*Journal of Fractional Calculus and Applications*,
Vol. 4(1) Jan. 2013, pp. 1- 7.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

---

# ON THE FRACTIONAL-ORDER MEMRISTOR MODEL

M. E. FOUDA, A. G. RADWAN

ABSTRACT. Fractional order calculus is the general expansion of linear integer-order calculus and is considered as one of the novel topics for modelling dynamical systems in different applications. In this paper, the generalized state equation of the nonlinear two-terminal element which is called memristor is discussed in the fractional-order sense. The effect of the added fractional-order parameter on the memristor characteristics and output behaviour are introduced. The fractional order resistance of the memristor in general for any applied voltage is derived. The generalized formulas and numerical analysis of the step response of the memristor including the instantaneous resistance, startup time and saturation time are studied which will be reduced to the conventional memristor response as $\alpha = 1$.

## 1. INTRODUCTION

Modelling using the concept of fractional calculus penetrates the basic fundamentals of many applications due to its advantages and also since the conventional integer-order modelling is only a narrow subset of the fractional-calculus. From the main advantages of fractional-order modelling is its long-memory dependency and also the ability to increase the degree of freedom for the system through the added fractional-order parameters. Many ground rules in many applications have been generalized in the fractional-order sense such as in the control theory [1]-[3], circuit theory [4]-[6] and in the fractional-order Smith chart [7]. In the circuit theory, the fractional-order element (FOE) is considered as a generalized element that covers the conventional three passive elements which are inductor, resistor and capacitor when the fractional order parameter equals to $-1, 0$ and $1$ respectively. One of the realizations of the half order capacitor can be obtained by dipping a capacitive type probe, coated with a porous film of polymer of particular thickness, into a polarizable medium [8]. Recently, the memristor (short for memory resistor) was postulated and theoretically proved by Leon Chua in a seminal paper in 1971[9]. Then he later generalized his theorem to all memristive systems in 1976 [10]. This new element represents the missing relation between the charge and flux among the conventional elements. Although the theoretical concepts related to this fourth

---

2000 *Mathematics Subject Classification.* 34A12, 34A30, 34D20.
*Key words and phrases.* Memristor, Memristor modeling, Fractional order applications, Memristive systems analysis, Fractional-Calculus.
Submitted Apr. 1, 2012. Accepted Aug. 7. Published Jan. 1, 2013.

passive two-terminal element has been postulated more than 40 years ago, the first passive realization in nanotechnology was introduced by HP-lab a few years ago with its pinched i-v hysteresis [11]-[13]. Since the existence of the passive memristor device, huge research interests and projects have been directed towards the new applications related to this element [11]. For example, the memristor can be used as a non-volatile memory instead of the capacitor and transistor circuits because it can memorize its previous state. Moreover, the memristor resistance can be changed between $R_{off}$ and $R_{on}$ which can represent logic 1 and 0 in the digital design circuits. Therefore building the memristor-based logic and digital circuits instead of transistors draws great attention due to its nano dimension size[14]-[16]. In addition, the time-varying property of the memristor resistance introduces many novel fundamentals in the analogy circuit design such as in the case of memristor-based oscillators [17]-[19].

Since the memristor is a nonlinear element, the relationship between the voltage$v$ and the current of this element is proportional to the resistance which is a function of the charge $q$. Generally, the ohmic relationship is given by:

$$v(t) = R_m(q, x, t)i(t) \tag{1}$$

$$R_m = xR_{on} + (1 - x)R_{off} \tag{2}$$

where $x$ represents the state variable of the memristor which physically represents the ratio between the length of doped region to the total length of the memristor $D$ . Also, $R_{on}$ and $R_{off}$ represent the minimum and maximum resistances of the memristor respectively. Furthermore the state equation is given by

$$\frac{dx}{dt} = \pm ki(t)f(x) \tag{3}$$

where $\pm$ represents the polarity of the memristor, $k = \mu_v R_{on}/D^2$ is the memristor constant which depends on dopant mobility $\mu_v$, and $f(x)$ is the dopant drift window function of the memristor which is given by:

$$f(x) = 1 - (2x - 1)^{2p} \tag{4}$$

This paper is organized as follows: In section 2 the fractional state equation of the memristor is presented with its solution in case of linear window function $f(x) = 1$. Then the step response of the memristor with its general formula are discussed. The last section discusses the conclusion and future work.

