Numerical Solution and Exponential Decay to Von Kármán System with Frictional Damping

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Abstract: In this work we consider the Von Kármán system with frictional damping acting on the displacement and using the Method of Nakao we prove the exponential decay of the solution. The numerical scheme is presented for calculate the solution and to verify the long-time decay energy.

Keywords: Von Kármán system, Method of Nakao, Decay of solutions, Numerical solution, Finite differences method.

1 Introduction

For several years the system of Theodor von Kármán [19] was studied in different situations and methods.

The exponential decay of the energy to the von Kármán equations with memory in noncylindrical domains was studied by Park and Kang [17] in 2009 using the same method as in [18].

To the models of von Kármán taking into account for rotational forces, Bradley and Lasiecka [6] in 1994 showed the uniform decay rates for the solutions in cylindrical domain.


For thermal damping Menzala and Zuazua [10] in 1998 proved the exponential decay by the semigroup properties.

For Viscoelastic plates with memory, using energy method, we cite Rivera and Menzala [15] in 1999, and the work of Rivera, Oquendo and Santos [16] in 2005 where was proved that the energy decays uniformly, exponentially or algebraically with the same rate of decay of the relaxation function.

Based on multipliers method, the exponential decay of solution for the full von Kármán System of Dynamic Thermoelasticity was proved by Benabdallah and Lasiecka [4] in 2000.


For the numerical scheme we mention for example Reinhart [14] in 1982 where was studied the approximation of the von Kármán equations stationary by the mixed finite element. The work of Yosibash, Kirby and Gottlieb [20] in 2004 where was studied the von Kármán system over rectangular domains and numerically solved using both the Chebyshev-collocation and Legendre-collocation methods for the spacial discretization and the implicit Newmark-β scheme combined with a non-linear fixed point algorithm for the temporal discretization, and Bilbao [5] in 2007 used numerical stability for numerical methods for the von Kármán system, through the use of energy-conserving methods.

What distinguishes this paper from other related works is that we apply the Method of Nakao in the von Kármán system to prove the exponential decay of the solution and we present an numerical scheme by finite

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differences method to numerical solution and the long-time decay energy, in this sense, there is few result in the literature.

The remainder of this paper is organized as follows. In section 2 we present the result of existence of weak solution, in the section 3 we prove the exponential decay of the solution, in the section 4 we applied the Finite Difference Method in the von Kármán system and finally in the section 5 we give the conclusion.

2 Existence of solution

We use the standard Lebesgue space and Sobolev Space with their usual properties as in Adams (1975) [1] and in this sense $⟨·,·⟩$ and $⟨·⟩$ denote the inner product in $L^2$ and $H^1_0$ respectively. By $|·|$ we denote the usual norm in $L^2$. Let $Ω ⊂ \mathbb{R}^2$ be a bounded domain of the plane with regular boundary $Γ$. For a real number $T > 0$ we denote $Q = Ω × (0, T)$ and $Σ = Γ × (0, T)$. Here $u = u(x, y, t)$ is the displacement, $v = v(x, y, t)$ the Airy stress function and $η$ is the unit normal external in $Ω$ and $u_0 = u_t$. With this notation we have the following system

\[
\begin{align*}
    u'' - Δ^2 u - [u, v] + u' = f & \quad \text{in} \ Ω, \tag{1} \\
    [u, v] = 0 & \quad \text{in} \ Ω, \tag{2}
\end{align*}
\]

and

\[
\begin{align*}
    u(0) &= u_0, \
    u'(0) &= u_1 & \text{in} \ Ω, \tag{3} \\
    u = v &= 0, \quad \frac{∂u}{∂η} = \frac{∂v}{∂η} &= 0 & \text{on} \ Σ, \tag{4}
\end{align*}
\]

where

\[
[u, v] = \frac{∂^2 u}{∂x^2} \frac{∂^2 v}{∂y^2} - 2 \frac{∂^2 u}{∂x∂y} \frac{∂^2 v}{∂y^2} + \frac{∂^2 u}{∂y^2} \frac{∂^2 v}{∂x^2}
\]

Now using the same idea as in [10] we have the following result of existence of solution.

