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Inextensible Flows of Spacelike Curves in De-Sitter Space $\mathbb{S}^{2,1}$

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Abstract: In this paper we study the relations between certain integrable equations and geometric motion of spacelike and timelike curves in 3-dimensional de-Sitter space $\mathbb{S}^{2,1}$. We give the associated evolution equations for curvature and torsion as a system of partial differential equations. In addition, we study inextensible flows of both spacelike and timelike curves in $S^{2,1}$, and we get necessary and sufficient conditions for the flows of those curves to be inextensible. We give explicit examples of the motion of inextensible spacelike curves in $\mathbb{S}^{2,1}$ and we determine the curves from their intrinsic equations (curvature and torsion), and then determine the surfaces that are generated by the motion of these curves and draw these surfaces in de-Sitter space $\mathbb{S}^{2,1}$ by using the hollow ball model.

Keywords: inextensible spacelike curves, inextensible flows, motion of curves, curves in de-Sitter space

1 Introduction

The connection between integrable equations (soliton equations) and the geometric motions of curves in spaces has been studied for a long time by many authors in different geometries. Applications of the motion of curves and surfaces in many areas, in applied sciences such as dynamics of vortex filaments, image processing, motions of interfaces, shape control of robot arms, propagation of flame front, supercoiled DNAs, magnetic fluxes, dynamics of proteins and deformation of membranes,

Schief and Rogers [1], studied the binormal motions of curves of constant curvature or torsion. Recently, Nassar et al [2,3,4,5], constructed new geometrical models of motion of plane curves. Also, they constructed Hashimoto surfaces from its fundamental coefficients via numerical integration of Gauss-Weingarten equations and fundamental theorem of surfaces. In addition, they derived the equations of motion for a general helix curve ($\tau = \beta k$). Moreover, they studied the kinematics of moving generalized curves in a n-dimensional Euclidean space in terms of intrinsic geometries.

Samah [6], studied the motions of inextensible curves in spherical space $\mathbb{S}^3$.

Rawya and Samah [7], studied the motion of curves in 3-dimensional Euclidean space $\mathbb{R}^3$. They gave new explicit examples of motions of inextensible curves in $\mathbb{R}^3$.

2 Preliminaries

The Minkowski space, or Lorentz space $\mathbb{R}^{3,1}$, is defined as a four-dimensional $\mathbb{R}$-vector space consisting of vectors $\{X = (x_0, x_1, x_2, x_3) \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}$, with the metric

$$g = dx_1^2 + dx_2^2 + dx_3^2 - dx_0^2.$$

Definition 1. *Let X, Y, Z be vectors in $\mathbb{R}^{3,1}$, where $X = (x_0, x_1, x_2, x_3)$, $Y = (y_0, y_1, y_2, y_3)$ and $Z = (z_0, z_1, z_2, z_3)$. The inner product is defined by*

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_0 y_0.$$

The pseudo vector product of X, Y and Z is defined as

$$X \times Y \times Z = \det \begin{pmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{pmatrix},$$

where $e_0 = (1,0,0,0), e_1 = (0,1,0,0), e_2 = (0,0,1,0)$ and $e_3 = (0,0,0,1)$.

Definition 2. *An arbitrary nonzero vector $v \in \mathbb{R}^{3,1}$ is*

- *Spacelike if $\langle v, v \rangle > 0$.*
- *Timelike if $\langle v, v \rangle < 0$.*
- *Null (lightlike) if $\langle v, v \rangle = 0$.*

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The signature of a vector v is

$$sign(v)=\begin{cases}1 & v \text{ is spacelike},\\ 0 & v \text{ is lightlike},\\ -1 & v \text{ is timelike}.\end{cases}$$

The norm of the vector v is $\|v\|=\sqrt{|\langle v,v\rangle|}$.

Definition 3. *The* $3-$*dimensional de-Sitter space* $\mathbb{S}^{2,1}$ *is defined by*

$$\mathbb{S}^{2,1}=\{(x_0,x_1,x_2,x_3)\in\mathbb{R}^{3,1}\mid\sum_{j=1}^{3}x_j^2-x_0^2=1\}.$$

The set of null vectors of $\mathbb{R}^{3,1}$ *forms the light cone*

$$L^3=\{(x_0,x_1,x_2,x_3)\mid x_0^2=x_1^2+x_2^2+x_3^2, x_0\neq 0\}.$$

Definition 4. *To visualize surfaces in de-Sitter space* $\mathbb{S}^{2,1}$, *we use the hollow ball model of* $\mathbb{S}^{2,1}$ *(it is a* $3-$*dimensional ball in* $\mathbb{R}^3$*), as in [9]. For any point* $(x_0,x_1,x_2,x_3)\in\mathbb{S}^{2,1}\longrightarrow\begin{pmatrix}x_0+x_3 & x_1+ix_2\\ x_1-ix_2 & x_0-x_3\end{pmatrix}$, *define*

$$y_k=e^{\arctan x_0}\frac{x_k}{\sqrt{1+x_0^2}},\qquad k=1,2,3.$$

Then $e^{-\pi}<y_1^2+y_2^2+y_3^2<e^{\pi}$. *The identification* $(x_0,x_1,x_2,x_3)\longleftrightarrow(y_1,y_2,y_3)$ *is then a bijection from* $\mathbb{S}^{2,1}$ *to the hollow ball*

$$\mathscr{H}=\{(y_1,y_2,y_3)\in\mathbb{R}^3\mid e^{-\pi}<y_1^2+y_2^2+y_3^2<e^{\pi}\}.$$

So $\mathbb{S}^{2,1}$ *is identified with the hollow ball* $\mathscr{H}$.

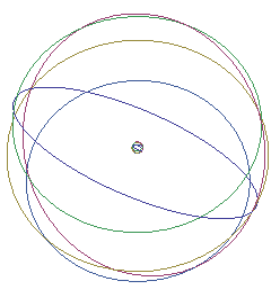

Fig. 1: Hollow ball model.

3 The differential geometry of curves in $\mathbb{S}^{2,1}$

Definition 5. *Consider the* $3-$*dimensional de-Sitter space* $\mathbb{S}^{2,1}$ *in* $\mathbb{R}^{3,1}$. *A regular parametrized curve* $\hat{\gamma}=\hat{\gamma}(u)$, $\hat{\gamma}: I\longrightarrow\mathbb{S}^{2,1}$ *is called:*

- *Spacelike if* $\langle\dot{\hat{\gamma}},\dot{\hat{\gamma}}\rangle>0$.
- *Timelike if* $\langle\dot{\hat{\gamma}},\dot{\hat{\gamma}}\rangle<0$.
- *Null (light-like) if* $\langle\dot{\hat{\gamma}},\dot{\hat{\gamma}}\rangle=0$.

for all $\hat{u}\in I$, *where* $\hat{u}$ *is the parameter of the curve* $\hat{\gamma}$ *and* $\dot{\hat{\gamma}}(\hat{u})$ *is the tangent vector to the curve* $\hat{\gamma}$ *and* $\dot{}=\frac{d}{d\hat{u}}$ *(see [8]).*

Definition 6. *Let* $\hat{\gamma}(\hat{s}(\hat{u})): I\to\mathbb{S}^{2,1}$ *be a regular spacelike or timelike curve in de-Sitter space* $\mathbb{S}^{2,1}$. *The arc-length of a spacelike or timelike curve* $\hat{\gamma}$ *with arbitrary parameter* $\hat{u}\in I$ *measured from* $\hat{\gamma}(0)$, $0\in I$ *is*

$$\hat{s}(\hat{u})=\int_0^{\hat{u}}\|\dot{\hat{\gamma}}(\hat{\sigma})\|d\hat{\sigma}.$$

Since $\hat{\gamma}$ *is regular, then we define* $\hat{g}>0$ *by* $\frac{d\hat{s}}{d\hat{u}}=\|\dot{\hat{\gamma}}\|=\sqrt{\hat{g}}$.

