

Exact Soliton Solutions for Generalized Equal Width Equation

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Abstract: The homogeneous balance method is used to construct exact travelling wave solutions of generalized equal width equation. Many exact travelling wave solutions of generalized equal width equation are successfully obtained, which contain soliton-like and periodic-like solutions. This method can be applied to nonintegrable equations as well as integrable ones.

Keywords: Homogeneous balance method; Generalized equal width equation; Riccati equation; Soliton-like solution; Periodic-like solution.

1 Introduction

The investigation of the traveling wave solutions of nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics, fluid dynamics. In recent years Wang [1, 2], Khalafallah [3, 4], used a useful homogeneous balance (HB) method for finding exact solutions of a given nonlinear partial differential equations. Fan [5] used HB method to search for Backlund transformation and similarity reduction of nonlinear partial differential equations. In this paper, we use the homogeneous balance method (HB) to solve the Riccati equation $\phi' = a\phi^2 + b\phi + c$ and the reduced nonlinear ordinary differential equation for generalized equal width (GEW) equation, respectively. It makes the HB method use more extensively.

2 Generalized equal width equation

For generalized equal width equation [7]

$$u_t + \varepsilon u^n u_x - \delta u_{xxt} = 0, \quad (1)$$

where ε and δ are positive constants and n is a positive integer.

Let us consider the travelling wave solutions

$$u(x, t) = u(\xi), \quad \xi = k(x + \lambda t), \quad (2)$$

where k and λ are constants.

Substituting (2) into (1), then (1) is reduced to the following nonlinear ordinary differential equation

$$\lambda u' + \varepsilon u^n u' - \delta k^2 \lambda u''' = 0, \quad (3)$$

where by integrating (3) and neglecting the constant of integration we obtain

$$\lambda u + \frac{\varepsilon}{n+1} u^{n+1} - \delta k^2 \lambda u'' = 0. \quad (4)$$

We now seek the solutions of Eq. (4) in the form

$$u = \left\{ \sum_{i=0}^m a_i \phi^i \right\}^{\frac{1}{n}}, \quad (5)$$

where a_i are constants to be determined later and ϕ satisfy the following Riccati equation

$$\phi' = a\phi^2 + b\phi + c \quad (6)$$

where a, b and c are constants.

Balancing u^{n+1} with u'' in (4), we find

$$(n+1)m = m + 2$$

so that

$$m = \frac{2}{n}.$$

To get a closed form solution, m should be an integer. To achieve our goal, we use the transformation

$$u(\xi) = v^{\frac{1}{n}}(\xi), \quad (7)$$

that will carry (4) into the nonlinear ordinary differential equation

$$\lambda n^2(n+1)v^2 + \varepsilon n^2v^3 - \delta \lambda k^2(1-n^2)(v')^2 - \delta \lambda k^2 n(n+1)vv'' = 0. \quad (8)$$

Balancing v^3 with vv'' in (8) give

$$3m = 2m + 2$$

so that

$$m = 2.$$

The homogeneous balance method admits the use of the finite expansion in the form

$$v = a_0 + a_1\phi + a_2\phi^2, \quad (9)$$

where a_0, a_1 and a_2 are constants to be determined and ϕ satisfy Eq. (6).

Substituting Eqs. (9) and (6) into Eq. (8), and equating the coefficients of like powers of

ϕ^i ($i = 0, 1, 2, 3, 4, 5, 6$) to zero yields a set of algebraic system for a_0, a_1, a_2, k and λ