Theorem 1. For $u_0 \in H^2_0(Ω)$, $u_1 \in L^2(Ω)$ and $f \in L^2_{loc}(\mathbb{R}^+; L^2(Ω))$ there exists $u, v : Ω \rightarrow \mathbb{R}$ such that

$u, v \in L^∞(0, T; H^2_0(Ω)), \quad u' \in L^∞(0, T; L^2(Ω)),$

and $u, v$ weak solution (1)-(4).

3 Asymptotic behavior

In this section, we will use the Method of Nakao (1978) [12] to prove the exponential decay of the solution. First we define

\[
E(t) = |u'(t)|^2 + |Δ u(t)|^2 + \frac{1}{2} |Δ v(t)|^2 \tag{5}
\]

Lemma 1. The functional of energy $E(t)$ is limited.

Proof. Multiplying (1) by $u'$ and integrating in $Ω$, we have

\[
 \frac{1}{2} \frac{d}{dt} \left[ |u'(t)|^2 + |Δ u(t)|^2 \right] - ⟨[u(t), v(t)], u'(t)⟩ + |u'(t)|^2 = (f(t), u'(t))
\]

Using (2) we obtain

\[
⟨[u(t), v(t)], u'(t)⟩ = ⟨[u(t), u'(t)], v(t)⟩ = \frac{1}{2} \frac{d}{dt} (u(t), u(t), v(t)) - \frac{1}{2} (Δ^2 v(t), v(t)) = \frac{1}{4} \frac{d}{dt} (Δ v(t))^2
\]

from where follows

\[
\frac{d}{dt} \left[ |u'(t)|^2 + |Δ u(t)|^2 + \frac{1}{2} |Δ v(t)|^2 \right] + 2 |u'(t)|^2 = 2 (f(t), u'(t)) \tag{6}
\]

Performing integration from 0 to $t$ follows by Cauchy-Schwarz inequality we obtain

\[
|u'(t)|^2 + |Δ u(t)|^2 + \frac{1}{2} |Δ v(t)|^2 + 2 \int_0^t |u'(s)|^2 ds ≤ \int_0^t |u'(s)|^2 ds + \int_0^t |f(s)|^2 ds + |u_1|^2 + |Δ u_0|^2 + \frac{1}{2} |Δ v_0|^2
\]

then

\[
E(t) + \int_0^t |u'(s)|^2 ds ≤ E(0) + \int_0^t |f(s)|^2 ds
\]

from where follows $E(t) ≤ C$ with $C$ constant independently of $t$.

Now we introduce a new functional.

Lemma 2. The functional

\[
F^2(t) = E(t) - E(t+1) + \int_0^{t+1} |f(s)|^2 ds
\]

satisfies

\[
\int_0^{t+1} |u'(s)|^2 ds ≤ E^2(t)
\]

Proof. Integrating (6) from $τ_1$ to $τ_2$ with $0 < τ_1 < τ_2$, we obtain

\[
E(τ_2) + 2 \int_{τ_1}^{τ_2} |u'(s)|^2 ds = E(τ_1) + 2 \int_{τ_1}^{τ_2} (f(s), u'(s)) ds \tag{7}
\]

and for all $t > 0$

\[
E(t+1) + 2 \int_0^{t+1} |u'(s)|^2 ds = E(t) + 2 \int_0^{t+1} (f(s), u'(s)) ds ≤ E(t) + \int_0^{t+1} |f(s)|^2 ds + \int_0^{t+1} |u'(s)|^2 ds
\]
then
\[
\int_{t_1}^{t_1+1} |u'(s)|^2 ds \leq E(t) - E(t+1) + \int_{t_1}^{t_1+1} |f(s)|^2 ds = F^2(t)
\] (8)