Definition 7. *If* $\|\dot{\hat{\gamma}}\|=1$ *for all* $\hat{u}\in I$, *then* $\hat{\gamma}=\hat{\gamma}(\hat{s})$ *is said to be an arc-length parametrized or unit speed parametrized curve.*

Now assume that the curve $\hat{\gamma}$ is parametrized by arc-length. Assume that $\langle\hat{\gamma}''(\hat{s}),\hat{\gamma}''(\hat{s})\rangle\neq 1$, where $'=\frac{d}{d\hat{s}}$. Let $\{\hat{\gamma},\hat{T},\hat{N},\hat{B}\}$ be the Serret-Frenet frame of the curve $\hat{\gamma}$, where $\hat{\gamma}(\hat{s})$ is the position vector of the curve $\hat{\gamma}$ and $\hat{T}$, $\hat{N}$ and $\hat{B}$ are, respectively, the unit tangent, unit principal normal and unit binormal vector field to the curve $\hat{\gamma}(\hat{s})$.

Definition 8. *The Frenet frame in* $\mathbb{S}^{2,1}$ *is defined by*

- $\langle\hat{\gamma},\hat{\gamma}\rangle=1$, *since the curve is in* $\mathbb{S}^{2,1}$.
- $\varepsilon_1=sign(\hat{T})$, $\varepsilon_2=sign(\hat{T}'+\varepsilon_1\hat{\gamma})$, *where* $\hat{\gamma}'=\hat{T}$, *and the constants* $\varepsilon_1,\varepsilon_2$ *are called the first and second causal characters of the curve* $\hat{\gamma}$.
- $\hat{B}$ *is chosen so that* $\{\hat{\gamma},\hat{T},\hat{N},\hat{B}\}$ *is an oriented orthonormal basis of* $\mathbb{R}^{3,1}$, *so* $\hat{B}=\hat{\gamma}\times\hat{T}\times\hat{N}$.

Definition 9. *The unit normal vector to the spacelike or timelike curve* $\hat{\gamma}(\hat{s})$ *is defined by*

$$\hat{N}=\frac{\hat{T}'+\varepsilon_1\hat{\gamma}}{\|\hat{T}'+\varepsilon_1\hat{\gamma}\|}.$$

Lemma 1. *The inner product and the vector product are given by:*

- $\langle\hat{\gamma},\hat{T}\rangle=\langle\hat{\gamma},\hat{N}\rangle=\langle\hat{\gamma},\hat{B}\rangle=\langle\hat{T},\hat{N}\rangle=\langle\hat{T},\hat{B}\rangle=\langle\hat{N},\hat{B}\rangle=0.$
- $\hat{B}\times\hat{T}\times\hat{N}=\varepsilon_1\varepsilon_2\hat{\gamma},\ \hat{\gamma}\times\hat{B}\times\hat{N}=\varepsilon_2\hat{T},\ \hat{\gamma}\times\hat{T}\times\hat{B}=\varepsilon_1\hat{N}.$

Lemma 2. *The inner product for the unit principal normal vector* $\hat{N}$ *and the unit binormal vector* $\hat{B}$ *can be determined in terms of* ε_1 *and* ε_2 *as follows:*

$$\langle\hat{N},\hat{N}\rangle=\varepsilon_2,\qquad\langle\hat{B},\hat{B}\rangle=-\varepsilon_1\varepsilon_2.$$

Definition 10. *The curvature of the spacelike or timelike curve* $\hat{\gamma}(\hat{s})$ *is defined by*

$$\hat{k}=\sqrt{\varepsilon_2\langle\hat{T}'+\varepsilon_1\hat{\gamma},\hat{T}'+\varepsilon_1\hat{\gamma}\rangle},$$

or

$$\hat{k}=\sqrt{|\langle\hat{T}'+\varepsilon_1\hat{\gamma},\hat{T}'+\varepsilon_1\hat{\gamma}\rangle|}=\|\hat{T}'+\varepsilon_1\hat{\gamma}\|.$$

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Lemma 3.
$$\hat{k} = \varepsilon_2 \langle \hat{T}' + \varepsilon_1 \hat{\gamma}, \hat{N} \rangle. \tag{1}$$

Definition 11. *The torsion $\hat{\tau}(\hat{s})$ of the spacelike or timelike curve $\hat{\gamma}(\hat{s})$ is defined by:*

$$\hat{\tau} = -\varepsilon_1 \varepsilon_2 \langle \hat{N}', \hat{B} \rangle. \tag{2}$$

Lemma 4.
$$\hat{\tau} = \frac{\varepsilon_1 \varepsilon_2}{\hat{k}^2} \det(\hat{\gamma}, \hat{\gamma}', \hat{\gamma}'', \hat{\gamma}'''). \tag{3}$$

Lemma 5. *The Frenet Frame for the curve in $\mathbb{S}^{2,1}$ satisfies the following:*
$$\hat{F}_{\hat{s}} = \hat{M} \cdot \hat{F}, \tag{4}$$

where $\hat{F} = \begin{pmatrix} \hat{\gamma} \\ \hat{T} \\ \hat{N} \\ \hat{B} \end{pmatrix}$ *and* $\hat{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\varepsilon_1 & 0 & \hat{k} & 0 \\ 0 & -\varepsilon_1\varepsilon_2\hat{k} & 0 & \hat{\tau} \\ 0 & 0 & \varepsilon_1\hat{\tau} & 0 \end{pmatrix}$.

Proof. The vectors $\{\hat{\gamma}', \hat{T}', \hat{N}', \hat{B}'\}$ can be uniquely decomposed as follows:

$$\begin{aligned} \hat{\gamma}' &= \hat{T}, \\ \hat{T}' &= a_{11}\hat{\gamma} + a_{12}\hat{T} + a_{13}\hat{N} + a_{14}\hat{B}, \\ \hat{N}' &= a_{21}\hat{\gamma} + a_{22}\hat{T} + a_{23}\hat{N} + a_{24}\hat{B}, \\ \hat{B}' &= a_{31}\hat{\gamma} + a_{32}\hat{T} + a_{33}\hat{N} + a_{34}\hat{B}. \end{aligned} \tag{5}$$

We will compute the coefficients a_{ij}, where $i = 1,2,3$ and $j = 1,2,3,4$.
Since $\langle \hat{T}, \hat{T} \rangle = \varepsilon_1$, then $\langle \hat{T}', \hat{T} \rangle = 0$. By the inner product of the second equation of (5) with $\hat{T}$, we have

$$\langle \hat{T}', \hat{T} \rangle = a_{12}\varepsilon_1,$$

then $a_{12} = 0$. Similarly, $a_{23} = a_{34} = 0$.
Using the first property of Lemma (1), then we have

$$a_{11} = -\varepsilon_1, \qquad a_{21} = 0, \qquad a_{31} = 0, \qquad a_{22} = -\varepsilon_1\varepsilon_2 a_{13},$$

$$a_{32} = \varepsilon_2 a_{14}, \qquad a_{33} = \varepsilon_1 a_{24}.$$

Since $\hat{B}' \perp \hat{B}$, and also $\hat{B}' \perp \hat{T}$, hence $a_{14} = 0$. Using (1), then we have $a_{13} = \hat{k}$. Using (2), so we have

$$a_{24} = \hat{\tau}.$$

Hence, the lemma holds.