$$\begin{aligned} n^2(n+1)\lambda a_0^2 + n^2\varepsilon a_0^3 - n(n+1)\delta \lambda k^2 a_0(a_1bc + 2c^2a_2) - \delta \lambda k^2(1-n^2)c^2a_1^2 &= 0, \\ -n(n+1)\delta \lambda k^2 a_0(a_1(2ac + b^2) + 6a_2bc) - n(n+1)\delta \lambda k^2 a_1(a_1bc + 2a_2c^2) + 2n^2(n+1)\lambda a_0a_1 &+ \\ 3n^2\varepsilon a_0^2a_1 - 2\delta \lambda k^2(1-n^2)a_1c(a_1b + 2a_2c) &= 0, \\ n^2(n+1)\lambda(2a_0a_1 + a_1^2) - n(n+1)\delta \lambda k^2 a_0(3aba_1 + 4a_2(2ac + b^2)) - n(n+1)\delta \lambda k^2 a_1 & \\ (a_1(2ac + b^2) + 6a_2bc) - n(n+1)\delta \lambda k^2 a_2(a_1bc + 2a_2c^2) - \delta \lambda k^2(1-n^2)(2a_1c(aa_1 + 2a_2b) & \\ + (a_1b + 2a_2c)^2) + n^2\varepsilon(a_0(2a_0a_2 + a_1^2) + 2a_0a_1^2 + a_2a_0^2) &= 0, \\ 2n^2(n+1)\lambda a_1a_2 - n(n+1)\delta \lambda k^2 a_0(2a_1a^2 + 10aba_2) - n(n+1)\delta \lambda k^2 a_1(3aba_1 + 4a_2(2ac + b^2)) & \\ -n(n+1)\delta \lambda k^2 a_2(a_1(2ac + b^2) + 6bca_2) - \delta \lambda k^2(1-n^2)(4caa_1a_2 + 2(a_1b + 2a_2c)(aa_1 + 2a_2b)) & \\ + n^2\varepsilon(4a_0a_1a_2 + a_1(2a_0a_2 + a_1^2)) &= 0, \\ -6n(n+1)\delta \lambda k^2 a^2 a_0a_2 - n(n+1)\delta \lambda k^2 a_1(2a^2a_1 + 10aba_2) - n(n+1)\delta \lambda k^2 a_2(3aba_1 & \\ + 4a_2(2ac + b^2)) + n^2(n+1)\lambda a_2^2 + n^2\varepsilon(a_0a_2^2 + 2a_1^2a_2 + a_2(2a_0a_2 + a_1^2)) - \delta \lambda k^2(1-n^2) & \\ (4(a_1b + 2a_2c)aa_2 + (aa_1 + 2a_2b)^2) &= 0, \end{aligned} \quad (10)$$

$$\begin{aligned}
 & -6n(n+1)\delta\lambda k^2 a_1 a_2 a^2 - n(n+1)\delta\lambda k^2 a_2 (2a^2 a_1 + 10aba_2) - 4\delta\lambda k^2 (1-n^2)(aa_1 + 2a_2 b)aa_2 \\
 & + 3n^2 \varepsilon a_1 a_2^2 = 0, \\
 & -4\delta\lambda k^2 (1-n^2)a^2 a_2^2 + n^2 \varepsilon a_2^3 - 6n(n+1)\delta\lambda k^2 a^2 a_2^2 = 0.
 \end{aligned}$$

For which, with the aid of Maple, we find

$$\begin{aligned}
 a_0 &= -\frac{2ac\lambda(n+1)(n+2)}{(4ac-b^2)\varepsilon}, \quad a_1 = -\frac{2ab\lambda(n+1)(n+2)}{(4ac-b^2)\varepsilon}, \quad a_2 = -\frac{2a^2\lambda(n+1)(n+2)}{(4ac-b^2)\varepsilon}, \\
 k &= \pm \frac{n}{\sqrt{\delta(b^2-4ac)}}.
 \end{aligned} \tag{11}$$

It is to be noted that the Riccati equation (6) can be solved using the homogeneous balance method as follows

Case:I. Let $\phi = \sum_{i=0}^m b_i \tanh^i \xi$. Balancing ϕ' with ϕ^2 leads to

$$\phi = b_0 + b_1 \tanh \xi. \tag{12}$$

Substituting Eq. (12) into Eq. (6), we obtain the following solution of Eq. (6)

$$\phi = -\frac{1}{2a}(b + 2 \tanh \xi), \quad ac = \frac{b^2}{4} - 1. \tag{13}$$

Substituting Eqs. (11) and (13) into (2), (7) and (9), we get the following traveling wave solutions of the generalized equal width (GEW) equation (1) :

$$\begin{aligned}
 u(x,t) &= \left\{ \frac{(n+1)(n+2)}{(4ac-b^2)\varepsilon} \left[\frac{b^2}{2} - 2ac - 2 \tanh^2 \left[\frac{n}{\sqrt{\delta(b^2-4ac)}}(x + \lambda t) \right] \right] \right\}^{\frac{1}{n}} \\
 &= \left\{ -\frac{(n+1)(n+2)}{2\varepsilon} \operatorname{sech}^2 \left[\frac{n}{2\sqrt{\delta}}(x + \lambda t) \right] \right\}^{\frac{1}{n}},
 \end{aligned} \tag{14}$$

where $ac = \frac{b^2}{4} - 1$.

