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New Exact Solutions of the MkdV-Sine-Gordon Equation

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Abstract: In this paper authors applied sub-equation method to obtain the new exact solutions of the MkdV-Sine-Gordon equation. It is understood that the sub-equation method is a reliable and efficient tool for solving the nonlinear partial differential equations (NPDEs) arising in mathematical physics. In solution procedure wave transform is used which changes partial differential equation into nonlinear ordinary differential equation.

Keywords: Sub-equation method, analytical solutions, conformable fractional derivative.

1 Introduction

Solving nonlinear partial differential equations [8,10,11,12] have a great importance because of its huge amount of application area. For example, in fluids dynamics (and more generally continuous media dynamics), electromagnetic theory, quantum mechanics, traffic flow the mathematical models are expressed with the help of nonlinear differential equations [13-22]. As a result of this huge application area, researchers have been paying great attention to obtain the analytical solutions of NPDEs [5,6,23,24]. For this aim many different analytical methods appeared in the last few decades. For instance, expansion method [2], method [9], F-expansion method [1], functional variable method [4], extended trial equation method [4] and etc. Almost all these methods depend on a wave transform. By the help of this wave transform NPDE can be turned into a nonlinear ordinary differential equation. Then this nonlinear ordinary differential equation can be solved by using the analytical methods mentioned above. In this article, we use the sub-equation method [7] to obtain the exact solutions of MkdV-Sine-Gordon equation. A brief description of the method is given below.

1.1 Sub-Equation Method

Regard the following partial differential equation below

$$P\left(u,u_t,u_x,u_{tt},u_{xx},\ldots\right)=0. \tag{1}$$

Now, let's summarize the sub- equation method [25] step by step.
Step 1. Define the wave transform

$$\xi = kx+wt, \tag{2}$$

Here, k and w are arbitrary constants to be examined later. So, the function $u(x,t)$ becomes a function with one variable $u(\xi)$ and Eq. (1) is turned into a nonlinear ordinary differential equation by using the chain rule.

$$G\left(U,U',U'',\ldots\right)=0 \tag{3}$$

where prime shows the known derivative with respect to ξ.
Step 2. Suppose that Eq. (3) has a solution in the following form

$$U(\xi)=\sum_{i=0}^{n} a_i\varphi^i(\xi), \quad a_n\neq 0, \tag{4}$$

where.$a_i\,(i=0,\ldots,n)$. are constant coefficients which are going to be evaluated later. The positive integer is going to be obtained by using balancing principle in Eq. (3) and $\varphi(\xi)$ is the solution of the following ordinary differential equation

$$\varphi'(\xi)=\sigma+(\varphi(\xi))^2 \tag{5}$$

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where σ is a constant. Some special solutions for the Eq. (5) are given as follows.

$$\varphi(\xi)=\begin{cases}-\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right), & \sigma<0\\ -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right), & \sigma<0\\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right), & \sigma>0\\ \sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right), & \sigma>0\\ -\frac{1}{\xi+\varpi},\ \varpi \text{ is a cons.}, & \sigma=0\end{cases} \tag{6}$$

Step 3. We can write the Eqs. (4) and (5) into Eq. (3) and equalize the coefficients of $\varphi^i(\xi)$to zero. As a result of this process we obtain an algebraic equations system with the variables $a_i\,(i=0,...,n)$, k,w and $.\sigma$. Solving the obtained algebraic equation, we get the values for these variables.

Step 4. At last, the obtained values of the variables from the nonlinear algebraic system at previous step and the solutions of Eq. (5) are placed into Eq. (4) with the help of formulas given in (6). This procedure processes the exact solutions for Eq. (1).

2 Main Results

In this section authors obtain the new exact solutions of MkdV-Sine-Gordon equation by using the considered method.

2.1 Exact Solutions of MkdV-Sine-Gordon Equation

Consider the MkdV-Sine-Gordon Equation [3]

$$u_{xt}+\frac{3}{2}u_x^2u_{xx}+u_{xxxx}=\sin(u). \tag{7}$$

Using the wave transform in Eq. (7), equations turns into a nonlinear ordinary differential equation

$$wku''+\frac{3}{2}k^4u'^2u''+k^4u^{(iv)}=\sin(u) \tag{8}$$

where prime shows the derivative with respect to variable ξ. Now regard the transformations

$$v=e^{iu},\ i=\sqrt{-1} \tag{9}$$

so

$$\sin u=\frac{v-v^{-1}}{2i},\ \cos u=\frac{v+v^{-1}}{2} \tag{10}$$

which gives

$$u=\arccos\left(\frac{v+v^{-1}}{2}\right). \tag{11}$$

Applying the transformation to Eq. (8), we get

$$-v^3+v^5+2kwv^2v'^2+9k^4v'^4-2kwv^3v'' \tag{12}$$
$$-21k^4vv'^2v''+6k^4v^2v''^2+8k^4v^2v'v'''-2k^4v^3v^{(iv)}=0 \tag{13}$$

