \in X | 0 * x = x \}$). Note that $G(X) \cap X_+ = \{ 0 \}$ (see [13]). If $X_+ = \{ 0 \}$, then $X$ is called a p-semisimple BCI-algebra. In a p-semisimple BCI-algebra $X$, the following hold:

(a8)$x * z * (y * z) = x * y$.
(a9)$0 * (0 * x) = x$ for all $x \in X$.
(a10)$x * (0 * y) = y * (0 * x)$.
(a11)$x * y = 0$ implies $x = y$.
(a12)$x * a = x * b$ implies $a = b$.
(a13)$x * b = x * b$ implies $a = b$.
(a14)$a * (a * x) = x$.
(a15)$x * (y * z) = (w * z) * (y * z)$.

Let $X$ be a p-semisimple BCI-algebra. We define addition “+” as $x + y = x * (0 * y)$ for all $x, y \in X$. Then $(X, +)$ is an abelian group with identity 0 and $x - y = x - y$. Conversely let $(X, +)$ be an abelian group with identity 0 and let $x + y = x - y$. Then $X$ is a p-semisimple BCI-algebra and $x + y = x * (0 * y)$ for all $x, y \in X$ (see [14]).

For a BCI-algebra $X$ we denote $x \wedge y = y * (y * x)$, in particular $0 * (0 * x) = a$, and $L_p(X) := \{ a \in X | x * a = 0 \Rightarrow x = a, \forall x \in X \}$. We call the elements of $L_p(X)$ the p-atoms of $X$. For any $a \in X$, let $V(a) := \{ x \in X | a * x = 0 \}$, which is called the branch of $X$ with respect to $a$. It follows that $x \wedge y \in V(a * b)$ whenever $x \in V(a)$ and $y \in V(b)$ for all $x, y \in X$ and all $a, b \in L_p(X)$. Note that $L_p(X) = \{ x \in X | x = a \}$, which is the p-semisimple part of $X$, and $X$ is a p-semisimple BCI-algebra if and only if $L_p(X) = X$ (see [12],[Proposition 3.2]). Note also that $a_e \in L_p(X)$, i.e., $0 * (0 * a_e) = a_e$, which implies that $a_e * y \in L_p(X)$ for all $y \in X$. It is clear that $G(X) \subset L_p(X)$, and $x * (x * a) = a$ and $a * x \in L_p(X)$ for all $a \in L_p(X)$ and all $x \in X$. For more details, refer to [2, 3, 11, 12, 13, 14].

3 Generalized Left Derivations

We introduce the notion of generalized left derivation of a BCI-algebra $X$ as follows:

**Definition 1.** Let $X$ be a BCI-algebra. Then a self map $H : X \rightarrow X$ is called a generalized left derivation of $X$ if there exists a left derivation $D : X \rightarrow X$ such that

$$D(x * y) = x * H(y) \wedge y * D(x)$$

for all $x, y \in X$.

Note that if $H = D$, then the generalized left derivation of a BCI-algebra $X$ is a left derivation of a BCI-algebra $X$.

**Example 1.** Let $X = \{ 0, 1, 2 \}$ a BCI-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

(1) We define a map

$$D : X \rightarrow X, x \mapsto \begin{cases} 2 & \text{if } x \in \{ 0, 1 \}, \\ 0 & \text{if } x = 2. \end{cases}$$

It can be easily verified that $D$ is a left derivation of $X$. Again, define a map

$$H : X \rightarrow X, x \mapsto \begin{cases} 0 & \text{if } x \in \{ 0, 1 \}, \\ 2 & \text{if } x = 2. \end{cases}$$

It is easy to check that $H$ is a generalized left derivation of $X$.

(2) Define a map

$$D : X \rightarrow X, x \mapsto \begin{cases} 0 & \text{if } x \in \{ 0, 2 \}, \\ 1 & \text{if } x = 1. \end{cases}$$

It is easy to check that $D$ is a left derivation of $X$.

(2.1) Define a map

$$H : X \rightarrow X, x \mapsto \begin{cases} 2 & \text{if } x \in \{ 0, 1 \}, \\ 0 & \text{if } x = 2. \end{cases}$$

It is easy to see that $H$ is a generalized left derivation of $X$. 
(2.2) If we define a map \( H : X \to X \) by \( H(x) = 2 \) for all \( x \in X \), then we can easily verify that \( H \) is generalized left derivation of \( X \).

(2.3) If we define a map \( H : X \to X \) by \( H(x) = 0 \) for all \( x \in X \), then we can easily verify that \( H \) is generalized left derivation of \( X \).

Theorem 1. Let \( H \) be a generalized left derivation of a BCI-algebra \( X \). Then

(1) \( x \in L_p(X) \Rightarrow H(x) \in L_p(X) \) for all \( x \in X \).