Lemma 6. *Consider the spacelike or timelike curve $\hat{\gamma}(\hat{s}(\hat{u}))$ with arbitrary parameter $\hat{u} \in I$. Then the Serret-Frenet frame satisfies*

$$\hat{F}_{\hat{u}} = \sqrt{\hat{g}} \hat{M} \cdot \hat{F}, \tag{6}$$

where $\hat{M}$ and $\hat{F}$ are given as in (4).

4 Motions of spacelike or timelike curves in $\mathbb{S}^{2,1}$

Let $\hat{\gamma}_0 : I \to \mathbb{S}^{2,1}$ be a regular spacelike or timelike curve in $\mathbb{S}^{2,1}$. Consider the family of curves $\hat{C}_t : \hat{\gamma}(\hat{s},t)$, where $\hat{\gamma}(\hat{s},t) : I \times [0,\infty) \longrightarrow \mathbb{S}^{2,1}$, with initial curve $\hat{\gamma}_0 = \hat{\gamma}(\hat{s},0)$. $\hat{\gamma}(\hat{s},t)$ denotes the position vector of a point on the spacelike or timelike curve at time t and $\hat{s}$ is an arc-length parameter for the curve. The time parameter t is the parameter for the deformation $\hat{C}_t$ of the curve.
The arc-length of a spacelike or timelike curve $\hat{C}_t$ is defined as

$$\hat{s}(\hat{u},t) = \int_0^{\hat{u}} \sqrt{\hat{g}(\hat{\sigma},t)}\, d\hat{\sigma},$$

where $\sqrt{\hat{g}} = \|\dot{\hat{\gamma}}(\hat{\sigma},t)\|$. Thus the element of arc-length is $d\hat{s} = \sqrt{\hat{g}(\hat{u},t)}\, d\hat{u}$, and

$$\frac{\partial}{\partial \hat{s}} = \frac{1}{\sqrt{\hat{g}}} \frac{\partial}{\partial \hat{u}}, \qquad \frac{\partial \hat{s}}{\partial \hat{u}} = \sqrt{\hat{g}}.$$

The time evolution of the curve or the curve flow specified by the velocity field

$$\frac{\partial \hat{\gamma}}{\partial t} = \hat{W}\hat{T} + \hat{U}\hat{N} + \hat{V}\hat{B}, \tag{7}$$

where $\{\hat{\gamma}, \hat{T}, \hat{N}, \hat{B}\}$ is the orthonormal Frenet frame to the curve $\hat{C}_t$, and $\hat{W}$, $\hat{U}$ and $\hat{V}$ are the velocity vectors in the direction of $\hat{T}$, $\hat{N}$ and $\hat{B}$ respectively. These velocities are functions of the curvature $\hat{k}(\hat{s},t)$, torsion $\hat{\tau}(\hat{s},t)$ of the curve, and the derivatives of $\hat{k}(\hat{s},t)$, $\hat{\tau}(\hat{s},t)$.

Remark.

- The derivatives with respect to $\hat{u}$ and t commute,

$$\frac{\partial}{\partial \hat{u}} \frac{d}{dt} = \frac{d}{dt} \frac{\partial}{\partial \hat{u}}.$$

- The derivatives with respect to $\hat{s}$ and t in general do not commute,

$$\frac{\partial}{\partial \hat{s}} \frac{d}{dt} = \frac{d}{dt} \frac{\partial}{\partial \hat{s}} + \frac{\hat{g}_t}{2\hat{g}} \frac{\partial}{\partial \hat{s}}. \tag{8}$$

5 Main results

Theorem 1. *The time evolution of the Serret-Frenet frame can be given in matrix form as follows:*

$$\hat{F}_t = \hat{Q} \cdot \hat{F}, \tag{9}$$

where

$$\hat{F} = \begin{pmatrix} \hat{\gamma} \\ \hat{T} \\ \hat{N} \\ \hat{B} \end{pmatrix} \quad \text{and} \quad \hat{Q} = \begin{pmatrix} 0 & \hat{W} & \hat{U} & \hat{V} \\ -\varepsilon_1\hat{W} & 0 & \hat{f}_1 & \hat{f}_2 \\ -\varepsilon_2\hat{U} & -\varepsilon_1\varepsilon_2\hat{f}_1 & 0 & \hat{\xi} \\ \varepsilon_1\varepsilon_2\hat{V} & \varepsilon_2\hat{f}_2 & \varepsilon_1\hat{\xi} & 0 \end{pmatrix}.$$

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The time evolution of the curvature and torsion of the curve $\hat{C}_t$ can be written as follows

$$\begin{pmatrix}\hat{k}\\ \hat{\tau}\end{pmatrix}_t = \begin{pmatrix} -\frac{\hat{g}_t}{2\hat{g}} & \varepsilon_1 \hat{f}_2 \\ \varepsilon_1\varepsilon_2 \hat{f}_2 & -\frac{\hat{g}_t}{2\hat{g}}\end{pmatrix}\begin{pmatrix}\hat{k}\\ \hat{\tau}\end{pmatrix} + \begin{pmatrix}\varepsilon_1 \hat{U} + \hat{f}_{1\hat{s}} \\ \hat{\xi}_{\hat{s}}\end{pmatrix}, \tag{10}$$

where

$$\hat{f}_1 = \hat{k}\hat{W} + \hat{U}_{\hat{s}} + \varepsilon_1 \hat{\tau}\hat{V},$$
$$\hat{f}_2 = \hat{V}_{\hat{s}} + \hat{\tau}\hat{U},$$
$$\hat{\xi} = \frac{1}{\hat{k}}\left(\varepsilon_1 \hat{V} + \frac{\partial \hat{f}_2}{\partial \hat{s}} + \hat{f}_1\hat{\tau}\right).$$

Proof. Take the $\hat{u}$ derivative of (7), then

$$\hat{\gamma}_{t\hat{u}} = \sqrt{\hat{g}}\Big((-\varepsilon_1\hat{W})\hat{\gamma} + (\frac{\partial \hat{W}}{\partial \hat{s}} - \varepsilon_1\varepsilon_2\hat{k}\hat{U})\hat{T} + (\hat{k}\hat{W} + \hat{U}_{\hat{s}} + \varepsilon_1\hat{\tau}\hat{V})\hat{N} + (\hat{\tau}\hat{U} + \hat{V}_{\hat{s}})\hat{B}\Big).$$

Choose $\hat{k}\hat{W} + \hat{U}_{\hat{s}} + \varepsilon_1\hat{\tau}\hat{V} = \hat{f}_1$ and $\hat{V}_{\hat{s}} + \hat{\tau}\hat{U} = \hat{f}_2$, so

$$\hat{\gamma}_{t\hat{u}} = \sqrt{\hat{g}}\Big((-\varepsilon_1\hat{W})\hat{\gamma} + (\frac{\partial \hat{W}}{\partial \hat{s}} - \varepsilon_1\varepsilon_2\hat{k}\hat{U})\hat{T} + \hat{f}_1\hat{N} + \hat{f}_2\hat{B}\Big). \tag{11}$$

Since $\hat{\gamma}_{\hat{u}} = \sqrt{\hat{g}}\hat{\gamma}_{\hat{s}} = \sqrt{\hat{g}}\hat{T}$, then

$$\hat{\gamma}_{\hat{u}t} = \sqrt{\hat{g}}\hat{T}_t + \frac{\hat{g}_t}{2\sqrt{\hat{g}}}\hat{T}. \tag{12}$$

Since the derivatives with respect to $\hat{u}$ and t commute, then

$$\hat{\gamma}_{\hat{u}t} = \hat{\gamma}_{t\hat{u}}. \tag{13}$$

Substituting from (11) and (12) into (13), then

$$\frac{\partial \hat{g}}{\partial t} = 2\hat{g}(\frac{\partial \hat{W}}{\partial \hat{s}} - \varepsilon_1\varepsilon_2\hat{k}\hat{U}), \tag{14}$$
$$\frac{\partial \hat{T}}{\partial t} = -\varepsilon_1\hat{W}\hat{\gamma} + \hat{f}_1\hat{N} + \hat{f}_2\hat{B}. \tag{15}$$