## 2. Fractional order memristor model

The fractional differential equation of memristor state can be given by

$$\frac{d^\alpha x}{dt^\alpha} = \pm ki(t)f(x) \tag{5}$$

By differentiating both sides of (2), then

$$\frac{d^\alpha R_m}{dt^\alpha} = -R_d \frac{d^\alpha x}{dt^\alpha} \tag{6}$$

Substituting by (6) into (5)

$$\frac{d^\alpha R_m}{dt^\alpha} = \mp k R_d i(t) f(x) \tag{7}$$

where $R_d$ is the difference between $R_{off}$ and $R_{on}$. For linear window function $f(x) = 1$ and substituting in (1)

$$R_m d^\alpha R_m = \mp k R_d v(t) dt^\alpha \tag{8}$$

By integrating both sides

$$J^\alpha R_m d^\alpha R_m = \mp k R_d J^\alpha v(t) dt^\alpha \tag{9}$$

Using the basic definition of the fractional integral proposed by Riemann Liouville [20], which was given by

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{10}$$

Therefore, the left hand side L.H.S of (9) can be calculated by parts as follows

$$L.H.S = J^\alpha R_m d^\alpha R_m = \frac{1}{\Gamma(\alpha)} \int_0^{R_m} (R_k - R)^{\alpha-1} R dR = \frac{R_k^{\alpha+1}}{\Gamma(\alpha+2)} \tag{11}$$

If the memristor's resistance $R_k$ changes from its initial value $R_{in}$ to $R_m$ then

$$\frac{R_m^{\alpha+1} - R_{in}^{\alpha+1}}{\Gamma(\alpha+2)} = \frac{\mp k R_d}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} v(\tau) d\tau \tag{12}$$

Therefore, the memristor resistance as a function of the input voltage and the time can be obtained by:

$$R_m = \left(R_{in}^{\alpha+1} \mp \alpha(\alpha+1) k R_d \int_0^t (t-\tau)^{\alpha-1} v(\tau) d\tau\right)^{\frac{1}{\alpha+1}} \tag{13}$$

When the fractional order memristor becomes a conventional memristor at $\alpha = 1$,then

$$R_m^2 = R_{in}^2 \mp 2kR_d \int_0^t v(\tau) d\tau = R_{in}^2 \mp 2kR_d \phi(t) \tag{14}$$

Where $\phi(t)$ represents the flux. It is clear that the above equation gives the same results which are proposed in [21]-[22]. In the next section, the step response of the memristor resistance will be discussed in two different cases as follows.

## 3. Step input voltage

In case of applying step input voltage across the memristor where the input signal is defined by

$$v(t) = V_{DC} u(t) \tag{15}$$

where $u(t)$ is the unit step function. By substituting from (15) into (14), then the resistance of the memristor is given by

$$R_m = \left(R_{in}^{\alpha+1} \mp (\alpha+1) k R_d V_{DC} t^\alpha\right)^{\frac{1}{\alpha+1}} \tag{16}$$

The positive or negative sign in (16) discusses the polarity effect for both the memristor and the applied voltage $V_{DC}$, consequently two cases will be discussed in the following subsections .

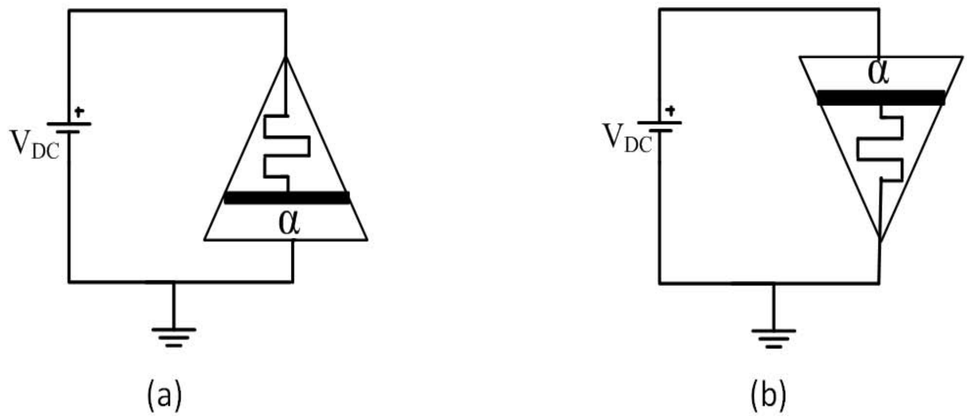


Figure 1. Memristor configurations with the input voltage

3.1. **Opposite polarity configuration.** The memristor circuit is connected as shown in Fig.1(a) where the positive of the supply is connected to the negative of the memristor then the resistance of the memristor is given by:

$$R_m = \left(R_{in}^{\alpha+1} + (\alpha+1)kR_dV_{DC}t^{\alpha}\right)^{\frac{1}{\alpha+1}} \tag{17}$$

