A Perishable Inventory System with Repeated Customers and Server Interruptions

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Abstract: In this article, we present a continuous review perishable inventory system with a service facility. The service facility consists of a single server and a finite waiting room. If the arriving primary customer finds the waiting hall is empty and the server is idle, he immediately joins the service. The demanded items are issued by a server after some random time due to service on it. We assume that the service may interrupted due to some physical phenomena and the service resumes after repair. An arriving primary customer finds the waiting room is full is permitted to enter into orbit otherwise the customer waits for his service in the waiting hall. The customers in the orbit are called repeated customers and they retry for their service after some random time. The inventory is replenished according to an (s, S) ordering policy. The joint probability distribution of the number of customers in the waiting area, the number of customers in the orbit and the inventory level is obtained for the steady state case. Some important system performance measures in the steady state are derived. Several numerical examples are presented to illustrate the effect of the system parameters.

Keywords: Continuous review inventory system, Perishable item, Service facility, (s, S) policy, Repeated customer, Service interruption, Repair.

1 Introduction

Inventory models considered in the literature, the demanded items are directly issued from the stock, if available. The demands that occurred during stock out period are either not satisfied (lost sales case) or satisfied only after the replenishment of the ordered items (backlog case). Latter it is assumed either all (full backlog case) or only a fixed number of demands (partial backlogging) that occurred during stock out period are satisfied. The case of inventories maintained at service facilities, the demanded items are issued to the customers only after some service is performed on it. In this situation the items are issued not immediately to the demand but after a random time of service. This forces the formation of queues in this type of models. This necessitates the study of both the inventory level and the queue length joint distributions. Study of such models is beneficial to organizations which

- Provide service to customers by using items from a stock.
- Maintain stock of items each of which needs service such as assembly or initialization or installation, etc.

Examples for the first type include firms that are engaged in servicing consumer products such as Television sets, Computers, etc., and for the second type include firms that supply bicycles which need assembly of its parts, that supply food items which need heating or garnishing and that computers which need installation of basic services.

The concept of retrial demands in inventory was introduced by Artalejo et al. [1]. They have assumed Poisson demand, exponential lead time and exponential retrial time. In that work, the authors proceeded with an algorithmic analysis of the system. Ushakumari [16] considered a retrial inventory system with classical retrial policy. As a variant, we consider a continuous review retrial inventory system with server interruptions in this paper. For a brief review of retrial queues, see Artalejo ([2], [3]), Artalejo et al. [4], Artalejo et al. [5], Falin [8] and Falin and Templeton [9].

Krishnamoorthy and Anbazhagan [11] analyzed a perishable queueing-inventory system with N policy, Poisson arrivals, exponential distributed lead times and service times. The joint probability distributions of the number of customers in the system and the inventory

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level were obtained in the steady state case. Two other papers where an inventory model with service time is considered, are by Krishnamoorthy et al. ([12], [13]). The article [12] is considered for an inventory model with instantaneous replenishment and the service process is subject to interruptions. The discussion in [13] is an inventory model with positive lead time, server interruptions and an orbit of infinite capacity, where no waiting space is provided for customers, other than for the one whose service gets interrupted.

In this paper, we consider a continuous review \((s,S)\) retrial inventory system with server interruptions, in which the server provides \(K\) types of heterogeneous service and each arriving customer has the option of choosing either type of service. The joint probability distribution of the number of customers in the waiting hall, number of customers in the orbit and the inventory level is obtained in the steady state case. Various system performance measures are derived and the total expected cost rate is calculated.

The rest of the paper is organized as follows. In the next section, we describe the mathematical model and the notations used in this paper are defined. Analysis of the model and the steady state solution of the model are dealt with in section 3. Some key system performance measures are derived in section 4. In section 5, we calculate the total expected cost rate. In section 6, we provide some interesting numerical examples. The last section is meant for conclusion.

2 Mathematical model

We consider a continuous review inventory system with a stock of maximum \(S\) units. The system consists of a single server and a finite waiting hall of size \(N\). The primary customers arrive according to a Poisson process with parameter \(\lambda(> 0)\). The items are issued by a server to the customer after some service time due to the service performed on the items. The service time follows a negative exponential distribution with parameter \(\mu\). An arriving primary customer finds the waiting hall is empty and the server is idle, he immediately joins the service. We assume \(K\) types of services are available at service facility. The customer chooses type \(i\) service with probability \(p_i, i = 1, 2, \ldots, K\) and \(\sum_{i=1}^{K} p_i = 1\). Any arriving customer, who finds the waiting room is full, are permitted to enter into orbit of finite size \(M\). The customers in the orbit are called repeated customers. They retry for their service after some random time. We assume the time between two successive retrials is an exponential random variable with parameter \(\theta\). In this article, we assumes the classical retrial policy. That is, the rate of retrial, when the repeated attempt in an interval \((t, t + dt)\) given that there are \(i\) customers in the orbit at time \(t\) is \(i \theta + o(dt)\).