To compute the time evolution equations for the curvature $\hat{k}$ and the unit normal vector $\hat{N}$ to the curve $\hat{C}_t$, we take the $\hat{u}$ derivative of the second equation of (14), then

$$\hat{T}_{t\hat{u}} = \sqrt{\hat{g}}\Big((-\varepsilon_1\hat{W}_{\hat{s}})\hat{\gamma} - \varepsilon_1(\hat{W} + \varepsilon_2\hat{k}\hat{f}_1)\hat{T} + (\frac{\partial \hat{f}_1}{\partial \hat{s}} + \varepsilon_1\hat{f}_2\hat{\tau})\hat{N} + (\frac{\partial \hat{f}_2}{\partial \hat{s}} + \hat{f}_1\hat{\tau})\hat{B}\Big). \tag{16}$$

Since

$$\hat{T}_{\hat{u}} = \sqrt{\hat{g}}\hat{T}_{\hat{s}} = \sqrt{\hat{g}}(-\varepsilon_1\hat{\gamma} + \hat{k}\hat{N}).$$

Taking the t derivative of this equation, then we have

$$\hat{T}_{\hat{u}t} = \sqrt{\hat{g}}\Big(\hat{k}\hat{N}_t - (\varepsilon_1\frac{\hat{g}_t}{2\hat{g}})\hat{\gamma} - (\varepsilon_1\hat{W})\hat{T} + (\hat{k}_t + \hat{k}\frac{\hat{g}_t}{2\hat{g}} - \varepsilon_1\hat{U})\hat{N} - (\varepsilon_1\hat{V})\hat{B}\Big). \tag{17}$$

Since

$$\hat{T}_{t\hat{u}} = \hat{T}_{\hat{u}t} \tag{18}$$

Substitute from (16) and (17) in (18) and put

$$\hat{\xi} = \frac{1}{\hat{k}}\left(\varepsilon_1\hat{V} + \frac{\partial \hat{f}_2}{\partial \hat{s}} + \hat{f}_1\hat{\tau}\right),$$

then

$$\frac{\partial \hat{k}}{\partial t} = \varepsilon_1\hat{U} + \frac{\partial \hat{f}_1}{\partial \hat{s}} + \varepsilon_1\hat{\tau}\hat{f}_2 - \frac{\hat{g}_t}{2\hat{g}}\hat{k}, \tag{19}$$
$$\frac{\partial \hat{N}}{\partial t} = (-\varepsilon_2\hat{U})\hat{\gamma} - \varepsilon_1\varepsilon_2\hat{f}_1\hat{T} + \hat{\xi}\hat{B}. \tag{20}$$

The time evolution equation for the unit binormal vector $\hat{B}$ to the curve $\hat{C}_t$ is given as follows:
Since $\hat{B} = \hat{\gamma}\times\hat{T}\times\hat{N}$, so

$$\hat{B}_t = \hat{\gamma}_t\times\hat{T}\times\hat{N} + \hat{\gamma}\times\hat{T}_t\times\hat{N} + \hat{\gamma}\times\hat{T}\times\hat{N}_t. \tag{21}$$

Substitute from (7) and the second equation of both (14) and (19) into (21), then

$$\frac{\partial \hat{B}}{\partial t} = \varepsilon_1\varepsilon_2\hat{V}\hat{\gamma} + \varepsilon_2\hat{f}_2\hat{T} + \varepsilon_1\hat{\xi}\hat{N}. \tag{22}$$

Take the $\hat{u}$ derivative of (22), then

$$\hat{B}_{t\hat{u}} = \sqrt{\hat{g}}\Big(-(\varepsilon_1\varepsilon_2\hat{\tau}\hat{U})\hat{\gamma} + \varepsilon_2(\varepsilon_1\hat{V} + \frac{\partial \hat{f}_2}{\partial \hat{s}} - \hat{k}\hat{\xi})\hat{T} + (\varepsilon_2\hat{k}\hat{f}_2 + \varepsilon_1\hat{\xi}_{\hat{s}})\hat{N} + (\varepsilon_1\hat{\xi}\hat{\tau})\hat{B}\Big). \tag{23}$$

Since

$$\hat{B}_{\hat{u}} = \sqrt{\hat{g}}\hat{B}_{\hat{s}} = \sqrt{\hat{g}}(\varepsilon_1\hat{\tau}\hat{N}).$$

Taking the t derivative of this equation, then we have

$$\hat{B}_{\hat{u}t} = \varepsilon_1\sqrt{\hat{g}}(\frac{\hat{g}_t}{2\hat{g}}\hat{\tau} + \hat{\tau}_t)\hat{N} + (\varepsilon_1\sqrt{\hat{g}}\hat{\tau})\hat{N}_t. \tag{24}$$

Since $\hat{B}_{t\hat{u}} = \hat{B}_{\hat{u}t}$, using (23) and (24), then we have the time evolution equation for the torsion $\hat{\tau}$ of the curve $\hat{C}_t$

$$\frac{\partial \hat{\tau}}{\partial t} = \varepsilon_1\varepsilon_2\hat{k}\hat{f}_2 + \hat{\xi}_{\hat{s}} - \frac{\hat{g}_t}{2\hat{g}}\hat{\tau}. \tag{25}$$

Theorem 2. *Consider the Serret-Frenet matrix $\hat{F}$ that satisfies (6) and (9), then we have the integrability condition:*

$$\hat{M}_t - \hat{Q}_{\hat{s}} + [\hat{M},\hat{Q}] = \frac{-\hat{g}_t}{2\hat{g}}\hat{M}, \tag{26}$$

where $[\hat{M},\hat{Q}] = \hat{M}\cdot\hat{Q} - \hat{Q}\cdot\hat{M}$ is the Lie bracket.

Proof. Since

$$\hat{F}_{\hat{u}t} = \hat{F}_{t\hat{u}}. \tag{27}$$

Differentiating (6) with respect to t, then

$$\hat{F}_{\hat{u}t} = \sqrt{\hat{g}}\left(\hat{M}_t + \hat{M}\cdot\hat{Q} + \frac{\hat{g}_t}{2\hat{g}}\hat{M}\right)\cdot\hat{F}. \tag{28}$$

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Differentiating (9) with respect to $\hat{u}$, then

$$\hat{F}_{t\hat{u}} = \sqrt{\hat{g}}\Big(\hat{Q}_{\hat{s}} + \hat{Q}\cdot\hat{M}\Big)\cdot\hat{F}. \tag{29}$$

Substitute from (28) and (29) into (27), then the theorem holds.