**Lemma 3.** The functional
\[
G^2(t) = 8 C \operatorname{ess sup}_{s \in [t, t+1]} |\Delta u(s)| F(t) + 2(1 + C^2) \int_{t_1}^{t_2} |u'(t)|^2 dt + 2 C^2 \int_{t_1}^{t_2} |f(t)|^2 dt
\]
satisfies
\[
\int_{t_1}^{t_2} \left( |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 \right) dt \leq G^2(t)
\]

**Proof.** First we note that
\[
\langle [u(t), v(t)], u(t) \rangle = \langle [u(t), u(t)], v(t) \rangle = -\langle \Delta^2 v(t), v(t) \rangle
\]
\[
= -|\Delta v(t)|^2
\]
\[
(9)
\]
From (8) there exists \( t_1 \in [t, t + \frac{1}{2}] \) and \( t_2 \in [t + \frac{1}{2}, t + 1] \) such that
\[
|u'(t_i)| \leq 2 F(t), \quad i = 1, 2
\]
Multiplying (1) by \( u \) and integrating in \( \Omega \), we have
\[
\frac{d}{dt} \langle [u(t), u(t)], u(t) \rangle = \langle [u(t), u(t)], v(t) \rangle = -\langle \Delta^2 v(t), v(t) \rangle
\]
Performing integration from \( t_1 \) to \( t_2 \) and using (9) we have
\[
\int_{t_1}^{t_2} (|\Delta u(t)|^2 + |\Delta v(t)|^2) dt = (u'(t_1), u(t_1))
\]
\[-(u'(t_2), u(t_2)) + \int_{t_1}^{t_2} (|u'(t)|^2 - |u(t), u(t)|) dt + \int_{t_1}^{t_2} |f(t)|^2 dt dt
\]
Now, choosing \( C \) such that \( |u| \leq C |\Delta u| \) and applying Cauchy-Schwarz inequality we get
\[
\int_{t_1}^{t_2} \left( \frac{1}{2} |\Delta u(t)|^2 + |\Delta v(t)|^2 \right) dt
\]
\[
\leq C \operatorname{ess sup}_{s \in [t, t+1]} (|\Delta u(s)|(|u'(t_1)| + |u'(t_2)|)) + (1 + C^2) \int_{t_1}^{t_2} |u'(t)|^2 dt + C^2 \int_{t_1}^{t_2} |f(t)|^2 dt,
\]
using (10),
\[
\int_{t_1}^{t_2} \left( \frac{1}{2} |\Delta u(t)|^2 + |\Delta v(t)|^2 \right) dt
\]
\[
\leq 8 C \operatorname{ess sup}_{s \in [t, t+1]} |\Delta u(s)| |F(t)| (2 + (1 + C^2) \int_{t_1}^{t_2} |u'(t)|^2 dt + 2 C^2 \int_{t_1}^{t_2} |f(t)|^2 dt
\]
and then
\[
\int_{t_1}^{t_2} \left( |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 \right) dt \leq G^2(t)
\] (11)

**Theorem 2.** For \( f \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega)) \) with \( \int_0^1 |f(s)| ds \leq \alpha_1 e^{-\alpha_2 t} \), for all \( t \geq 1 \) and \( \alpha_1, \alpha_2 > 0 \), then the solution \((u, v)\) satisfies
\[
|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 + \int_{t_1}^{t_1+1} |u'(s)|^2 ds \leq k_1 e^{-k_2 t}
\]
for almost every \( t \geq 1 \), with \( k_1, k_2 > 0 \), constants independently from \( t \).