Assume that the solution of the Eq. (12) is in the following form

$$v(\xi)=\sum_{i=0}^{n}a_i\varphi^i(\xi),\quad a_n\neq0. \tag{14}$$

Using the balancing procedure, we get $n=2$. So

$$v(\xi)=a_0+a_1\varphi(\xi)+a_2\varphi^2(\xi). \tag{15}$$

Substituting Eq. (15) into Eq. (12), we have the following algebraic equation system.

$$\begin{aligned}
&-a_0{}^3+a_0{}^5+2a_0{}^2a_1{}^2kw\sigma^2-4a_0{}^3a_2kw\sigma^2+16a_0{}^2a_1{}^2k^4\sigma^3\\
&-32a_0{}^3a_2k^4\sigma^3+9a_1{}^4k^4\sigma^4-42a_0a_1{}^2a_2k^4\sigma^4+24a_0{}^2a_2{}^2k^4\sigma^4=0,\\
&-3a_0{}^2a_1+5a_0{}^4a_1-4a_0{}^3a_1kw\sigma-32a_0{}^3a_1k^4\sigma^2+4a_0a_1{}^3kw\sigma^2\\
&-4a_0{}^2a_1a_2kw\sigma^2-10a_0a_1{}^3k^4\sigma^3+112a_0{}^2a_1a_2k^4\sigma^3\\
&+30a_1{}^3a_2k^4\sigma^4-120a_0a_1a_2{}^2k^4\sigma^4=0,\\
&-3a_0a_1{}^2+10a_0{}^3a_1{}^2-3a_0{}^2a_2+5a_0{}^4a_2-8a_0{}^2a_1{}^2kw\sigma\\
&-16a_0{}^3a_2kw\sigma+8a_0{}^2a_1{}^2k^4\sigma^2-272a_0{}^3a_2k^4\sigma^2+2a_1{}^4kw\sigma^2\\
&+8a_0a_1{}^2a_2kw\sigma^2-4a_0{}^2a_2{}^2kw\sigma^2+10a_1{}^4k^4\sigma^3\\
&-68a_0a_1{}^2a_2k^4\sigma^3+352a_0{}^2a_2{}^2k^4\sigma^3\\
&+30a_1{}^2a_2{}^2k^4\sigma^4-120a_0a_2{}^3k^4\sigma^4=0,\\
&-a_1{}^3+10a_0{}^2a_1{}^3-6a_0a_1a_2+20a_0{}^3a_1a_2-4a_0{}^3a_1kw\\
&-80a_0{}^3a_1k^4\sigma-4a_0a_1{}^3kw\sigma-44a_0{}^2a_1a_2kw\sigma\\
&-14a_0a_1{}^3k^4\sigma^2-64a_0{}^2a_1a_2k^4\sigma^2+8a_1{}^3a_2kw\sigma^2\\
&+8a_0a_1a_2{}^2kw\sigma^2+34a_1{}^3a_2k^4\sigma^3-56a_0a_1a_2{}^2k^4\sigma^3=0,\\
&5a_0a_1{}^4-3a_1{}^2a_2+30a_0{}^2a_1{}^2a_2-3a_0a_2{}^2+10a_0{}^3a_2{}^2\\
&-10a_0{}^2a_1{}^2kw-12a_0{}^3a_2kw-80a_0{}^2a_1{}^2k^4\sigma\\
&-480a_0{}^3a_2k^4\sigma-32a_0a_1{}^2a_2kw\sigma-32a_0{}^2a_2{}^2kw\sigma\\
&-112a_0a_1{}^2a_2k^4\sigma^2+608a_0{}^2a_2{}^2k^4\sigma^2+14a_1{}^2a_2{}^2kw\sigma^2\\
&+4a_0a_2{}^3kw\sigma^2+52a_1{}^2a_2{}^2k^4\sigma^3-208a_0a_2{}^3k^4\sigma^3=0,\\
&a_1{}^5+20a_0a_1{}^3a_2-3a_1a_2{}^2+30a_0{}^2a_1a_2{}^2-48a_0{}^3a_1k^4\\
&-8a_0a_1{}^3kw-40a_0{}^2a_1a_2kw-46a_0a_1{}^3k^4\sigma-608a_0{}^2a_1a_2k^4\sigma\\
&-4a_1{}^3a_2kw\sigma-44a_0a_1a_2{}^2kw\sigma-14a_1{}^3a_2k^4\sigma^2\\
&+296a_0a_1a_2{}^2k^4\sigma^2+12a_1a_2{}^3kw\sigma^2-24a_1a_2{}^3k^4\sigma^3=0,\\
&5a_1{}^4a_2+30a_0a_1{}^2a_2{}^2-a_2{}^3+10a_0{}^2a_2{}^3-72a_0{}^2a_1{}^2k^4\\
&-240a_0{}^3a_2k^4-2a_1{}^4kw-40a_0a_1{}^2a_2kw-28a_0{}^2a_2{}^2kw\\
&-10a_1{}^4k^4\sigma-380a_0a_1{}^2a_2k^4\sigma+160a_0{}^2a_2{}^2k^4\sigma-8a_1{}^2a_2{}^2kw\sigma\\
&-16a_0a_2{}^3kw\sigma+80a_1{}^2a_2{}^2k^4\sigma^2+16a_0a_2{}^3k^4\sigma^2\\
&+4a_2{}^4kw\sigma^2-16a_2{}^4k^4\sigma^3=0,\\
&10a_1{}^3a_2{}^2+20a_0a_1a_2{}^3-42a_0a_1{}^3k^4-432a_0{}^2a_1a_2k^4-12a_1{}^3a_2kw\\
&-52a_0a_1a_2{}^2kw-90a_1{}^3a_2k^4\sigma\\
&+40a_0a_1a_2{}^2k^4\sigma-4a_1a_2{}^3kw\sigma+40a_1a_2{}^3k^4\sigma^2=0,\\
&10a_1{}^2a_2{}^3+5a_0a_2{}^4-9a_1{}^4k^4-294a_0a_1{}^2a_2k^4-120a_0{}^2a_2{}^2k^4\\
&-22a_1{}^2a_2{}^2kw-20a_0a_2{}^3kw-20a_1{}^2a_2{}^2k^4\sigma+80a_0a_2{}^3k^4\sigma=0,\\
&5a_1a_2{}^4-72a_1{}^3a_2k^4-192a_0a_1a_2{}^2k^4-16a_1a_2{}^3kw+40a_1a_2{}^3k^4\sigma=0,\\
&a_2{}^5-78a_1{}^2a_2{}^2k^4-24a_0a_2{}^3k^4-4a_2{}^4kw+16a_2{}^4k^4\sigma=0,\\
&-24a_1a_2{}^3k^4=0.
\end{aligned}$$