Lemma 7. *If the integrability condition (26) is satisfied, then we have the PDE system (10).*

Proof. Since

$$\hat{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\varepsilon_1 & 0 & \hat{k} & 0 \\ 0 & -\varepsilon_1\varepsilon_2\hat{k} & 0 & \hat{\tau} \\ 0 & 0 & \varepsilon_1\hat{\tau} & 0 \end{pmatrix},$$

and

$$\hat{Q} = \begin{pmatrix} 0 & \hat{W} & \hat{U} & \hat{V} \\ -\varepsilon_1\hat{W} & 0 & \hat{f}_1 & \hat{f}_2 \\ -\varepsilon_2\hat{U} & -\varepsilon_1\varepsilon_2\hat{f}_1 & 0 & \hat{\xi} \\ \varepsilon_1\varepsilon_2\hat{V} & \varepsilon_2\hat{f}_2 & \varepsilon_1\hat{\xi} & 0 \end{pmatrix}.$$

Then

$$\hat{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{k}_t & 0 \\ 0 & -\varepsilon_1\varepsilon_2\hat{k}_t & 0 & \hat{\tau}_t \\ 0 & 0 & \varepsilon_1\hat{\tau}_t & 0 \end{pmatrix}, \tag{30}$$

and

$$\hat{Q}_{\hat{s}} = \begin{pmatrix} 0 & \hat{W}_{\hat{s}} & \hat{U}_{\hat{s}} & \hat{V}_{\hat{s}} \\ -\varepsilon_1\hat{W}_{\hat{s}} & 0 & \hat{f}_{1\hat{s}} & \hat{f}_{2\hat{s}} \\ -\varepsilon_2\hat{U}_{\hat{s}} & -\varepsilon_1\varepsilon_2\hat{f}_{1\hat{s}} & 0 & \hat{\xi}_{\hat{s}} \\ \varepsilon_1\varepsilon_2\hat{V}_{\hat{s}} & \varepsilon_2\hat{f}_{2\hat{s}} & \varepsilon_1\hat{\xi}_{\hat{s}} & 0 \end{pmatrix}. \tag{31}$$

The Lie bracket $[\hat{M},\hat{Q}]$ is given by:

$$[\hat{M},\hat{Q}] = \begin{pmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ -\varepsilon_1\hat{a}_{12} & 0 & \hat{a}_{23} & \hat{a}_{24} \\ -\varepsilon_2\hat{a}_{13} & -\varepsilon_1\varepsilon_2\hat{a}_{23} & 0 & \hat{a}_{34} \\ \varepsilon_1\varepsilon_2\hat{a}_{14} & \varepsilon_2\hat{a}_{24} & \varepsilon_1\hat{a}_{34} & 0 \end{pmatrix}, \tag{32}$$

where

$$\hat{a}_{12} = \varepsilon_1\varepsilon_2\hat{k}\hat{U}, \quad \hat{a}_{13} = \hat{f}_1 - \hat{k}\hat{w} - \varepsilon_1\hat{\tau}\hat{V}, \quad \hat{a}_{14} = \hat{f}_2 - \hat{\tau}\hat{U},$$

$$\hat{a}_{23} = -\varepsilon_1(\hat{U} + \hat{\tau}\hat{f}_2), \quad \hat{a}_{24} = \hat{k}\hat{\xi} - \hat{\tau}\hat{f}_1 - \varepsilon_1\hat{V},$$

$$\hat{a}_{34} = -\varepsilon_1\varepsilon_2\hat{k}\hat{f}_2.$$

Substitute from (5), (30), (31), (32) into (26), then we get the PDE system (10).

6 Motions of inextensible spacelike or timelike curves in $\mathbb{S}^{2,1}$

Definition 12. *The curve $\hat{\gamma}(\hat{s},t)$ and its flow $\frac{\partial\hat{\gamma}(\hat{s},t)}{\partial t}$ in $\mathbb{S}^{2,1}$ are said to be inextensible if $\frac{\partial}{\partial t}\|\dot{\hat{\gamma}}(\hat{s},t)\| = 0$, i.e, $\hat{g}_t = 0$.*

Remark. If $c = 1$, then the curve $\hat{\gamma}(\hat{s},t)$ is a unit speed curve.

Lemma 8. *If the curve $\hat{\gamma}(\hat{s},t)$ is inextensible, then the arclength of the curve $\hat{\gamma}(\hat{s},t)$ is preserved.*

Proof. Since

$$\hat{s}(\hat{u},t) = \int_0^{\hat{u}} \|\dot{\hat{\gamma}}(\hat{\sigma},t)\| d\hat{\sigma} = \int_0^{\hat{u}} \sqrt{\hat{g}}\, d\hat{\sigma}.$$

Then the variation of the arclength is

$$\dot{\hat{s}} = \frac{\partial\hat{s}}{\partial t} = \int_0^{\hat{u}} \frac{\partial}{\partial t}\sqrt{\hat{g}}\, d\hat{\sigma} = \int_0^{\hat{u}} \frac{1}{2\sqrt{\hat{g}}}\hat{g}_t\, d\hat{\sigma}. \tag{33}$$

Since the curve is inextensible ($\hat{g}_t = 0$), then $\dot{\hat{s}} = 0$ and $\hat{s} = constant$, hence $\hat{s}$ is independent of t. So the curve has the property that its arclength is preserved.

Lemma 9. *If the curve $\hat{\gamma}(\hat{s},t)$ is inextensible, then the derivatives with respect to $\hat{s}$ and t commute.*

Lemma 10. *The curve $\hat{\gamma}(\hat{s},t)$ is inextensible if and only if*

$$\frac{\partial\hat{W}}{\partial\hat{s}} = \varepsilon_1\varepsilon_2\hat{k}\hat{U}.$$

Proof. ($\Rightarrow$) Assume that the curve is inextensible. Since

$$\hat{s}(\hat{u},t) = \int_0^{\hat{u}} \|\dot{\hat{\gamma}}(\hat{\sigma},t)\| d\hat{\sigma} = \int_0^{\hat{u}} \sqrt{\hat{g}}\, d\hat{\sigma}.$$

The variation of the arclength is

$$\dot{\hat{s}} = \frac{\partial\hat{s}}{\partial t} = \int_0^{\hat{u}} \frac{1}{2\sqrt{\hat{g}}}\hat{g}_t\, d\hat{\sigma}. \tag{34}$$

Substitute from the first equation of (14) into (34), then

$$\dot{\hat{s}} = \int_0^{\hat{u}} \left(\frac{\partial\hat{W}}{\partial\hat{s}} - \varepsilon_1\varepsilon_2\hat{k}\hat{U}\right) d\hat{s}.$$

Since the curve is inextensible, so $\dot{\hat{s}} = 0$, hence

$$\frac{\partial\hat{W}}{\partial\hat{s}} = \varepsilon_1\varepsilon_2\hat{k}\hat{U}.$$

($\Leftarrow$) Assume that

$$\frac{\partial\hat{W}}{\partial\hat{s}} = \varepsilon_1\varepsilon_2\hat{k}\hat{U}. \tag{35}$$

Substitute from (35) into the first equation of (14), so $\hat{g}_t = 0$, hence the curve is inextensible.

Lemma 11. *If the curve $\hat{\gamma}(\hat{s},t)$ is inextensible, then the integrability condition (26)(or in this case it is called the zero curvature condition) is*

$$\hat{M}_t - \hat{Q}_{\hat{s}} + [\hat{M},\hat{Q}] = 0. \tag{36}$$

Lemma 12. *If the curve $\hat{\gamma}(\hat{s},t)$ is inextensible (the zero curvature condition (36) is satisfied), then the evolution equations for curvature and torsion (10) are*

$$\begin{pmatrix} \hat{k} \\ \hat{\tau} \end{pmatrix}_t = \begin{pmatrix} 0 & \varepsilon_1\hat{f}_2 \\ \varepsilon_1\varepsilon_2\hat{f}_2 & 0 \end{pmatrix}\begin{pmatrix} \hat{k} \\ \hat{\tau} \end{pmatrix} + \begin{pmatrix} \varepsilon_1\hat{U} + \hat{f}_{1\hat{s}} \\ \hat{\xi}_{\hat{s}} \end{pmatrix}. \tag{37}$$