**Proof.** From (8) and (11) we concludes
\[
\int_{t_1}^{t_2} \left( |u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 \right) dt \leq F^2(t) + G^2(t)
\]
There is \( t^* \in [t_1, t_2] \) such that
\[
E(t^*) = |u'(t^*)|^2 + |\Delta u(t^*)|^2 + \frac{1}{2} |\Delta v(t^*)|^2 \leq 2(F^2(t) + G^2(t))
\] (13)
From (7) we get
\[
E(t_1) = E(t^*) + 2 \int_{t_1}^{t^*} |u'(s)|^2 ds - 2 \int_{t_1}^{t^*} |f(s), u(s)| ds
\]
Then
\[
E(t) = E(t^*) + 2 \int_{t_1}^{t^*} |u'(s)|^2 ds - 2 \int_{t_1}^{t^*} |f(s), u(s)| ds
\]
\[
\leq E(t^*) + 3 \int_{t_1}^{t^*+1} |u'(s)|^2 ds + \int_{t_1}^{t^*+1} |f(s)|^2 ds,
\]
and
\[
\operatorname{ess sup}_{s \in [t^*, t+1]} E(s) \leq E(t^*) + 3 \int_{t}^{t^*+1} |u'(s)|^2 ds + \int_{t}^{t^*+1} |f(s)|^2 ds
\]
Now using (8) and (13) we obtain
\[
\operatorname{ess sup}_{s \in [t^*, t+1]} E(s) \leq 2(F^2(t) + G^2(t)) + 3\bar{F}^2(t) + \int_{t}^{t^*+1} |f(s)|^2 ds
\]
\[
\leq 5\bar{F}^2(t) + 16 \operatorname{ess sup}_{s \in [t^*, t+1]} |\Delta u(s)| |F(t)|
\]
\[
+ 4(1 + C^2) \int_{t^*}^{t^*+1} |u'(s)|^2 ds + (1 + 4C^2) \int_{t^*}^{t^*+1} |f(s)|^2 ds
\]
\[
\leq (9 + 4C^2)\bar{F}^2(t) + \frac{1}{2} \operatorname{ess sup}_{s \in [t^*, t+1]} E(s)
\]
\[
+ 128C^2 \bar{F}^2(t) + (1 + 4C^2) \int_{t}^{t^*+1} |f(s)|^2 ds,
\]
from where follows
\[
\text{ess sup}_{x \in [t, t+1]} E(s) \leq (274 + 8C^2)E^2(t) + (2 + 8C^2) \int_t^{t+1} |f(s)|^2 ds,
\]
and then
\[
E(t) \leq C_1(E(t) - E(t+1)) + C_2 \int_t^{t+1} |f(s)|^2 ds,
\]
where \(C_1, i = 1, 2\), constants independently from \(t\).

Without lost of generality, we can suppose \(C_1 > 1\) and for \(0 < \beta = \frac{1}{t_1} < 1\) we have
\[
E(t+1) \leq (1 - \beta)E(t) + \beta C_2 \int_t^{t+1} |f(s)|^2 ds
\]
For \(t \geq 1 \) and \( n \in \mathbb{N} \) such that \( n \leq t \leq n + 1 \)
\[
E(t) \leq (1 - \beta)E(t-1) + \beta C_2 \int_{t-1}^{t} |f(s)|^2 ds
\]
\[
\leq (1 - \beta)^n E(t-n) + \beta C_2 \int_{t-n}^{t} |f(s)|^2 ds
\]
Now
\[
(1 - \beta)^n < (1 - \beta)^{t-n} \implies (1 - \beta)^n < (1 - \beta)^{t-n-1}
\]
Then
\[
E(t) \leq (1 - \beta)^{t-n-1} \text{ess sup}_{x \in [0,1]} E(s) + \beta C_2 \int_0^{t-n} |f(s)|^2 ds
\]
\[
= \frac{m_0}{1 - \beta} (1 - \beta)^{t-n} + \beta C_2 \int_0^{t-n} |f(s)|^2 ds,
\]
with \(m_0 = \text{ess sup}_{x \in [0,1]} E(s) < \infty\).