While the server serves a customer, the service may get interrupted with the interruption process governed by a Poisson process with parameter \(\nu_1\). It is assumed that the server is under interruption, no further interruption can befall the server. On completion of an interruption the service resumes, with the duration of an interruption exponentially distributed with parameter \(\nu_2\). The demanding customers receive their service one by one and they demand a single item. The operating policy is \((s,S)\) ordering policy. According to that, an order for \(Q(= S - s > s + 1)\) items are placed whenever the inventory level drops to \(s\) and the items are received only after a random time which has exponential distribution with parameter \(\beta(> 0)\). The life time of each item has negative exponential distribution with parameter \(\gamma(> 0)\).

We have assumed that an item cannot be perished while in service. The customer finds both the waiting hall and the orbit full, is considered to be lost. Various stochastic processes involved in the system are independent.

2.1 Notations

\[
\begin{align*}
0 & \text{: Zero matrix} \\
[A]_{ij} & \text{: entry at } (i,j)^{th} \text{ position of a matrix A} \\
H(x) & \text{: } \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{otherwise}. \end{cases} \\
\delta_j & \text{: } \begin{cases} 1, & \text{if } j = i \\ 0, & \text{otherwise}. \end{cases} \\
\bar{\delta}_j & \text{: } 1 - \delta_j \\
k \in V^i_j & \text{: } k = i, i+1, \ldots, j \\
Y(t) & \text{: } \begin{cases} 0, & \text{if the server is idle at time } t \\ 1, & \text{if the server is busy at time } t \\ 2, & \text{if the server is on interruption at time } t \end{cases} \\
\Omega & \text{: } \begin{cases} c_j c_{j-1} \cdots c_k & \text{if } j \geq k \\ 1 & \text{if } j < k \end{cases} \\
\mathbf{e}^T & \text{: } (1, 1, \ldots, 1)
\end{align*}
\]

3 Analysis

Let \(L(t), Y(t), X(t)\) and \(Z(t)\) respectively, denote the inventory level, the server status, the number of customers (waiting and being served) in the waiting room and the number of customers in the orbit at time \(t\). From the assumptions made on the input and output processes, it can be shown that the stochastic process \(\{(L(t), Y(t), X(t), Z(t)), t \geq 0\}\) is a continuous time Markov chain with state space given by \(E\).
where \( E = E_1 \cup E_2 \cup E_3 \) with
\[
E_1 : \{ (0, 0, i_3, i_4) \mid i_3 = 0, 1, 2, \ldots, N, i_4 = 0, 1, 2, \ldots, M \},
\]
\[
E_2 : \{ (i_1, 0, i_4) \mid i_1 = 1, 2, \ldots, S, i_4 = 0, 1, 2, \ldots, M \},
\]
\[
E_3 : \{ (i_1, i_2, i_3, i_4) \mid i_1 = 1, 2, \ldots, S, i_2 = 1, 2, i_3 = 1, 2, \ldots, N, i_4 = 0, 1, 2, \ldots, M \}.
\]

The infinitesimal generator of this process is
\[
\Theta = \{ a((i_1, i_2, i_3, i_4); (j_1, j_2, j_3, j_4)) \mid (i_1, i_2, i_3, i_4), (j_1, j_2, j_3, j_4) \in E \}
\]

The infinitesimal generator \( \Theta \) can be written in terms of sub matrices \( \Theta_{i_1j_1} \), namely,
\[
\Theta = (\Theta_{i_1j_1}),
\]
where
\[
\Theta_{i_1j_1} = \begin{pmatrix}
A_{i_1} & j_1 = i_1, & i_1 = 0, 1, 2, \ldots, S \\
B_{i_1} & j_1 = i_1 - 1, & i_1 = 1, 2, \ldots, S - 1, S \\
C & j_1 = i_1 + 0, & i_1 = 1, 2, \ldots, s, \\
0 & \text{Otherwise.}
\end{pmatrix}
\]