7 Motions of inextensible spacelike curves in $\mathbb{S}^{2,1}$

We consider the motion of inextensible spacelike curves in 3−dimensional de-Sitter space. We restrict our study to inextensible spacelike curves with spacelike principal normal vector, so in this case $\varepsilon_1 = 1$, $\varepsilon_2 = 1$ and $\frac{\partial \hat{W}}{\partial \hat{s}} = \hat{k}\hat{U}$. Then the PDE system (37) can be written explicitly in the following form:

$$\begin{aligned} \hat{k}_t &= (1+\hat{k}^2+\hat{\tau}^2)\hat{U} + \hat{U}_{\hat{s}\hat{s}} + \hat{k}_{\hat{s}}\hat{W} + \hat{\tau}_{\hat{s}}\hat{V} + 2\hat{\tau}\hat{V}_{\hat{s}}, \\ \hat{\tau}_t &= \hat{k}(\hat{V}_{\hat{s}} + \hat{\tau}\hat{U}) + \frac{\partial}{\partial \hat{s}}\bigg(\frac{1}{\hat{k}}(1+\hat{\tau}^2)\hat{V} \\ &\quad + \frac{\hat{\tau}}{\hat{k}}(\hat{k}\hat{W} + 2\hat{U}_{\hat{s}}) + \frac{1}{\hat{k}}\hat{V}_{\hat{s}\hat{s}} + \frac{\hat{\tau}_{\hat{s}}}{\hat{k}}\hat{U}\bigg). \end{aligned} \tag{38}$$

Now, we give some examples of motions of inextensible spacelike curves with spacelike principal normal vector in $\mathbb{S}^{2,1}$:

*Example 1.*If

$$\hat{W} = a, \quad \hat{U} = 0 \quad and \quad \hat{V} = \hat{k}/a. \tag{39}$$

Then (38) takes the form:

$$\begin{aligned} \hat{k}_t &= \frac{a^2+2\hat{\tau}}{a}\hat{k}_{\hat{s}} + \frac{1}{a}\hat{k}\hat{\tau}_{\hat{s}}, \\ \hat{\tau}_t &= \frac{1}{a}\hat{k}\hat{k}_{\hat{s}} + \frac{\partial}{\partial \hat{s}}\left(a\hat{\tau} + \frac{1}{a}(1+\hat{\tau}^2) + \frac{1}{a\hat{k}}\hat{k}_{\hat{s}\hat{s}}\right). \end{aligned} \tag{40}$$

One solution of this system is

$$\begin{aligned} \hat{k}(\hat{s},t) &= -2c_1 \operatorname{sech}(c_1\hat{s} + c_2 t + c_3), \\ \hat{\tau}(\hat{s},t) &= -\frac{a(ac_1 - c_2)}{2c_1}, \end{aligned} \tag{41}$$

where c_1, c_2 and c_3 are constants. Substitute from (39), (41) into (4), (9) and solve the system (4) and (9) numerically. Then we can get the family of curves $\hat{C}_t = \hat{\gamma}(\hat{s},t)$, so we can determine the surface that is generated by this family of curves. To visualize this surface in de-Sitter space $\mathbb{S}^{2,1}$, we use the hollow ball model of $\mathbb{S}^{2,1}$ (Fig.1).

*Example 2.*If

$$\hat{W} = \hat{U} = 0 \quad and \quad \hat{V} = \hat{k}. \tag{42}$$

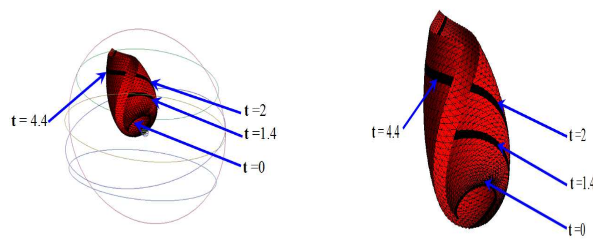
In this case, the curve moves with binormal velocity $\hat{V}$ equals the curvature of the curve, then (38) takes the form:

$$\begin{aligned} \hat{k}_t &= \hat{k}\hat{\tau}_{\hat{s}} + 2\hat{\tau}\hat{k}_{\hat{s}}, \\ \hat{\tau}_t &= \hat{k}\hat{k}_{\hat{s}} + \frac{\partial}{\partial \hat{s}}\left(1 + \hat{\tau}^2 + \frac{\hat{k}_{\hat{s}\hat{s}}}{\hat{k}}\right). \end{aligned}$$

One solution of this system is

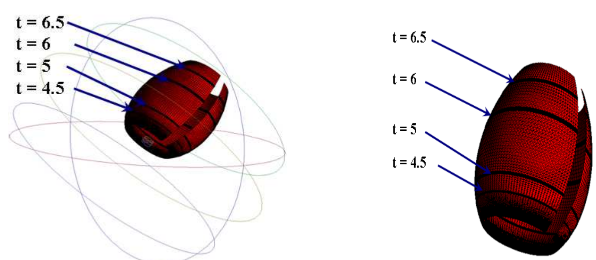
$$\hat{k}(\hat{s},t) = 2c_1 \operatorname{sech}(c_1\hat{s} + c_2 t + c_3), \quad \hat{\tau}(\hat{s},t) = \frac{c_2}{2c_1}. \tag{43}$$

where c_1, c_2 and c_3 are constants. Substitute from (42), (43) into (4), (9) and solve the system (4) and (9) numerically. Then we can get the family of curves $\hat{C}_t = \hat{\gamma}(\hat{s},t)$, so we can determine the surface that is generated by this family of curves (Fig.2).



(a) (b) Close up

Fig. 2: The surface that is generated by motion of the family of curves $\hat{C}_t$ for $\hat{s} \in [0,3]$, $t \in [0,5]$, $a = 1.02$, $c_1 = 0.3, c_2 = 0.06$ and $c_3 = -0.1$. The bold black curves in the surface represent the family of curves $\hat{C}_t$ for $t = 0, 1.4, 2, 4.4$.



(a) (b) Close up

Fig. 3: The surface that is generated by motion of the family of curves $\hat{C}_t$ for $\hat{s} \in [0,6], t \in [0,7], c_1 = 0.003, c_2 = 0.0001$ and $c_3 = 0.01$. The bold black curves in the surface represent the family of curves $\hat{C}_t$ for $t = 4.5, 5, 6, 6.5$.

7.1 *Special motions of inextensible spacelike curves in* $\mathbb{S}^{2,1}$

For special choices for velocities $\hat{W}, \hat{U}$ and $\hat{V}$, it is very difficult to solve the PDE system (38), so we will introduce

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new frame $\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4$, where

$$\begin{aligned}
\hat{E}_1 &= \hat{\gamma}, \\
\hat{E}_2 &= \hat{T}, \\
\hat{E}_3 &= \frac{1}{\sqrt{2}}(\hat{N}+\hat{B})e^{-\int_0^{\hat{s}} \hat{\tau}(\hat{s},t)d\hat{s}}, \\
\hat{E}_4 &= \frac{1}{\sqrt{2}}(\hat{N}-\hat{B})e^{\int_0^{\hat{s}} \hat{\tau}(\hat{s},t)d\hat{s}}.
\end{aligned} \tag{44}$$

And we put $\hat{r} = \frac{\hat{k}}{\sqrt{2}}e^{\int_0^{\hat{s}} \hat{\tau}(\hat{s},t)d\hat{s}}$ and $\hat{q} = \frac{\hat{k}}{\sqrt{2}}e^{-\int_0^{\hat{s}} \hat{\tau}(\hat{s},t)d\hat{s}}$.