Now we have
\[
E(t) < \frac{m_0}{1 - \beta} e^{\ln(1 - \beta)} + \beta C_2 \int_0^{t-n} |f(s)|^2 ds
\]
\[
< \frac{m_0}{1 - \beta} e^{-\beta t} + \beta C_2 \alpha_1 e^{-\alpha t},
\]
for almost every \( t \geq 1 \), with \( \beta_1 = -\ln(1 - \beta) > 0 \) and then
\[
|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 < \gamma_1 e^{-\gamma_1 t}
\]
From (8), we have
\[
\int_t^{t+1} |u'(s)|^2 ds \leq E(t) + \int_t^{t+1} |f(s)|^2 ds < \gamma e^{-\gamma t}
\]
with \( \gamma > 0 \) constants. Finally we concludes from (14) and (15) that there is constants \( k_1, k_2 > 0 \) such that
\[
|u'(t)|^2 + |\Delta u(t)|^2 + \frac{1}{2} |\Delta v(t)|^2 + \int_t^{t+1} |u'(s)|^2 ds < k_1 e^{-k_2 t}
\]
This completes the proof.

### 4 Numerical solution

For a given small constant \( \varepsilon > 0 \) we define a thin plate by
\[
\Omega \times (-\varepsilon, \varepsilon) = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \Omega, z \in (-\varepsilon, \varepsilon)\}
\]
whose midsurface is identified with \( \Omega \).

We resolve the von Kármán system in a square thin elastic plate by Finite Difference Method, subjected to a perpendicular load \( f \) and boundary condition of clamped type.

#### 4.1 Discrete formulation

Consider the discrete domain the midsurface of the square plate,
\[
\Omega_h = (0, \pi)^2 \quad \text{with uniform grid}
\]
\[
x_i = ih, y_j = jh, i, j = 0, ..., N + 1, h = \pi/(N + 1).
\]
The internal point are \( x_i = ih, y_j = jh, 1 \leq i, j \leq N \). The boundary of \( \Omega_h \) is denoted \( \Gamma_h \).

The temporal discretization of interval \( I_k = (0, T) \) is given by \( t_n = nk, n = 0, ..., M + 1, k = \pi/(M + 1) \).

Denote by \( u^n_{i,j} \) and \( v^n_{i,j} \), the functions \( u \) and \( v \) evaluate in the point \((x_i, y_j)\) and at the instant \( t_n \), respectively. It also, denoted by \( \Omega_h^i = \Omega_h \times I_i \) and \( \Sigma_h^i = \Gamma_h \times I_i \).

We show in Figure 1 the pattern mesh of \( \Omega \) with its points: internal (circles), boundary (squares) and ghost (diamonds). We define the following discrete differential operators:
\[
\delta u^n_{i,j} = \frac{1}{k} (u^n_{i+1,j} - u^n_{i,j}),
\]
\[
\delta^2 u^n_{i,j} = \frac{1}{k^2} (u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j})
\]

![Fig. 1: The pattern mesh of \( \Omega \) with internal, boundary and ghost points.](image)
\[
\begin{align*}
\delta^2 u^{n}_{i,j} &= \frac{1}{h^2} \left( u^{n}_{i+1,j} - 2u^{n}_{i,j} + u^{n}_{i-1,j} \right), \\
\delta^2 u^{n}_{j,h} &= \frac{1}{h^2} \left( u^{n}_{i,j+1} - 2u^{n}_{i,j} + u^{n}_{i,j-1} \right), \\
\delta_0 u^{n}_{i,j} &= \delta_y(\delta_x u^{n}_{i,j}) = \frac{1}{4h^2} \left( u^{n}_{i+1,j+1} - u^{n}_{i+1,j-1} - u^{n}_{i-1,j+1} + u^{n}_{i-1,j-1} \right), \\
\delta_2 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_3 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_5 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_3 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_5 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_3 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_5 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_3 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_5 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_3 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right), \\
\delta_5 u^{n}_{i,j} &= \delta_x(\delta_x u^{n}_{i,j}) = \frac{1}{h^2} \left( u^{n}_{i+2,j} - 4u^{n}_{i+1,j} + 6u^{n}_{i,j} - 4u^{n}_{i-1,j} + u^{n}_{i-2,j} \right).
\end{align*}
\]