(2) \( H(x) = 0 + H(x) \) for all \( x \in X \).

(3) \( H(x + y) = x + H(y) \) for all \( x, y \in L_p(X) \).

(4) \( x \in G(X) \Rightarrow H(x) \in G(X) \) for all \( x \in X \).

Proof. (1) For any \( x \in L_p(X) \), we have

\[
H(x) = H(0 \ast (0 \ast x)) \\
= (0 \ast H(0 \ast x)) \land ((0 \ast x) \ast D(0)) \\
= ((0 \ast x) \ast D(0)) \ast ((0 \ast x) \ast D(0)) \ast (0 \ast H(0 \ast x)) \\
= 0 \ast H(0 \ast x) \in L_p(X).
\]

(2) By (1), we have \( H(x) \in L_p(X) \). Then

\[
H(x) = 0 \ast (0 \ast H(x)) = 0 + H(x).
\]

(3) For any \( x, y \in L_p(X) \), we have

\[
H(x + y) = H(x \ast (0 \ast y)) \\
= (x \ast H(0 \ast y)) \land ((0 \ast y) \ast D(x)) \\
= ((0 \ast y) \ast D(x)) \ast ((0 \ast y) \ast D(x)) \ast (x \ast H(0 \ast y)) \\
= x \ast H(y) \\
= x \ast (0 \ast H(y)) \land (y \ast D(0)) \\
= x \ast (0 \ast H(y)) \\
= x \ast H(y).
\]

(4) Let \( x \in G(X) \). Then \( 0 \ast x = x \), and so

\[
H(x) = H(x \ast 0) \\
= (0 \ast H(x)) \land (x \ast D(0)) \\
= (x \ast D(0)) \ast ((x \ast D(0)) \ast (0 \ast H(x)) \\
= 0 \ast H(x)
\]

since \( 0 \ast H(x) \in L_p(X) \). Hence \( H(x) \in G(X) \). This completes the proof.

If we take \( H = D \) in Theorem 1, then we have the following corollary.

Corollary 1(11). Let \( D \) be a left derivation of a BCI-algebra \( X \). Then

(1) \( x \in L_p(X) \Rightarrow D(x) \in L_p(X) \) for all \( x \in X \).

(2) \( D(x) = 0 + D(x) \) for all \( x \in X \).

(3) \( D(x + y) = x + D(y) \) for all \( x, y \in L_p(X) \).

(4) \( x \in G(X) \Rightarrow D(x) \in G(X) \) for all \( x \in X \).

Theorem 2. Let \( H \) be a generalized left derivation of a BCI-algebra \( X \). Then

(1) \( x \in L_p(X) \Rightarrow H(x) = x \ast H(0) = x + H(0) \) for all \( x \in X \).

(2) \( H(x + y) = H(x) + H(y) - H(0) \) for all \( x, y \in L_p(X) \).

(3) \( H \) is identity on \( L_p(X) \) if and only if \( H(0) = 0 \).

(4) \( H(x \ast y) \leq x \ast H(y) \) for all \( x, y \in X \).

Proof. (1) For any \( x \in L_p(X) \), we have

\[
H(x) = H(x \ast 0) = (x \ast H(0)) \land (0 \ast D(x)) \\
= (0 \ast D(x)) \ast ((0 \ast D(x)) \ast (x \ast H(0))) \\
= 0 \ast (0 \ast (x \ast H(0))) \\
= x \ast H(0) = x \ast (0 \ast H(0)) \\
= x + H(0)
\]

since \( x \ast H(0) \in L_p(X) \) and \( H(0) \in G(X) \).

(2) If \( x, y \in L_p(X) \), then \( x + y \in L_p(X) \). Using (1), we have

\[
H(x + y) = (x + y) + H(0) \\
= x + H(0) + y + H(0) - H(0) \\
= H(x) + H(y) - H(0).
\]

(3) It follows from (1).

(4) For any \( x, y \in X \), we have

\[
H(x \ast y) = (x \ast H(y)) \land (y \ast D(x)) \\
= (y \ast D(x)) \ast ((y \ast D(x)) \ast (x \ast H(y))) \\
\leq x \ast H(y).
\]

This completes the proof.

Definition 2. A generalized left derivation \( H \) of a BCI-algebra \( X \) is said to be regular if \( H(0) = 0 \).

Example 2. (1) The generalized left derivation \( H \) of \( X \) in Examples 1 (1) and 1 (2.3) are regular.

(2) The generalized left derivation \( H \) of \( X \) in Examples 1 (2.1) and 1 (2.2) are not regular.

Theorem 3. If \( X \) is a BCK-algebra, then every generalized left derivation of \( X \) is regular.