The curvature and torsion under the transformation (44) can be given by:

$$\begin{aligned}
\hat{k} &= \sqrt{2\hat{r}\hat{q}}, \\
\hat{\tau} &= \frac{\hat{q}\hat{r}_{\hat{s}} - \hat{r}\hat{q}_{\hat{s}}}{2\hat{r}\hat{q}}.
\end{aligned} \tag{45}$$

Lemma 13. *The frame $\{\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4\}$ has the following properties:*

- $\langle \hat{E}_1, \hat{E}_1 \rangle = \langle \hat{E}_2, \hat{E}_2 \rangle = \langle \hat{E}_3, \hat{E}_4 \rangle = 1.$
- $\langle \hat{E}_1, \hat{E}_2 \rangle = \langle \hat{E}_1, \hat{E}_3 \rangle = \langle \hat{E}_1, \hat{E}_4 \rangle = 0$
- $\langle \hat{E}_2, \hat{E}_3 \rangle = \langle \hat{E}_2, \hat{E}_4 \rangle = \langle \hat{E}_3, \hat{E}_3 \rangle = \langle \hat{E}_4, \hat{E}_4 \rangle = 0.$
- $\hat{E}_1 = -\hat{E}_2 \times \hat{E}_3 \times \hat{E}_4.$
- $\hat{E}_2 = \hat{E}_1 \times \hat{E}_3 \times \hat{E}_4.$
- $\hat{E}_3 = \hat{E}_1 \times \hat{E}_2 \times \hat{E}_3.$
- $\hat{E}_4 = -\hat{E}_1 \times \hat{E}_2 \times \hat{E}_4.$

Lemma 14. *The Serret-Frenet frame (4) for spacelike curves ($\varepsilon_1 = 1$ and $\varepsilon_2 = 1$) under the transformation (44) can be given by*

$$\hat{E}_{\hat{s}} = \hat{A} \cdot \hat{E}, \tag{46}$$

$$\text{where } E = \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \\ \hat{E}_4 \end{pmatrix} \quad \text{and} \quad \hat{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \hat{r} & \hat{q} \\ 0 & -\hat{q} & 0 & 0 \\ 0 & -\hat{r} & 0 & 0 \end{pmatrix}.$$

Now the equation of motion (7) will transform to:

$$\hat{E}_{1t} = \hat{W}\hat{E}_2 + \hat{\eta}\hat{E}_3 + \hat{\delta}\hat{E}_4, \tag{47}$$

where

$$\hat{\eta} = \frac{\hat{r}}{\hat{k}}(\hat{U}+\hat{V}) \quad \text{and} \quad \hat{\delta} = \frac{\hat{q}}{\hat{k}}(\hat{U}-\hat{V}). \tag{48}$$

Theorem 3. *The time evolution equations for the frame $\{\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4\}$ can be written in matrix form*

$$\hat{E}_t = \hat{R} \cdot \hat{E}, \tag{49}$$

where $\hat{R} = \begin{pmatrix} 0 & \hat{W} & \hat{\eta} & \hat{\delta} \\ -\hat{W} & 0 & \hat{r}\hat{W}+\hat{\eta}_{\hat{s}} & \hat{q}\hat{W}+\hat{\delta}_{\hat{s}} \\ -\hat{\delta} & -(\hat{q}\hat{W}+\hat{\delta}_{\hat{s}}) & \hat{f} & 0 \\ -\hat{\eta} & -(\hat{r}\hat{W}+\hat{\eta}_{\hat{s}}) & 0 & -\hat{f} \end{pmatrix}$ *and* $\hat{f} = \hat{f}(\hat{s},t)$

Proof. Take the $\hat{u}$ derivative of (47), then

$$\begin{aligned}
\hat{E}_{1t\hat{u}} = \sqrt{\hat{g}}\Big(& (-\hat{W})\hat{E}_1 + (\hat{W}_{\hat{s}} - 2\frac{\hat{r}\hat{q}}{\hat{k}}\hat{U})\hat{E}_2 + (\hat{r}\hat{W}+\hat{\eta}_{\hat{s}})\hat{E}_3 \\
& + (\hat{q}\hat{W}+\hat{\delta}_{\hat{s}})\hat{E}_4\Big).
\end{aligned} \tag{50}$$

Since $\hat{E}_{1\hat{u}} = \sqrt{\hat{g}}\hat{E}_{1\hat{s}} = \sqrt{\hat{g}}\hat{E}_2$, then by taking the t derivative of this equation we have

$$\hat{E}_{1\hat{u}t} = \sqrt{\hat{g}}(\hat{E}_{2t} + \frac{\hat{g}_t}{2\hat{g}}\hat{E}_2). \tag{51}$$

Since $\hat{E}_{1\hat{u}t} = \hat{E}_{1t\hat{u}}$, then we get

$$\begin{aligned}
\hat{E}_{2t} &= (-\hat{W})\hat{E}_1 + (\hat{r}\hat{W}+\hat{\eta}_{\hat{s}})\hat{E}_3 + (\hat{q}\hat{W}+\hat{\delta}_{\hat{s}})\hat{E}_4, \\
\hat{g}_t &= 2\hat{g}(\hat{W}_{\hat{s}} - 2\frac{\hat{r}\hat{q}}{\hat{k}}\hat{U}).
\end{aligned} \tag{52}$$

The vectors $\hat{E}_{3t}$ and $\hat{E}_{4t}$ can be uniquely decomposed as follows:

$$\begin{aligned}
\hat{E}_{3t} &= b_{11}\hat{E}_1 + b_{12}\hat{E}_2 + b_{13}\hat{E}_3 + b_{14}\hat{E}_4, \\
\hat{E}_{4t} &= b_{21}\hat{E}_1 + b_{22}\hat{E}_2 + b_{23}\hat{E}_3 + b_{24}\hat{E}_4.
\end{aligned} \tag{53}$$

By using the the properties of the frame $\{\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4\}$ in lemma(13), then we can determined the factors b_{ij}, where $i = 1,2$ and $j = 1,2,3,4$ as follows:
Since $\langle \hat{E}_1, \hat{E}_3 \rangle = 0$, $\langle \hat{E}_2, \hat{E}_3 \rangle = 0$ and $\langle \hat{E}_3, \hat{E}_3 \rangle = 0$. Then we get respectively:

$$b_{11} = -\hat{\delta}, b_{12} = -(\hat{\delta}_{\hat{s}} + \hat{q}\hat{W}), \text{and } b_{14} = 0.$$

Hence,

$$\hat{E}_{3t} = -\hat{\delta}\hat{E}_1 - (\hat{\delta}_{\hat{s}} + \hat{q}\hat{W})\hat{E}_2 + b_{13}\hat{E}_3.$$

And since $\langle \hat{E}_1, \hat{E}_4 \rangle = 0$, $\langle \hat{E}_2, \hat{E}_4 \rangle = 0$, $\langle \hat{E}_4, \hat{E}_4 \rangle = 0$, and $\langle \hat{E}_3, \hat{E}_4 \rangle = 1$. Then we get respectively:

$$b_{21} = -\hat{\eta}, b_{22} = -\hat{\eta}_{\hat{s}} - \hat{r}\hat{W}, b_{23} = 0, \text{and } b_{24} = -b_{13}.$$

Hence

$$\hat{E}_{4t} = -\hat{\eta}\hat{E}_1 + (-\hat{\eta}_{\hat{s}} - \hat{r}\hat{W})\hat{E}_2 - b_{13}\hat{E}_4.$$

Choose

$$b_{13} = f(\hat{s},t),$$

where $f(s,t)$ will be determined by the integrability conditions. Hence the theorem holds.

Theorem 4. *Consider the Serret-Frenet matrix $\hat{E}$ that satisfies (46) and (49), then we have the integrability condition:*

$$\hat{A}_t - \hat{R}_{\hat{s}} + [\hat{A}, \hat{R}] = \frac{-\hat{g}_t}{2\hat{g}}\hat{A}, \tag{54}$$

where $[\hat{A}, \hat{R}] = \hat{A} \cdot \hat{R} - \hat{R} \cdot \hat{A}$ is the Lie bracket.