Similarly, let $\phi = \sum_{i=0}^m b_i \coth^i \xi$, then we obtain the following new traveling wave soliton solutions of the generalized equal width (GEW) equation (1):

$$\begin{aligned}
 u(x,t) &= \left\{ \frac{(n+1)(n+2)}{(4ac-b^2)\varepsilon} \left[\frac{b^2}{2} - 2ac - 2 \coth^2 \left[\frac{n}{\sqrt{\delta(b^2-4ac)}}(x + \lambda t) \right] \right] \right\}^{\frac{1}{n}} \\
 &= \left\{ \frac{(n+1)(n+2)}{2\varepsilon} \operatorname{csch}^2 \left[\frac{n}{2\sqrt{\delta}}(x + \lambda t) \right] \right\}^{\frac{1}{n}},
 \end{aligned} \tag{15}$$

where $ac = \frac{b^2}{4} - 1$.

Case: II. From [6], when $a=1, b=0$, the Riccati Eq. (6) has the following solutions

$$\phi = -\sqrt{-c} \tanh(\sqrt{-c} \xi), \quad c < 0,$$

$$\begin{aligned}\phi &= -\frac{1}{\xi}, & c &= 0, \\ \phi &= \sqrt{c} \tan(\sqrt{c}\xi), & c &> 0.\end{aligned}\tag{16}$$

From (7), (9), (11) and (16), we have the following new traveling wave solutions of generalized equal width (GEW) equation (1) which contain traveling wave solutions, periodic wave solutions and rational solutions as follows

When $c < 0$, we have the following traveling wave (soliton-like) solutions of generalized equal width (GEW) equation (1) :

$$u(x,t) = \left\{ -\frac{2\lambda(n+1)(n+2)}{(4ac-b^2)\varepsilon} \left(ac - ab\sqrt{-c} \tanh(\sqrt{-c}\xi) \right) - a^2c \tanh^2(\sqrt{-c}\xi) \right\}^{\frac{1}{n}},\tag{17}$$

$$\text{where } \xi = \frac{n}{\sqrt{\delta(b^2-4ac)}}(x + \lambda t).$$

When $c = 0$, we have

$$u(x,t) = \left\{ -\frac{2\lambda(n+1)(n+2)}{(4ac-b^2)\varepsilon} \left(ac - \frac{ab}{\xi} - \frac{a^2}{\xi^2} \right) \right\}^{\frac{1}{n}},\tag{18}$$

$$\text{where } \xi = \frac{n}{\sqrt{\delta(b^2-4ac)}}(x + \lambda t).$$

When $c > 0$, we have the following traveling wave (periodic-like) solutions of generalized equal width (GEW) equation (1) :

$$u(x,t) = \left\{ -\frac{2\lambda(n+1)(n+2)}{(4ac-b^2)\varepsilon} \left(ac + ab\sqrt{c} \tan(\sqrt{c}\xi) \right) + a^2c \tan^2(\sqrt{c}\xi) \right\}^{\frac{1}{n}},\tag{19}$$

$$\text{where } \xi = \frac{n}{\sqrt{\delta(b^2-4ac)}}(x + \lambda t).$$

Case: III. We suppose that the Riccati equation (6) has the following solutions of the form

$$\phi = A_0 + \sum_{i=1}^m (A_i f^i + B_i f^{i-1} g),\tag{20}$$

with

$$f = \frac{1}{\cosh \xi + r}, \quad g = \frac{\sinh \xi}{\cosh \xi + r},$$

which satisfy

$$\begin{aligned}f'(\xi) &= -f(\xi)g(\xi), & g'(\xi) &= 1 - g^2(\xi) - rf(\xi), \\ g^2(\xi) &= 1 - 2rf(\xi) + (r^2 - 1)f^2(\xi).\end{aligned}$$

Balancing ϕ' with ϕ^2 leads to

$$\phi = A_0 + A_1 f + B_1 g.\tag{21}$$

Substituting Eq. (21) into (6), collecting the coefficient of the same power $f^i(\xi)g^j(\xi)$ ($i = 0, 1, 2; j = 0, 1$) and setting each of the obtained coefficients to zero yield the following set of algebra equations

$$\begin{aligned} aA_0^2 + aB_1^2 + bA_0 + c &= 0, \\ 2aA_0A_1 - 2arB_1^2 - rB_1 + bA_1 &= 0, \\ aA_1^2 + a(r^2 - 1)B_1^2 + (r^2 - 1)B_1 &= 0, \\ 2aA_0B_1 + bB_1 &= 0, \\ 2aA_1B_1 + A_1 &= 0, \end{aligned}$$

which have solutions

$$A_0 = -\frac{b}{2a}, \quad A_1 = \pm \sqrt{\frac{(r^2 - 1)}{4a^2}}, \quad B_1 = -\frac{1}{2a}, \quad c = \frac{b^2 - 1}{4a}. \tag{22}$$