Proof. Let \( H \) be a generalized left derivation of a BCK-algebra \( X \). Then

\[
H(0) = H(0 \ast x) \\
= (0 \ast H(0)) \land (x \ast D(0)) \\
= 0 \land (x \ast D(0)) = 0.
\]

Hence \( H \) is regular.

In a BCI-algebra, Theorem 3 is not true as seen in the following example:
Example 3. In Example 1 (2.1), H is a generalized left derivation of a BCI-algebra X which is not regular.

Theorem 4. Let H be a regular generalized left derivation of a BCI-algebra X. Then

(1) Both x and H(x) belong to the same branch for all x ∈ X.
(2) H(x) ≤ x for all x ∈ X.
(3) H(x) * y ≤ x * H(y) for all x, y ∈ X.

Proof. (1) Let x ∈ X. Then we have

\[ 0 = H(0) = H(a_x * x) = (a_x * H(x)) \land (x * D(a_x)) = (x * D(a_x)) * ((x * D(a_x)) * (a_x * H(x))) = a_x * H(x) \]

since \( a_x * H(x) \in L_p(X) \). Hence \( a_x \leq H(x) \), and so \( H(x) \in V(a_x) \). Obviously, \( x \in V(a_x) \).

(2) Since H is regular, \( H(0) = 0 \). Then

\[ H(x) = H(x * 0) = (x * H(0)) \land (0 * D(x)) = (x * 0) \land (0 * D(x)) = (0 * D(x)) * ((0 * D(x)) * x) \leq x. \]

(3) Since \( H(x) \leq x \) for all \( x \in X \) by (2). Using (a3), we have

\[ H(x) * y \leq x * y \leq x * H(y). \]

This completes the proof.

Theorem 5. For any generalized left derivation H of a BCI-algebra X, the set

\[ H^{-1}(0) := \{ x \in X \mid H(x) = 0 \} \]

is a subalgebra of X if \( x = 0 \) for all \( x \in X \). Moreover, \( H^{-1}(0) \subseteq X_+ \).

Proof. Assume that \( x = 0 \) for all \( x \in X \). Let \( x, y \in H^{-1}(0) \). Then \( H(x) = 0 = H(y) \), and so

\[ H(x * y) \leq x * H(y) = 0 * 0 = 0 \]

by Theorem 2(4). Hence \( H(x * y) = y * H(y) \) for all \( x, y \in X \).

Example 4. (1) Let H be a generalized left derivation of X which is described in Example 1 (2.1). We know that \( I := \{ 0, 1 \} \) is an ideal of X which is not \( H \)-invariant.

(2) Let H be a generalized left derivation of X which is described in Example 1 (1). We know that \( I := \{ 0, 1 \} \) is a \( H \)-invariant ideal of X.

Theorem 6. Let H be a generalized left derivation of a BCI-algebra X. Then H is regular if and only if every ideal of X is \( H \)-invariant.

Proof. Let I be an ideal of X. Suppose H is regular, then it follows from Theorem 4 (2) that \( H(x) \leq x \) for all \( x \in X \) implying thereby \( H(x) * x = 0 \). Let \( y \in X \) be such that \( y \in H(I) \). Then \( y = H(x) \) for some \( x \in I \). Thus

\[ y * x = H(x) * x = 0 \in I. \]

Since I is an ideal of X, it follows that \( y \in A \) so that \( H(I) \subseteq I \). Therefore I is \( H \)-invariant.

Conversely, suppose that every ideal of X is \( H \)-invariant. Since the zero ideal \( \{ 0 \} \) is clearly \( H \)-invariant, we have \( H(\{ 0 \}) \subseteq \{ 0 \} \), and so \( H(0) = 0 \). Hence H is regular.

If we take \( H = D \) in Theorem 6, then we have the following corollary.

Corollary 2([1]). Let D be a left derivation of a BCI-algebra X. Then D is regular if and only if every ideal of X is \( D \)-invariant.

Next, we prove some results in a p-semisimple BCI-algebra.

Theorem 7. Let H be a generalized left derivation of a p-semisimple BCI-algebra X, we have the following assertions:

(1) \( x * H(x) = y * H(y) \) for all \( x, y \in X \).
(2) \( H(x * y) = x * H(y) \) for all \( x, y \in X \).
(3) \( H(x) * x = H(y) * y \) for all \( x, y \in X \).
(4) \( H(x) * x = y * H(y) \) for all \( x, y \in X \).

Proof. (1) Let X be a p-semisimple BCI-algebra. Then for any \( x, y \in X \), we have

\[ H(0) = H(x * x) = (x * H(x)) \lor (x * D(x)) = x * H(x). \]

Also,

\[ H(0) = H(y * y) = (y * H(y)) \lor (y * D(y)) = y * H(y). \]

Henceforth, we get \( x * H(x) = y * H(y) \).