More explicitly,
\[
\Theta = \begin{pmatrix}
A_S & B_S & A_{S-1} & B_{S-1} & \ldots & A_1 & B_1 \\
C & A_S & B_S & A_{S-1} & \ldots & C & A_1 \\
S-1 & C & A_S & B_S & A_{S-1} & \ldots & C & A_1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & \ldots & C & A_1 & A_0
\end{pmatrix}
\]

where
\[
[C^{(0)}]_{i_2j_2} = \begin{cases}
W^{(0)} & j_3 = i_3, \quad i_3 \in V_1^N \\
0 & \text{otherwise.}
\end{cases}
\]
\[
[C^{(1)}]_{i_2j_2} = \begin{cases}
C^{(0)} & j_2 = 0, \quad i_2 = 1 \\
C^{(0)} & j_2 = 1, \quad i_2 = 0 \\
0 & \text{otherwise.}
\end{cases}
\]
\[
[C^{(2)}]_{i_2j_2} = \begin{cases}
W^{(0)} & j_3 = 0, \quad i_3 = 0 \\
0 & \text{otherwise.}
\end{cases}
\]
\[
[C^{(1)}]_{i_2j_2} = \begin{cases}
C^{(0)} & j_2 = 0, \quad i_2 = 0 \\
C^{(0)} & j_2 = 1, \quad i_2 = 1 \\
0 & \text{otherwise.}
\end{cases}
\]
\[
[C^{(2)}]_{i_2j_2} = \begin{cases}
W^{(0)} & j_3 = i_3, \quad i_3 \in V_1^N \\
0 & \text{otherwise.}
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\[
[C^{(2)}]_{i_2j_2} = \begin{cases}
W^{(0)} & j_3 = i_3, \quad i_3 \in V_1^N \\
0 & \text{otherwise.}
\end{cases}
\]
\[
[C^{(2)}]_{i_2j_2} = \begin{cases}
W^{(0)} & j_3 = 0, \quad i_3 = 0 \\
0 & \text{otherwise.}
\end{cases}
\]
\[ [T_{i,i-1}]_{i,i} = \begin{cases} (i_1 - 1)\gamma, & j_4 = i_4, \quad i_4 \in V_0^M \\ 0, & \text{otherwise.} \end{cases} \]

\[ [A_0]_{i,i} = \begin{cases} A_0^{(i)} j_2 = 0, & i_2 = 0 \\ 0, & \text{otherwise.} \end{cases} \]

\[ [A_0^{(0)}]_{i,i} = \begin{cases} C_3 j_3 = i_3 + 1, & i_3 \in V_0^{N-1} \\ C_4 j_3 = i_3, & i_3 \in V_0^N \\ C_5 j_3 = i_3, & i_3 = N \\ 0, & \text{otherwise.} \end{cases} \]

\[ [C_0]_{i,i} = \begin{cases} \lambda, & j_4 = i_4, \quad i_4 \in V_0^M \\ -\delta m \lambda + (i_1 - 1)\gamma, \quad j_4 = i_4 + 1, i_4 \in V_0^{M-1} \\ 0, & \text{otherwise.} \end{cases} \]

\[ [C_1]_{i,i} = \begin{cases} \lambda, & j_4 = i_4, \quad i_4 \in V_0^M \\ -\delta m \lambda + (i_1 - 1)\gamma, \quad j_4 = i_4 + 1, i_4 \in V_0^{M-1} \\ 0, & \text{otherwise.} \end{cases} \]

For \( i_1 = 1, 2, 3, \ldots, S \),

\[ [A_i]_{i,i} = \begin{cases} A_0^{(i)} j_2 = 0, & i_2 = 0 \\ A_0^{(i)} j_2 = 1, & i_2 = 0 \\ A_1^{(i)} j_2 = 1, & i_2 = 1 \\ A_2^{(i)} j_2 = 2, & i_2 = 1 \\ A_2^{(i)} j_2 = 2, & i_2 = 2 \\ 0, & \text{otherwise.} \end{cases} \]

\[ [D^{(i)}]_{i,i} = \begin{cases} -\lambda + i_1 \gamma + i_1 \theta + \beta, & j_4 = i_4, i_4 \in V_0^M \\ 0, & \text{otherwise.} \end{cases} \]

\[ [C]_{i,i} = \begin{cases} C_3 j_3 = i_3 + 1, & i_3 \in V_0^{N-1} \\ C_4 j_3 = i_3, & i_3 \in V_0^N \\ C_5 j_3 = i_3, & i_3 = N \\ 0, & \text{otherwise.} \end{cases} \]

\[ [A_0^{(i)}]_{i,i} = \begin{cases} D^{(i)} j_3 = i_3, & i_3 = 0 \\ 0, & \text{otherwise.} \end{cases} \]

\[ [D^{(i)}]_{i,i} = \begin{cases} -\lambda + i_1 \gamma + i_1 \theta + \beta, & j_4 = i_4, i_4 \in V_0^M \\ 0, & \text{otherwise.} \end{cases} \]