It is clear from the previous equation that the resistance of the memristor increases from the initial value until it reaches its maximum $R_{off}$ in a certain time period which is called the saturation time $t_{sat}$. Figure 2 shows the memristor behaviour when the applied step input voltage and the memristor parameters $\mu_v$, $D$, $V_{DC}$, $R_{off}$, $R_{on}$ are equal to $10^{-10}cm^2s^{-1}V^{-1}$, $10nm$ , $1V$, $38k\Omega$, $100\Omega$ respectively for different values of $\alpha$. From Fig. 2(b), the saturation time depends on the value of the fractional-order $\alpha$ where the saturation time increases as $\alpha$ increases for certain $V_{DC}$.

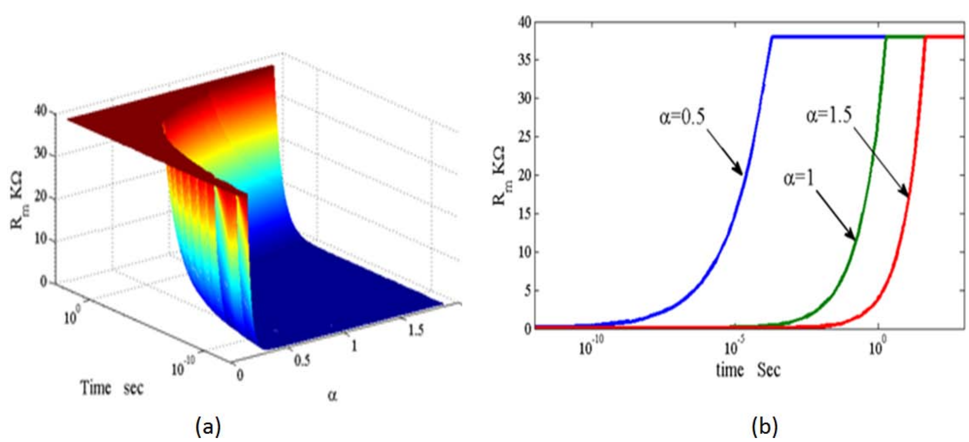


Figure 2. a) Memristor resistance versus $\alpha$ and time, and b) Memristor resistance versus time for different $\alpha$

The general formula of the saturation time $t_{sat}$ in the fractional-order case at which the memristor resistance increases from its initial value $R_{in}$ up to $R_{off}$ is given by:

$$t_{sat} = \left(\frac{R_{off}^{\alpha+1} - R_{in}^{\alpha+1}}{(\alpha+1)kR_d|V_{DC}|}\right)^{\frac{1}{\alpha}} \tag{18}$$

The maximum saturation time can be obtained when $R_{in} = R_{on}$ as follows:

$$t_{sat}|_{max} = \left(\frac{R_{off}^{\alpha+1} - R_{on}^{\alpha+1}}{(\alpha+1)kR_d|V_{DC}|}\right)^{\frac{1}{\alpha}} \tag{19}$$

For the conventional model of the memristor $\alpha = 1$ the saturation time will be reduced to the formula given in [21]-[22].