(2) Let X be a p-semisimple BCI-algebra. Then for any \( x, y \in X \), we have

\[ H(x * y) = (x * H(y)) \lor (y * D(x)) = x * H(y). \]
(3) Using (I), we have

\[(x \ast y) \ast (x \ast H(y)) \leq H(y) \ast y\]

and

\[(y \ast x) \ast (y \ast H(x)) \leq H(x) \ast x\]

these above inequalities can be rewritten as

\[((x \ast y) \ast (x \ast H(y))) \ast (H(y) \ast y) = 0\]

and

\[((y \ast x) \ast (y \ast H(x))) \ast (H(x) \ast x) = 0\]

Consequently, we get

\[((x \ast y) \ast (x \ast H(y))) \ast (H(y) \ast y) = ((y \ast x) \ast (y \ast H(x))) \ast (H(x) \ast x)\]

Also, using (1) and (2), we obtain

\[(x \ast y) \ast H(x \ast y) = (y \ast x) \ast H(y \ast x)\]

\[\implies (x \ast y) \ast (x \ast H(y)) = (y \ast x) \ast (y \ast H(x))\] \hspace{1cm} (3.2)

Since, \(X\) is a \(p\)-semisimple \(BCI\)-algebra. Hence, by using equation (3.2) and (a12), the above equation (3.1) yields \(H(x) \ast x = y \ast H(y)\).

(4) We know that \(H(0) = x \ast H(x)\). Using (3), we get \(H(0) \ast 0 = H(y) \ast y\) implies \(H(0) = H(y) \ast y\). Therefore \(H(y) \ast y = x \ast H(x)\) implying thereby \(H(x) \ast x = y \ast H(y)\). This completes the proof.

If we take \(H = D\) in Theorem 7, then we have the following corollary.

**Corollary 3([1]).** Let \(D\) be a left derivation of a \(p\)-semisimple \(BCI\)-algebra \(X\), we have the following assertions:

1. \(D(x \ast y) = x \ast D(y)\) for all \(x, y \in X\).
2. \(D(x \ast y) = y \ast D(x)\) for all \(x, y \in X\).
3. \(D(x \ast y) = y \ast D(x)\) for all \(x, y \in X\).

**Theorem 8.** Let \(H\) be a self map in a \(p\)-semisimple \(BCI\)-algebra \(X\). Then \(H\) is a generalized left derivation if and only if it is a derivation on \(X\).

**Proof.** Suppose that \(H\) is a generalized left derivation on \(X\). First, we show that \(H\) is a \((r,l)\)-derivation on \(X\). Let \(x, y \in X\). Using (a14), we have

\[H(x \ast y) = x \ast H(y)\]

\[= (H(x) \ast y) \ast ((H(x) \ast y) \ast (x \ast H(y)))\]

\[= (x \ast H(y)) \ast (H(x) \ast y)\]

Hence \(H\) is a \((r,l)\)-derivation on \(X\).

Again, we show that \(H\) is a \((l,r)\)-derivation on \(X\). Let \(x, y \in X\). Using Theorem 7(4) and (a15), we have

\[H(x \ast y) = x \ast H(y)\]

\[= (x \ast 0) \ast H(y)\]

\[= (x \ast (H(0) \ast H(0))) \ast H(y)\]

\[= (x \ast ((x \ast H(x))) \ast (H(y) \ast y)) \ast H(y)\]

\[= (x \ast H(y)) \ast ((x \ast H(x)) \ast (H(y) \ast y))\]

\[= (x \ast H(y)) \ast ((x \ast H(y)) \ast (H(x) \ast y))\]

\[= (H(x) \ast y) \land (x \ast H(y))\]

Conversely, suppose that \(H\) is a derivation of \(X\). As \(H\) is a \((r,l)\)-derivation on \(X\). Then for any \(x, y \in X\), we have

\[H(x \ast y) = (x \ast H(y)) \land (H(x) \ast y)\]

\[= (H(x) \ast y) \ast ((H(x) \ast y) \ast (x \ast H(y)))\]

\[= x \ast H(y)\]

\[= (y \ast D(x)) \ast ((y \ast D(x)) \ast (x \ast H(y)))\]

\[= (x \ast H(y)) \land (y \ast D(x))\].

Hence \(H\) is a generalized left derivation. This completes the proof.

If we take \(H = D\) in Theorem 8, then we have the following corollary.

**Corollary 4([1]).** Let \(D\) be a self map in a \(p\)-semisimple \(BCI\)-algebra \(X\). Then \(D\) is a left derivation if and only if it is a derivation on \(X\).

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