\[ [A_0^{(10)}]_{i,i} = \begin{cases} C_6 j_3 = i_3, & i_3 = 0 \\ 0, & \text{otherwise.} \end{cases} \]

\[ [C_0^{(i)}]_{i,i} = \begin{cases} \lambda, & j_4 = i_4, \quad i_4 \in V_0^M \\ i_4 \theta + j_4 = i_4 - 1, i_4 \in V_0^M \\ 0, & \text{otherwise.} \end{cases} \]

\[ [C_3]_{i,i} = \begin{cases} C_3 j_3 = i_3 + 1, & i_3 \in V_0^{N-1} \\ E^{(i)} j_3 = i_3, & i_3 \in V_0^{N-1} \\ G^{(i)} j_3 = i_3, & i_3 = N \\ 0, & \text{otherwise.} \end{cases} \]

\[ [E^{(i)}]_{i,i} = \begin{cases} -\lambda + (i_1 - 1)\gamma + H(s - i)\beta + v_1 + \mu, & j_4 = i_4, i_4 \in V_0^M \\ 0, & \text{otherwise.} \end{cases} \]

It may be noted that the matrices \( A_i, i_1 = 1, 2, \ldots, S, \) \( C \) and \( B_i, i_1 = 2, 3, \ldots, S, B_1, B_2 \) and \( C \) are square matrices of size \((2N+1)(M+1)\). \( C_0^{(i)}, W_0^{(i)} \), \( C_0^{(10)} \), \( C_2, C_3, C_4, C_5, C_6, C_7, C_8, B_0^{(0)}, W_0^{(0)}, D^{(i)} \), \( A_0^{(i)}, E^{(i)} \), \( G^{(i)} \), \( H^{(i)} \), \( J^{(i)} \), \( i_1 = 1, 2, \ldots, S \) and \( T_{i_1-1} \), \( i_1 = 1, 2, \ldots, S \) are square matrices of size \( (M+1) \). \( C_0, B_1, B_1, A_1, C_1, C_2, A_1, A_2, C_2, C_2, A_1, A_2, B_1^{(10)} \), and \( B_2^{(1)} \), \( i_1 = 1, 2, \ldots, S \) are square matrices of size \( N(M+1) \). \( A_0 \) and \( A_0^{(0)} \) are square matrices of size \((N+1)(M+1)\). \( B_1 \), \( C_1 \), \( C_1^{(i)} \), \( B_1^{(10)} \) and \( A_0^{(1)} \) are matrices of size \((2N+1)(M+1) \times (N+1)(M+1)\), \((N+1)(M+1) \times (2N+1)(M+1)\), \((N+1)(M+1) \times N(M+1)\), \( N(M+1) \times M+1 \) and \( M+1 \times N(M+1) \) respectively.

### 3.1 Steady state analysis

It can be seen from the structure of \( \Theta \) that the homogeneous Markov process \( \{L(t), Y(t), X(t), Z(t)\}, t \geq 0 \) on the finite space \( E \) is
irreducible. Hence the limiting distribution
\[
\phi(i_1,i_2,i_3,i_4) = \lim_{t \to \infty} P_r[L(t) = i_1, Y(t) = i_2, X(t) = i_3, Z(t) = i_4|L(0), Y(0), X(0), Z(0)],
\]
exists. Let
\[
\Phi = (\phi(0), \phi(1), \ldots, \phi(S))
\]
partitioning the vector, \(\phi(i_1)\) as follows:
\[
\phi(0) = (\phi(0,0,0), \phi(0,0,1), \ldots, \phi(0,0,N)), \\
\phi(1) = (\phi(1,0,0), \phi(1,0,1), \phi(1,0,2)), \quad i_1 = 1, 2, \ldots, S; \\
\phi(i_1) = (\phi(i_1,0,0), \phi(i_1,0,1), \phi(i_1,0,2), \ldots, \phi(i_1,0,M)), \quad i_1 = 1, 2, \ldots, S; \\
\phi(i_1,i_2) = (\phi(i_1,i_2,0), \phi(i_1,i_2,1), \ldots, \phi(i_1,i_2,M)), \quad i_1 = 1, 2, \ldots, S; \\
\phi(i_1,i_2,i_3) = (\phi(i_1,i_2,i_3,0), \phi(i_1,i_2,i_3,1), \ldots, \phi(i_1,i_2,i_3,M)), \quad i_1 = 1, 2, \ldots, S; \\
\phi(i_1,i_2,i_3,i_4) = (\phi(i_1,i_2,i_3,i_4,0), \phi(i_1,i_2,i_3,i_4,1), \ldots, \phi(i_1,i_2,i_3,i_4,M)), \quad i_1 = 1, 2, \ldots, S,
\]
where \(\phi(i_1,i_2,i_3,i_4)\) denotes the steady state probability for the state \((i_1,i_2,i_3,i_4)\) of the process, exists and is given by
\[
\Phi \Theta = 0 \quad \text{and} \quad \sum_{i_1,i_2,i_3} \phi(i_1,i_2,i_3,i_4) = 1. \tag{1}
\]