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Proof. Since

$$\hat{E}_{\hat{u}t} = \hat{E}_{t\hat{u}}. \tag{55}$$

Differentiating (46) with respect to t, then

$$\hat{E}_{\hat{u}t} = \sqrt{\hat{g}}\left(\hat{A}_t + \hat{A}\cdot\hat{R} + \frac{\hat{g}_t}{2\hat{g}}\hat{A}\right)\cdot\hat{E}. \tag{56}$$

Differentiating (49) with respect to $\hat{u}$, then

$$\hat{E}_{t\hat{u}} = \sqrt{\hat{g}}\left(\hat{R}_{\hat{s}} + \hat{R}\cdot\hat{A}\right)\cdot\hat{E}. \tag{57}$$

Substitute from (56) and (57) into (55), then the theorem holds.

Lemma 15. *If the curve $\hat{C}_t$ is inextensible, then the integrability condition (or in this case it is called the zero curvature condition) is*

$$\hat{A}_t - \hat{R}_{\hat{s}} + [\hat{A},\hat{R}] = 0. \tag{58}$$

Lemma 16. *If the integrability condition (58) is satisfied, then we have the following PDE system*

$$\begin{aligned} \hat{r}_t &= \hat{\eta}_{\hat{s}\hat{s}} + \hat{\eta} + \hat{r}_{\hat{s}}\hat{W} + \hat{r}(\hat{r}\hat{\delta} + \hat{q}\hat{\eta} - f),\\ \hat{q}_t &= \hat{\delta}_{\hat{s}\hat{s}} + \hat{\delta} + \hat{q}_{\hat{s}}\hat{W} + \hat{q}(\hat{r}\hat{\delta} + \hat{q}\hat{\eta} + f),\\ \hat{f}_{\hat{s}} &= \hat{r}\hat{\delta}_{\hat{s}} - \hat{q}\hat{\eta}_{\hat{s}}. \end{aligned} \tag{59}$$

By solving this system, then we can obtain the curvature and torsion of the curve. (45).

Example 3. If

$$\hat{W} = -\frac{1}{2}\hat{k}^2, \qquad \hat{U} = -\hat{k}_{\hat{s}} \quad and \quad \hat{V} = -\hat{k}\hat{\tau}. \tag{60}$$

Using (45), so

$$\hat{W} = -\hat{r}\hat{q}, \qquad \hat{U} = -\frac{\hat{q}\hat{r}_{\hat{s}} + \hat{r}\hat{q}_{\hat{s}}}{\sqrt{2\hat{r}\hat{q}}}, \qquad \hat{V} = \frac{\hat{q}\hat{r}_{\hat{s}} - \hat{r}\hat{q}_{\hat{s}}}{\sqrt{2\hat{r}\hat{q}}}.$$

From (48), we have $\hat{\eta} = -\hat{r}_{\hat{s}}$ and $\hat{\delta} = -\hat{q}_{\hat{s}}$, then the system (59) is

$$\begin{aligned} \hat{r}_t + \hat{r}_{\hat{s}\hat{s}\hat{s}} + \hat{r}_{\hat{s}}(1+2\hat{r}\hat{q}) + \hat{r}^2\hat{q}_{\hat{s}} + \hat{r}f &= 0,\\ \hat{q}_t + \hat{q}_{\hat{s}\hat{s}\hat{s}} + \hat{q}_{\hat{s}}(1+2\hat{r}\hat{q}) + \hat{q}^2\hat{r}_{\hat{s}} - \hat{q}f &= 0. \end{aligned} \tag{61}$$

$$\hat{f}_{\hat{s}} + \hat{r}\hat{q}_{\hat{s}\hat{s}} - \hat{q}\hat{r}_{\hat{s}\hat{s}} = 0. \tag{62}$$

Integrate (62), then

$$\hat{f} = \hat{q}\hat{r}_{\hat{s}} - \hat{r}\hat{q}_{\hat{s}} + C(t). \tag{63}$$

Substitute (63) into the (61), then

$$\begin{aligned} \hat{r}_t + \hat{r}_{\hat{s}\hat{s}\hat{s}} + \hat{r}_{\hat{s}} + 3\hat{r}\hat{q}\hat{r}_{\hat{s}} + \hat{r}C(t) &= 0,\\ \hat{q}_t + \hat{q}_{\hat{s}\hat{s}\hat{s}} + \hat{q}_{\hat{s}} + 3\hat{r}\hat{q}\hat{q}_{\hat{s}} - \hat{q}C(t) &= 0. \end{aligned} \tag{64}$$

To solve this system of PDE, we use the following transformation:

$$\begin{aligned} \hat{r} &= \alpha e^{-\int_0^t C(\hat{s})d\hat{s}},\\ \hat{q} &= \beta e^{\int_0^t C(\hat{s})d\hat{s}}. \end{aligned} \tag{65}$$

Hence

$$\begin{aligned} \hat{\alpha}_t + \hat{\alpha}_{\hat{s}\hat{s}\hat{s}} + \hat{\alpha}_{\hat{s}}(1+3\hat{\alpha}\hat{\beta}) &= 0,\\ \hat{\beta}_t + \hat{\beta}_{\hat{s}\hat{s}\hat{s}} + \hat{\beta}_{\hat{s}}(1+3\hat{\alpha}\hat{\beta}) &= 0. \end{aligned} \tag{66}$$

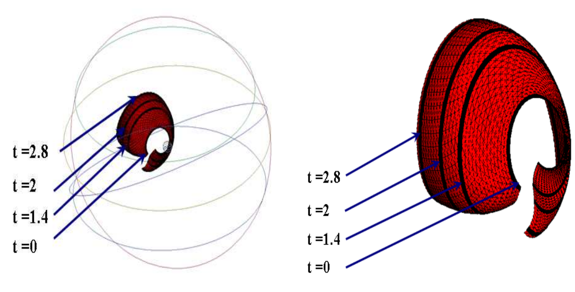
One solution of this system is

$$\begin{aligned} \hat{\alpha}(\hat{s},t) &= \frac{-\sqrt{c_1}}{3c_4}\left(-\sqrt{6c_5} + 6c_1^{3/2}\tanh(c_1\hat{s} + c_2t + c_3)\right),\\ \hat{\beta}(\hat{s},t) &= \frac{c_4}{6c_1^{3/2}}\left(\sqrt{6c_5} + 6c_1^{3/2}\tanh(c_1\hat{s} + c_2t + c_3)\right),\\ c_5 &= 2c_1^3 - c_2 - c_1. \end{aligned} \tag{67}$$

where c_1, c_2, and c_3 are constants. Using (45), (65) and (67), hence the curvature and torsion are

$$\begin{aligned} \hat{k}(\hat{s},t) &= \sqrt{\frac{2}{3c_1}}\sqrt{\hat{\rho}(\hat{s},t)},\\ \hat{\tau}(\hat{s},t) &= -\frac{c_1^{5/2}\sqrt{6c_5}}{\hat{\rho}(\hat{s},t)}\operatorname{sech}^2(c_1\hat{s} + c_2t + c_3),\\ \hat{\rho}(\hat{s},t) &= c_5 - 6c_1^3\tanh^2(c_1\hat{s} + c_2t + c_3). \end{aligned} \tag{68}$$

Substitute from (60), (68) into (4), (9) and solve the system (4) and (9) numerically. Then we can get the family of curves $\hat{C}_t$, so we can determine the surface that is generated by this family of curves (Fig.3).



(a) (b) Close up

Fig. 4: The surface that is generated by motion of the family of curves $\hat{C}_t$ for $\hat{s} \in [0,6]$, $t \in [0,3]$, $c_1 = -\frac{1}{2}, c_2 = 1$ and $c_3 = 0.1$. The bold black curves in the surface represent the family of curves $\hat{C}_t$ for $t = 0, 1.4, 2, 2.8$.

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