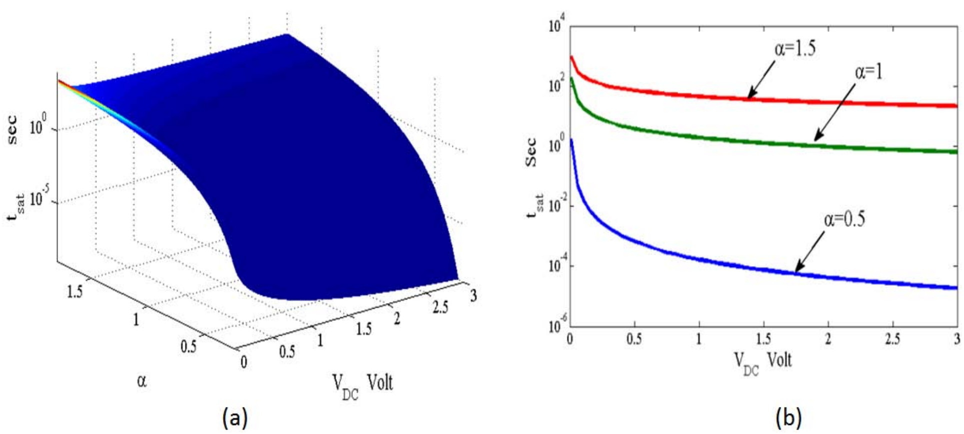


FIGURE 3. a) Saturation time versus $\alpha$ and $V_{DC}$, and b) Saturation time versus $V_{DC}$ for different $\alpha$

The saturation time surface as a function of the $\alpha - V_{DC}$ plane and three different cases of $\alpha = 0.5, 1$ and 1.5 are shown in Fig. 3(a) and Fig. 3(b) respectively. It is clear from the above response that the saturation time can be controlled through the fractional-order where it can be less than 1 Sec when $\alpha < 0.5$ up to higher values when $\alpha > 0.5$. It is worthy to note that, the memristor will act as a linear resistor as $\alpha$ tends to 0 with resistance $R_{in}$.

3.2. **Same polarity configuration.** This case discusses the memristor behaviour when the positive of the supply voltage is connected to the positive of the memristor as shown in Fig.1(b). Then the resistance of memristor is given by

$$R_m = \left(R_{in}^{\alpha+1} - (\alpha+1)kR_dV_{DC}t^{\alpha}\right)^{\frac{1}{\alpha+1}} \tag{20}$$

The resistance of the memristor decreases from the initial value $R_{in}$ until it reaches its minimum $R_{on}$ through the saturation time $t_{sat}$ which is given by:

$$t_{sat} = \left(\frac{R_{in}^{\alpha+1} - R_{on}^{\alpha+1}}{(\alpha+1)kR_d|V_{DC}|}\right)^{\frac{1}{\alpha}} \tag{21}$$

The maximum saturation time when $R_{in} = R_{off}$ can be obtained by:

$$t_{sat}|_{max} = \left(\frac{R_{off}^{\alpha+1} - R_{on}^{\alpha+1}}{(\alpha+1)kR_d|V_{DC}|}\right)^{\frac{1}{\alpha}} \quad (22)$$

This is similar to the previous case given by equation (19). Figure 4 shows the saturation time surface and curves for three different cases. For the conventional case $\alpha = 1$, the saturation time is given by the same relation in [21] which also matches the results in [12].

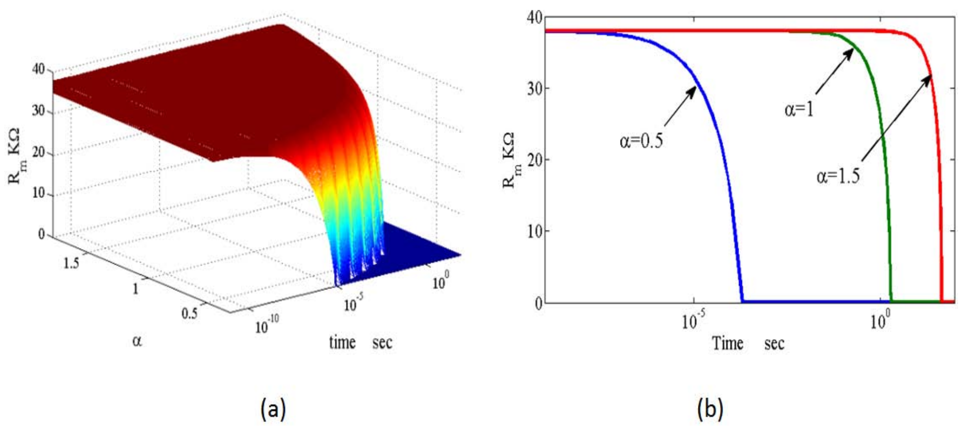


FIGURE 4. a) memristor resistance versus $\alpha$ and time, and b) memristor resistance versus time for different $\alpha$

## 4. Conclusion

This paper introduces the analysis of the fractional order memristor state equation for step input voltage supply. The generalized formulas of the memristor's resistance and the saturation time for the two different cases of memristor polarities are derived. The maximum saturation time which is needed for digital circuit design is also introduced. Moreover, it is clear from the above discussion that the fractional-order parameter can be used to control the saturation time from a part of a second up to several minutes under the same input voltage.