The first equation of the above yields the following set of equations:
\[
\phi(i_1)B_1 + \phi(i_1-1)A_{i_1-1} = 0, \quad i_1 = 1, 2, \ldots, Q, \\
\phi(i_1)B_1 + \phi(i_1-1)A_{i_1-1} + \phi(i_1-1-0)C_1 = 0, \quad i_1 = Q + 1, \quad (*) \\
\phi(i_1)B_1 + \phi(i_1-1)A_{i_1-1} + \phi(i_1-1-Q)C_1 = 0, \quad i_1 = Q + 2, Q + 3, \ldots, S, \\
\phi(S)A_S + \phi(S)C = 0.
\]

After lengthy simplifications, the above equations, except (*), yields
\[
\phi(i_1) = (-1)^{Q-i_1} \phi(Q-i_1+1) \Omega^1 B_iA_{i_1-1}, \\
i_1 = Q - 1, Q - 2, \ldots, 0 \\
= (-1)^{Q-i_1+1} \phi(Q) \times \\
\sum_{j=0}^{S-i_1} \left[ \left( \Omega^{i_1-j} B_iA_{i_1-1} \right) C_A^{-1} \left( \Omega^{Q-i_1+1-j} B_iA_{i_1-1} \right) \right], \\
i_1 = S, S - 1, \ldots, Q + 1
\]
where \(\phi(Q)\) can be obtained by solving,
\[
\phi(Q+1)B_{Q+1} + \phi(Q)A_Q + \phi(0)C_1 = 0 \quad \text{and} \quad \sum_{i_1=0}^{S} \phi(i_1)e = 1,
\]
\[
\lambda = 1.2, \beta = 2, \gamma = 0.3, \theta = 3, \mu = 2, \nu_1 = 1, \nu_2 = 0.5, \quad p_1 = 0.2, p_2 = 0.5, p_3 = 0.3, K = 3, N = 6, M = 4, \quad c_k = 0.01, c_s = 50, c_p = 0.5, c_f = 0.3, c_l = 2, c_t = 6, \quad c_r = 2, c_w = 0.4
\]

\[\text{Fig. 1: A three dimensional plot of the cost function} \quad TC(S,s,N,M)\]

\[
\text{where} \quad \phi(Q) \left[ (-1)^Q \sum_{j=0}^{Q} \left( \Omega^{i_1-j} B_iA_{i_1-1} \right) C_A^{-1} \right] \times \\
B_{Q+1} + A_Q + \left( (-1)^Q \sum_{j=0}^{Q} \left( \Omega^{i_1-j} B_iA_{i_1-1} \right) C \right] = 0,
\]

and
\[
\phi(Q) \left[ \sum_{i_1=0}^{Q} \left( (-1)^Q \Omega^{i_1} B_iA_{i_1-1} \right) + I \right] + \\
\sum_{i_1=0}^{Q} \sum_{j=0}^{S-i_1} \left[ (-1)^{Q-i_1+1-j} \Omega^{i_1-j} B_iA_{i_1-1} \right] C \times \\
A_{S-j}^{-1} \left( \Omega^{i_1-j} \Omega^{Q-i_1+1-j} B_iA_{i_1-1} \right) \right] e = 1.
\]

\[\text{4 System performance measures}\]

In this section some performance measures of the system under consideration in the steady state are derived.
γ = 0.3, θ = 3, μ = 2, v1 = 1, v2 = 0.5, p1 = 0.2, p2 = 0.5, p3 = 0.3, K = 3, N = 6, M = 4, c6 = 0.01, c3 = 50, cP = 0.5, c1 = 0.3, c0 = 2, c1 = 6, c7 = 2, cω = 0.4

Fig. 2: Variation of β vs λ on the cost function

4.1 Expected inventory level

Let ηIL denote the average inventory level in the steady state. Then

\[ η_{IL} = \sum_{i_1=1}^{S} \sum_{i_4=0}^{M} i_1 \phi^{(i_1,0,i_4)} + \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \sum_{i_4=0}^{M} i_1 \phi^{(i_1,i_3,i_4)} \]