For calculate \( u \) in the level 2, we first calculate
\[
\Delta^n_{h} u^{n}_{i,j} = \frac{1}{h^2} \left( u^{n}_{i+1,j} - 2u^{n}_{i,j} + u^{n}_{i-1,j} \right) + g^{n}_{i,j}, \quad n = 1, \ldots, M. \tag{31}
\]

Using (22) for function \( v \), the equation (31) is given by
\[
\begin{align*}
\Delta^n_{h} v^{n}_{i,j} &= \frac{1}{h^2} \left( v^{n}_{i+1,j} - 2v^{n}_{i,j} + v^{n}_{i-1,j} \right) + \frac{1}{h^2} \left( u^{n}_{i+1,j} - 2u^{n}_{i,j} + u^{n}_{i-1,j} \right) + g^{n}_{i,j}, \quad n = 1, \ldots, M \tag{32}
\end{align*}
\]

where \( A = (a_{i,j})_{N^2 \times N^2} \) is a symmetric matrix, \( B = (b_{i,j})_{N^2 \times N^2} \) and \( D = (d_{i,j})_{N^2 \times N^2} \), where \( N_0 \) is the number the boundary points, \( N_2 \) is the number the ghost points and, \( \nu \) and \( \nu \) denote the function \( v \) evaluate in the boundary and ghost points, respectively. Thus, for all \( n = 1, \ldots, M \)
\[
\begin{align*}
\nu^n &= 0 \tag{33} \\
\nu^n &= \nu^n \tag{34}
\end{align*}
\]

The equations (33) and (34) are due to a clamped boundary condition, given by (27) and (28), and because the exterior normal coincides with canonical vectors. Thus, the linear system (32) is resolved by the SOR method.

Once known \( v \) we can calculate \( u \) for all \( n = 1, \ldots, M \)
\[
\begin{align*}
\Delta^{n+1}_{h} u^{n+1}_{i,j} &= \mu_1 \delta_1 u^{n+1}_{i,j} + \mu_2 \left( \omega^{n+1}_{i,j} - (1/8) \omega^{n+1}_{i+1,j} + \omega^{n+1}_{i,j} \right) \\
&+ \mu_3 u^{n+1}_{i,j} + \mu_4 u^{n+1}_{i,j} + \mu_5 f_{i,j}^{n} \tag{35}
\end{align*}
\]

where,
\[
\begin{align*}
\mu_1 &= a_1^2 k^2 / h^4, \quad \mu_2 = a_2^2 k^2 / h^4, \quad \mu_3 = 2 - a_2^2 k \\
\mu_4 &= a_3^2 k - 1, \quad \mu_5 = k^2,
\end{align*}
\]
\[
\begin{align*}
\delta_1 u^{n+1}_{i,j} &= u^{n+1}_{i+1,j} + 2u^{n+1}_{i,j} - 2u^{n+1}_{i-1,j} - 2u^{n+1}_{i,j+1} - 2u^{n+1}_{i,j-1} + 4u^{n+1}_{i+1,j} + 4u^{n+1}_{i-1,j} + 4u^{n+1}_{i,j+1} + 4u^{n+1}_{i,j-1}, \\
\delta_2 u^{n+1}_{i,j} &= u^{n+1}_{i,j+1} + 2u^{n+1}_{i,j} - 2u^{n+1}_{i,j-1} + 4u^{n+1}_{i,j+1} + 4u^{n+1}_{i,j-1}, \\
\omega^{n+1}_{i,j} &= u^{n+1}_{i+1,j} - 2u^{n+1}_{i,j} + u^{n+1}_{i-1,j} \left( v^{n+1}_{i,j} - 2v^{n+1}_{i,j} + v^{n+1}_{i,j} \right), \\
\omega^{n+1}_{i,j} &= \left( u^{n+1}_{i+1,j} - 2u^{n+1}_{i,j} + u^{n+1}_{i-1,j} \right) \left( v^{n+1}_{i,j} - 2v^{n+1}_{i,j} + v^{n+1}_{i,j} \right). 
\end{align*}
\]