## References

[1] Y. Q. Chen, I. Petras, D. Xue, Fractional order control-A tutorial, proc. of the American Control Conference, 1397-1411, 2009.
[2] L. Dorcak, I. Petras, I. Kostial, J. Terpak, Fractional-order state space models, Proc. of the International Carpathian Control Conference, 193-198, 2002.
[3] R. Caponette, G. Dongola, L. Fortuna, I. Petras, Fractional order systems: Modeling and Control Applications, Singapore: World Scientific, 2010.
[4] A. G. Radwan, A. S. Elwakil, A. M. Soliman, Fractional-Order Sinusoidal Oscillators: Design: Procedure and Practical Examples, IEEE Trans. Circuits Syst. I, Vol. 55, No. 7, 20512063, 2008.
[5] A. G. Radwan, A. M. Soliman, and A. S. Elwakil, Design equations for fractional-order sinusoidal oscillators: Practical circuit examples, International Conference on Microelectronics(ICM), 91-94, 2007.
[6] A. G. Radwan, and A. M. Soliman, A. S. Elwakil, First-order filters generalized to the fractional domain, Journal of Circuits Systems Computers, 17, 55-66, 2008.

[7] A. G. Radwan, A. Shamim, K. N. Salama, Theory of fractional order elements based impedance matching networks, IEEE Microwave Wireless Components Letters, 21, 3, 120-122, 2011.
[8] M. S. Krishna, S. Das, K. Biswas, B. Goswami, Fabrication of a Fractional Order Capacitor With Desired Specifications: A Study on Process Identification and Characterization, IEEE Trans. Electron Devices, Vol. 58, No. 11, 4067-4073, 2011.
[9] L. O. Chau, Memristor-the missing circuit element, IEEE Trans. Circuit Theory, Vol. 18, 507519, 1971.
[10] L.O. Chua, S.M. Kang, Memristive devices and systems, Proc. Of the IEEE, Vol. 64, No. 2, 209-223, 1976.
[11] I. Petras, Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulations, Springer, 2010.
[12] D. B. Strukov, G. S. Snider, D. R. Stewart, R. S. Williams, The missing memristor found, Nature, Vol. 453, 8083, 2008.
[13] R. S. Williams, How we found the missing memristor, IEEE Spectrum, 1-11, 2008.
[14] E. Lehtonen, M. Laiho, Stateful Implication Logic with Memristors, Symposium A Quarterly Journal In Modern Foreign Lit., 33-36, 2009.
[15] K. Bickerstaff, E. Swartzlander, Memristor-based Arithmetic, ASILOMAR, 1173-1177, 2010.
[16] S. Shin, K. Kim, S. M. Kang, Memristive XOR for resistive multiplier, Electronics Letters , Vol. 48, No. 2, 78-80, 2012.
[17] A. Talukdar, A. G. Radwan, K. N. Salama, Time domain oscillating poles: stability rede ned in memristor based Wien-oscillator, The 22nd International Conference on Microelectronics (ICM), 288-291, 2010.
[18] M. Z. Affan, H. Omran, A. G. Radwan, K. N. Salama, Memristor-based reactance-less oscillator, Electronics Letters , Vol. 47, No.22, 1220-1221, 2011.
[19] Y. V. pershin, M. D. Ventra, Practical approach to programmable analog circuits with memristors, ASILOMAR, Vol. 57, No.8, 1857-1864, 2010.
[20] K. S. Miller, B. Ross, An introduction to fractional calculus and fractional differential equations. New York: Wiley, 1993.
[21] A. G. Radwan, M. A. Zidan, K. N. Salama, on the mathematical modeling of memristors, ICM, 284287, 2010.
[22] A. G. Radwan, M. A. Zidan and K. N. Salama,HP memristor mathematical model for periodic signals and DC, 53rd IEEE International Midwest Symposium on Circuits and Systems, USA,861-864, 2010.

M. E. Fouda, A. G. Radwan
Engineering Mathematics Department, Faculty of Engineering, Cairo University, Giza, 12613, Egypt.
Nanoelectronics Integrated Systems Center (NISC), Nile University, Cairo, Egypt.
*E-mail address*: m_elneanaei@ieee.org, agradwan@ieee.org