4.2 Expected reorder rate

Let ηRR denote the expected reorder rate in the steady state. Then

\[ η_{RR} = \sum_{i_1=0}^{S} (s+1)y_0^{(s+1,0,i_1)} + \sum_{i_2=1}^{S} \sum_{i_1=1}^{S} \sum_{i_4=0}^{M} s y_0^{(s+1,i_2,i_4)} + \sum_{i_4=0}^{M} \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \sum_{i_4=0}^{M} (p,μ) y_0^{(s+1,1,i_3,i_4)} \]

4.3 Expected perishable rate

Let ηPR denote the expected perishable rate in the steady state. Then

\[ η_{PR} = \sum_{i_1=1}^{S} \sum_{i_4=0}^{M} i_1 y_0^{(i_1,0,i_4)} + \sum_{i_1=1}^{S} \sum_{i_2=1}^{S} \sum_{i_3=1}^{N} \sum_{i_4=0}^{M} (i_1 - 1) y_0^{(i_1,i_2,i_3,i_4)} \]

4.4 Expected interruption rate

Let ηINTR denote the effective interruption rate in the steady state. Then

\[ η_{INTR} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \sum_{i_4=0}^{M} v_1 \phi^{(i_1,1,i_3,i_4)} \]

4.5 Expected repair rate

Let ηREP denote the effective repair rate is given by

\[ η_{REP} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \sum_{i_4=0}^{M} v_2 \phi^{(i_1,2,i_3,i_4)} \]

Fig. 3: Variation of β vs μ on the cost function

4.6 Expected number of customers lost

Let ηLL denote the expected number of customers lost in the steady-state. Then

\[ η_{LL} = \lambda \phi^{(0,0,N,M)} + \sum_{i_1=1}^{S} \sum_{i_2=1}^{S} \lambda \phi^{(i_1,i_2,N,M)} \]

4.7 Expected number of customers in the orbit

Let ηOO denote the expected number of customers in the orbit. Then

\[ η_{OO} = \sum_{i_3=0}^{N} \sum_{i_4=1}^{M} i_3 \phi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{S} \sum_{i_4=1}^{M} i_1 \phi^{(i_1,0,0,i_4)} + \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \sum_{i_4=1}^{M} i_3 \phi^{(i_1,0,i_3,i_4)} \]
4.8 Expected waiting time in the waiting room

Let $\eta_{WW}$ denote the expected waiting time of the primary customers in the waiting room. Then

$$\eta_{WW} = \frac{\Gamma}{\eta_{AR}},$$

where

$$\Gamma = \sum_{i_3=0}^{N-1} \sum_{i_4=0}^{M} \lambda \phi(i_1,0,i_3,i_4) + \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_4=0}^{M} \lambda \phi(i_1,1,i_2,i_4)$$

and the effective arrival rate (Ross [15]), $\eta_{AR}$ is given by

$$\eta_{AR} = \sum_{i_3=0}^{N-1} \sum_{i_4=0}^{M} \lambda \phi(0,0,i_3,i_4) + \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_4=0}^{M} \lambda \phi(i_1,0,0,i_4)$$

$$+ \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_4=0}^{M} \lambda \phi(i_1,1,1,i_4)$$

4.9 Probability that server is busy

Let $P_B$ denote probability that server is busy is given by

$$P_B = \sum_{i_1=1}^{S} \sum_{i_2=1}^{N} \sum_{i_4=0}^{M} \phi(i_1,1,i_2,i_4)$$

4.10 Probability that server is idle

Let $P_{ID}$ denote probability that server is idle is given by

$$P_{ID} = \sum_{i_1=1}^{S} \sum_{i_2=1}^{N} \sum_{i_4=0}^{M} \phi(i_1,0,0,i_4) + \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_3=0}^{M} \phi(i_1,0,i_3,i_4)$$

4.11 Probability that server is on interruption

Let $P_{IN}$ denote probability that server is on interruption is given by

$$P_{IN} = \sum_{i_1=1}^{S} \sum_{i_2=1}^{N} \sum_{i_3=0}^{M} \phi(i_1,2,i_2,i_3)$$

4.12 The overall rate of retrials

The overall rate of retrials at which the orbiting customers request his demand is given by

$$\eta_{RT} = \sum_{i_3=0}^{N-1} \sum_{i_4=1}^{M} (i_4)\phi(i_3,0,i_3,i_4) + \sum_{i_1=1}^{S} \sum_{i_4=1}^{M} (i_4)\phi(i_1,0,0,i_4)$$

$$+ \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_3=0}^{M} (i_4)\phi(i_1,i_2,i_3,i_4)$$