### 4.2 Numerical tests

Consider the following analytical solution of the equations (24) - (28)
\[
\begin{align*}
\omega_{u}(x,y,t) &= \sin^2 x \sin^2 y e^{-t} \tag{36} \\
v_{u}(x,y,t) &= \sin^2 x \sin^2 y \tag{37}
\end{align*}
\]

with loads given by
\[
\begin{align*}
f(x,y,t) &= e^{-t} \left( (1 - a_1^2) \sin^2 \sin^2 - 8a_1(\cos 2x \sin^2 \sin^2 \\
&- \cos 2x \cos 2y + \cos 2y \sin^2) \\
&- 2a_2^2 (4 \cos 2x \sin^2 \sin^2 - \sin^2 \sin^2 \sin^2) \right) \tag{38} \\
g(x,y,t) &= -8(\cos 2x \sin^2 \sin^2 - \cos 2x \cos 2y + \cos 2y \sin^2 \\
&+ \cos 2x \sin^2 \cos 2y \sin^2 \sin^2 \sin^2 \sin^2) \tag{39}
\end{align*}
\]
We use for size of mesh $N = 19$. For case $g = 0$ the mechanical energy is given by

$$E(t) = |u'(t)|^2 + a_1^2|\Delta u(t)|^2 + \frac{1}{2}a_2^2|\Delta v(t)|^2$$

(40)

**Example 1.** The first problem we consider the following data: $a_1 = 0.01, a_2 = 0.5$ and $a_3 = 1.25$. $f$ and $g$ are given in (38) and (39), respectively. $u_0$ and $u_1$ are obtained from (36) and $u$ and $v$ satisfies the clamping conditions. For constant $C_0 = 0.4$ a convergence is attained in $T = k(M + 1) \approx 0.031416 \times (830) \approx 26.075$ s. In Figure 2 we present for all $t \in (0, T)$, the absolute error defined by $|u(t) - u_0(t)|$. In Figure 3, we show the long-time behavior of the transversal displacement in the point $(\pi/2, \pi/2)$. In Figure 4, we show in 3D the transversal displacement of plate for different time steps: $t_0 = 0$, $t_{25} \approx 0.785$, $t_{70} \approx 2.199$, $t_{830} \approx 26.075$. We initially observe that the deflections are larger in the corners of the plate and then they reduce smoothly and expand rapidly near the boundary.

![Fig. 2](image1.png)

**Fig. 2:** The absolute error at $L^2$ norm the long-time.

![Fig. 3](image2.png)

**Fig. 3:** Transversal displacement in the point $(\pi/2, \pi/2)$.

![Fig. 4](image3.png)

**Fig. 4:** Transversal displacement of plate. (a) $t = 0$ s, (b) $t_{25} \approx 0.785$ s, (c) $t_{70} \approx 2.199$ s and (d) $t_{830} \approx 26.075$ s.

**Example 2.** In this example we consider $a_1 = 0.01, a_2 = 1$ and $a_3 = 1$. $f$ is given in (38) and $g = 0$. 
\( u_0 \) and \( u_1 \) are obtained from (36) and \( u \) and \( v \) satisfy the clamping conditions. For constant \( C_0 = 0.2 \), convergence is attained in \( T = k(M+1) \approx 0.0314159(1862) \approx 58.496 \) s. For calculate of energy of system, given by (40), we have computed in all the plate using the Composite Simpson’s rule. In Figure 5 and 6 we show the long-time behavior of the solution in the point \((\pi/2, \pi/2)\) and energy of system, respectively. Note that energy converge to 0. In Figure 7, we show the transversal displacement of the plate, in 3D, for different time steps: \( t_0 = 0, t_{45} \approx 1.414, t_{140} \approx 4.398, t_{1843} \approx 58.496 \). We initially observe that the deflections are larger near the boundary, but in long-time these deflections disappear.

5 Conclusion

The Nakao’s method proved to be an efficient method for the demonstration of the exponential decay of the solution of the system of von Kármán. Numerical tests have shown the decay of the mechanical energy of the system.

References