From Eqs. (21), (22), we obtain

$$\phi = -\frac{1}{2a} \left(b + \frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right). \tag{23}$$

From Eqs. (7), (9), (11) and (23), we obtain the new solutions of generalized equal width (GEW) equation (1):

$$u(x, t) = \left\{ \frac{\lambda(n+1)(n+2)}{(4ac - b^2)\epsilon} \left(\frac{b^2}{2} - 2ac - \frac{1}{2} \left(\frac{\sinh \xi \mp \sqrt{(r^2 - 1)}}{\cosh \xi + r} \right)^2 \right) \right\}^{\frac{1}{n}}, \tag{24}$$

where $\xi = \frac{n}{\sqrt{\delta(b^2 - 4ac)}}(x + \lambda t)$.

Case: IV. We take ϕ in the Riccati equation (6) being of the form

$$\phi = e^{p_1\xi} \rho(z) + p_4(\xi), \tag{25}$$

where

$$z = e^{p_2\xi} + p_3,$$

where p_1, p_2 and p_3 are constants to be determined.

Substituting (25) into (6) we find that when $c = \frac{-p_1^2 + b^2}{4a}$, we have

$$\phi = -\frac{p_1 e^{p_1\xi}}{a(e^{p_1\xi} + p_3)} + \frac{p_1 - b}{2a}. \tag{26}$$

If $p_3 = 1$ in (26), we have

$$\phi = -\frac{p_1}{2a} \tanh\left(\frac{1}{2} p_1 \xi\right) - \frac{b}{2a}. \tag{27}$$

If $p_3 = -1$ in (26), we have

$$\phi = -\frac{p_1}{2a} \coth\left(\frac{1}{2} p_1 \xi\right) - \frac{b}{2a}. \tag{28}$$

From (7), (9), (11) and (26), we obtain the following new traveling wave solutions of generalized equal width (GEW) equation (1):

$$u(x, t) = \left\{ -\frac{2\lambda(n+1)(n+2)}{(4ac - b^2)\epsilon} \left(ac + \left(\frac{p_1 + b}{2} - \frac{p_1 e^{\frac{p_1}{\sqrt{\delta(b^2 - 4ac)}}(x + \lambda t)}}}{e^{\frac{p_1}{\sqrt{\delta(b^2 - 4ac)}}(x + \lambda t)} + p_3} \right) \left(\frac{p_1 - b}{2} - \frac{p_1 e^{\frac{p_1}{\sqrt{\delta(b^2 - 4ac)}}(x + \lambda t)}}}{e^{\frac{p_1}{\sqrt{\delta(b^2 - 4ac)}}(x + \lambda t)} + p_3} \right) \right) \right\}^{\frac{1}{n}}. \tag{29}$$

When $p_3 = 1$, we have the following traveling wave (soliton-like) solutions of generalized equal width (GEW) equation (1) :

$$u(x,t) = \left\{ \frac{(n+1)(n+2)}{(4ac-b^2)\varepsilon} \left(\frac{b^2}{2} - 2ac - \frac{p_1^2}{2} \tanh^2 \left(\frac{p_1}{2} \left(\frac{n}{\sqrt{\delta(b^2-4ac)}} (x + \lambda t) \right) \right) \right) \right\}^{\frac{1}{n}}. \quad (30)$$

When $p_3 = -1$, we have the following traveling wave (periodic-like) solutions of equal width wave equation (1) :

$$u(x,t) = \left\{ \frac{(n+1)(n+2)}{(4ac-b^2)\varepsilon} \left(\frac{b^2}{2} - 2ac - \frac{p_1^2}{2} \coth^2 \left(\frac{p_1}{2} \left(\frac{n}{\sqrt{\delta(b^2-4ac)}} (x + \lambda t) \right) \right) \right) \right\}^{\frac{1}{n}}. \quad (31)$$

Case: V. We suppose that the Riccati equation (6) have the following solutions of the form:

$$\phi = A_0 + \sum_{i=1}^m \sinh^{i-1}(A_i \sinh w + B_i \cosh w), \quad (32)$$

where $\frac{dw}{d\xi} = \sinh w$ or $\frac{dw}{d\xi} = \cosh w$. It is easy to find that $m = 1$ by balancing ϕ' and ϕ^2 .