4.13 The successful retrial rate

The rate at which the orbiting customers successfully receive his demands is given by

$$\eta_{SR} = \sum_{i_1=1}^{S} \sum_{i_4=1}^{M} (i_4)\pi(i_1,0,0,i_4)$$

4.14 Fraction of successful rate of retrials

The fraction of successful rate of retrials is given by

$$\eta_{FR} = \frac{\eta_{SR}}{\eta_{OR}}$$
The long run total expected cost rate is given by

$$TC(S,s,N) = \sum_{i=1}^{M} c_s \sum_{i=1}^{N} s \gamma \phi^{(s+1,i,1,i_1)} + \sum_{i_1=1}^{N} s \gamma \phi^{(s+1,i_2,i_3,i_4)} +$$

Substituting the values of $\eta$’s we get $TC(S,s,N,M)=$

$$c_s \sum_{i=1}^{M} i \phi^{(i+1,0,0,i_1)} + c_s \sum_{i=1}^{M} s \gamma \phi^{(s+1,i_2,i_3,i_4)} +$$

$$c_s \sum_{i_1=1}^{N} \sum_{i_1=1}^{N} i_2 \phi^{(i_2,i_3,i_4)} + c_s \sum_{i_1=1}^{N} \sum_{i_1=1}^{N} i_4 \phi^{(i_2,i_3,i_4)} +$$

$$c_s \sum_{i_1=1}^{N} \sum_{i_1=1}^{N} i_2 \phi^{(i_2,i_3,i_4)} + c_s \sum_{i_1=1}^{N} \sum_{i_1=1}^{N} i_4 \phi^{(i_2,i_3,i_4)} +$$

The long run total expected cost rate is given by

$$TC(S,s,N) = c_s \eta_{IL} + c_i \eta_{RR} + c_p \eta_{PR} + c_w \eta_{WW} + c_o \eta_{OO} + c_f \eta_{LL} + c_r \eta_{INT} + c_r \eta_{REP}$$

5 Cost analysis

To compute the total expected cost per unit time (total expected cost rate), the following costs are considered,

- $c_b$: The inventory carrying cost per unit item per unit time
- $c_p$: Set up cost per unit item per unit time
- $c_f$: Failure cost per unit item per unit time
- $c_w$: Waiting time cost of a primary customer per unit time
- $c_o$: Waiting time cost of a orbiting customer per unit time
- $c_l$: Cost due to loss of customers per unit per unit time
- $c_r$: Cost per interruption per unit time,

$\lambda = 1.2, \gamma = 0.3, \theta = 3, \nu_1 = 1, \nu_2 = 0.5, p_1 = 0.2, p_2 = 0.5, p_3 = 0.3, K = 3, N = 6, M = 4, c_h = 0.01, c_r = 50, c_p = 0.5,$

$\lambda_1 = 0.3, c_0 = 2, c_6 = 6, c_r = 2, c_w = 0.4$

Fig. 6: Variation of $\beta$ vs $\mu$ on $\eta_{FR}$

$\beta = 2, \gamma = 0.3, \theta = 3, \mu = 2, \nu_1 = 1, \nu_2 = 0.5,$

$p_1 = 0.2, p_2 = 0.5, p_3 = 0.3, K = 3, N = 6, c_h = 0.01,$

$c_r = 50, c_p = 0.5, c_2 = 0.3, c_0 = 2, c_6 = 6, c_r = 2, c_w = 0.4$

Fig. 8: Variation of $\lambda$ vs $M$ on $\eta_{OO}$

$\lambda = 1.2, \beta = 2, \gamma = 0.3, \theta = 3, \nu_1 = 1, \nu_2 = 0.5,$

$p_1 = 0.2, p_2 = 0.5, p_3 = 0.3, K = 3, N = 6, c_h = 0.01,$

$c_r = 50, c_p = 0.5, c_2 = 0.3, c_0 = 2, c_6 = 6, c_r = 2, c_w = 0.4$

Fig. 9: Variation of $\mu$ vs $M$ on $\eta_{OO}$
Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost function analytically. Hence, a detailed computational study of the cost function is carried out numerically.

\[ \beta = 2, \gamma = 0.3, \theta = 3, v_1 = 1, v_2 = 0.5, p_1 = 0.2, p_2 = 0.5, p_3 = 0.3, K = 3, N = 6, M = 4, c_h = 0.01, c_f = 0.5, c_t = 0.3, c_0 = 2, c_1 = 6, c_r = 2, c_w = 0.4 \]