Solving the above systems yields

$$a_0=0,\ a_1=0,\ a_2=\pm\frac{1}{\sigma},\ w=\frac{\pm1+16k^4\sigma^2}{4k\sigma}. \tag{16}$$

Hence using the equations (16), (15), (11), (2) and (6), analytical solutions of the MkdV-Sine-Gordon can be obtained as follows.

$$u_{1,2}(x,t)=arc\cos\left(\mp\frac{1}{2}\coth\left(\sqrt{-\sigma}\left(kx+\frac{t(\pm1+16k^4\sigma^2)}{4k\sigma}\right)\right)^2\right.$$
$$\left.\mp\frac{1}{2}\tanh\left(\sqrt{-\sigma}\left(kx+\frac{t(\pm1+16k^4\sigma^2)}{4k\sigma}\right)\right)^2\right),$$

$$u_{3,4}(x,t) = arccos\left(\pm\frac{1}{2}\cot\left(\sqrt{\sigma}\left(kx+\frac{t(\pm1+16k^4\sigma^2)}{4k\sigma}\right)\right)^2\right.$$
$$\left.\pm\frac{1}{2}\tan\left(\sqrt{\sigma}\left(kx+\frac{t(\pm1+16k^4\sigma^2)}{4k\sigma}\right)\right)^2\right).$$

3 Conclusion

In this study the sub-equation method is employed to get the new exact solutions of MkdV-Sine-Gordon equation which describes the propagation of few-cycle-pulse optical solitons in Kerr media. Also this equation was studied in the supersonic motion of a crowdion, nonlinear wave propagation in an infinite one-dimensional non-atomic lattice. The results show that considered method is a reliable, applicable and efficient technique for obtaining the solutions of NPDEs. We believe our manuscript is very timely and will interest the broad range of scientists who study on analytical solutions of nonlinear partial differential equations.

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