So we choose

$$\phi = A_0 + A_1 \sinh w + B_1 \cosh w, \quad (33)$$

when $\frac{dw}{d\xi} = \sinh w$, we substitute (33) and $\frac{dw}{d\xi} = \sinh w$, into (6) and set the coefficients of

$\sinh^i w \cosh^j w$ ($i = 0, 1, 2; j = 0, 1$) to zero. A set of algebraic equations is obtained as follows:

$$\begin{aligned} aA_0^2 + aB_1^2 + bA_0 + c &= 0, \\ 2aA_0A_1 + bA_1 &= 0, \\ aA_1^2 + aB_1^2 - B_1 &= 0, \\ 2aA_0B_1 + bB_1 &= 0, \\ 2aA_1B_1 + A_1 &= 0, \end{aligned}$$

for which, we have the following solutions:

$$A_0 = -\frac{b}{2a}, \quad A_1 = 0, \quad B_1 = \frac{1}{a}, \quad (34)$$

where $c = \frac{b^2-4}{4a}$, and

$$A_0 = -\frac{b}{2a}, \quad A_1 = \pm \sqrt{\frac{1}{2a}}, \quad B_1 = \frac{1}{2a}, \quad (35)$$

where $c = \frac{b^2-1}{4a}$.

To $\frac{dw}{d\xi} = \sinh w$, we have

$$\sinh w = -\operatorname{csc} h \xi, \quad \cosh w = -\operatorname{coth} \xi. \quad (36)$$

From (34) – (36), we obtain

$$\phi = -\frac{b + 2\operatorname{coth} \xi}{2a}, \quad (37)$$

where $c = \frac{b^2-4}{4a}$, and

$$\phi = -\frac{b \pm \csc h \xi + \coth \xi}{2a}, \tag{38}$$

where $c = \frac{b^2 - 1}{4a}$.

From Eqs. (7), (9), (11), (37) and (38) we get the following traveling wave solutions of the generalized equal width (GEW) equation (1):

$$u(x, t) = \left\{ \frac{(n+1)(n+2)}{2\varepsilon} \csc h^2 \left(\frac{n}{2\sqrt{\delta}}(x + \lambda t) \right) \right\}^{\frac{1}{n}}, \tag{39}$$

where $ac = \frac{b^2}{4} - 1$. and

$$u(x, t) = \left\{ \frac{(n+1)(n+2)}{(4ac - b^2)\varepsilon} \left(ac - \frac{b}{2}(b \pm \csc h \xi + \coth \xi) + \frac{1}{4}(b \pm \csc h \xi + \coth \xi)^2 \right) \right\}^{\frac{1}{n}}, \tag{40}$$

where $c = \frac{b^2 - 1}{4a}$ and $\xi = \frac{n}{\sqrt{\delta(b^2 - 4ac)}}(x + \lambda t)$.

Similarly, when $\frac{dw}{d\xi} = \cosh w$, we obtain the following traveling wave (periodic-like) solutions of the generalized equal width (GEW) equation (1) :

$$u(x, t) = \left\{ \frac{(n+1)(n+2)}{2\varepsilon} \csc^2 \left(\frac{n}{2\sqrt{\delta}}(x + \lambda t) \right) \right\}^{\frac{1}{n}}, \tag{41}$$

where $ac = \frac{b^2}{4} - 1$. and

$$u(x, t) = \left\{ \frac{(n+1)(n+2)}{(4ac - b^2)\varepsilon} \left(ac - \frac{b}{2}(b \pm \csc \xi + \cot \xi) + \frac{1}{4}(b \pm \csc \xi + \cot \xi)^2 \right) \right\}^{\frac{1}{n}}, \tag{42}$$

where $c = \frac{b^2 - 1}{4a}$.

In summary we have used the extended homogeneous balance method to obtain many traveling wave solutions of the generalized equal width (GEW) equation.

We now summarize the key steps as follows:

Step 1: For a given nonlinear evolution equation

$$F(u, u_t, u_x, u_{xt}, u_{tt}, \dots) = 0, \tag{43}$$

we consider its travelling wave solutions $u(x, t) = u(\xi)$, $\xi = k(x + lt) + d$ then Eq. (43) is reduced to an nonlinear ordinary differential equation

$$Q(u, u', u'', u''', \dots) = 0, \tag{44}$$

where a prime denotes $\frac{d}{d\xi}$.

Step 2: For a given ansatz equation (for example, the ansatz equation is $\phi' = a\phi^2 + b\phi + c$ in this paper), the form of u is decided and the homogeneous balance method is used on Eq. (44) to find the coefficients of u .

Step 3: The homogeneous balance method is used to solve the ansatz equation.

Step 4: Finally, the travelling wave solutions of Eq. (43) are obtained by combining steps 2 and 3.

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