**Fig. 10:** Variation of \( \mu \) vs \( \lambda \) on \( \eta_{LL} \)

\[ \lambda = 1.2, \beta = 2, \gamma = 0.3, \theta = 3, \mu = 2, v_1 = 1, v_2 = 0.5, p_1 = 0.2, p_2 = 0.5, p_3 = 0.3, K = 3, N = 6, M = 4, c_h = 0.01, c_f = 0.5, c_t = 0.3, c_0 = 2, c_1 = 6, c_r = 2, c_w = 0.4 \]

**Fig. 12:** Variation of \( \lambda \) vs \( N \) on \( \eta_{WW} \)

### 6 Numerical Illustrations

In this section, we discuss some numerical examples that reveal the possible convexity of the total expected cost rate. A typical three dimensional plot of the total expected cost function is given in Figure 1. We have studied the effect of varying the cost and other system parameters on the optimal values and the system performance measures and also the results agreed with what one would expect.

**Example 1.** First, we explore the behaviour of the cost function by considering as the function of two variables by fixing the others at a constant level. Tables 1 – 5, give the total expected cost rate as a function of \( TC(S,s,6,4) \), \( TC(S,3,4,M) \), \( TC(34,s,N,9) \), \( TC(S,3,N,4) \) and \( TC(34,3,4,M) \). Towards this end, we first fix the parameter and cost values as \( \lambda = 1.2, \beta = 2, \gamma = 0.3, \theta = 3, \mu = 2, v_1 = 1, v_2 = 0.5, p_1 = 0.2, p_2 = 0.5, p_3 = 0.3, c_h = 0.01, c_f = 0.5, c_t = 0.3, c_0 = 2, c_1 = 6, c_r = 2, c_w = 0.4 \). In each table, underlined value denotes the row minimum and in **bold** faced value denotes the column minimum. Hence the both underlined and **bold** faced value refers the optimal value of the function.

**Example 2.** Here, we study the effect of the primary demand rate \( \lambda \), the service time \( \mu \), the lead time rate \( \beta \), the retrial demand rate \( \theta \) and the perishable rate \( \gamma \) on the total expected cost rate. From figure 2 to figure 4, we observe the following:

- The optimal expected cost rate increases when \( \lambda \) increases.
- As is to be expected, \( \beta \) increases the total expected cost rate decreases.
- Again the optimal expected cost rate increases when \( \mu \) and \( \gamma \) increase.
Example 3. In this example, we look at the impact of the primary demand rate $\lambda$, the service time $\mu$, the lead time rate $\beta$, the retrial demand rate $\theta$ and the perishable rate $\gamma$ on the fraction, $\eta_{RR}$ of successful retrial from the orbit. From figure 5 to figure 7, we observe the following:

- The fraction of successful rate of retrial, $\eta_{RR}$ decreases when the primary arrival rate increases.
- As is to be expected as the mean retrial time decreases, $\eta_{RR}$ decreases.
- When $\beta$ and $\mu$ increase $\eta_{RR}$ increases.

Example 4. In this example, we illustrate the effect of the primary demand rate $\lambda$, the service time $\mu$, the lead time rate $\beta$, the retrial demand rate $\theta$ and the perishable rate $\gamma$ on $\eta_{OO}$. From figure 8 and figure 9, we observe the following:

- As is to be expected as the primary arrival rate increases, $\eta_{OO}$ increases.
- When $\mu$ increase $\eta_{OO}$ decreases.

Example 5. In this example, we monitored the effect of the primary demand rate $\lambda$, the service time $\mu$, the lead time rate $\beta$, the retrial demand rate $\theta$ and the perishable rate $\gamma$ on $\eta_{LL}$. From figure 10 and figure 11, we observe the following:

- As is to be expected as the primary arrival rate increases, we lost more customers.
- And also we restrict our customer loss by increasing the service rate
- The customer loss direct proportional to the life time of the item and inversely to the $\beta$.

Example 6. In this example, we calculate the effect of the primary demand rate $\lambda$ on $\eta_{low}$. From figure 12, we observe the following:

- The expected waiting time increases when as the primary arrival rate increases.
- As is to be expected the waiting time increases when $N$ increases.

7 Conclusion

The stochastic model discussed here is useful in studying a perishable inventory system at a service facility with server interruptions, repeated customers and $(s, S)$ policy. The joint probability distribution of the number of customers in the waiting hall, number of customers in the orbit and the inventory level is derived in the steady state. Various system performance measures are derived and the long-run total expected cost rate is calculated. By assuming a suitable cost structure on the inventory system, we have presented extensive numerical illustrations to show the effect of change of values for constants on the total expected cost rate. The authors are working in the direction of MAP (Markovian arrival process) arrival for the customers and service times follow PH-distributions.
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References


