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A Mathematical Structure of Processes for Generating Rankings Through the Use of Nonnegative Irreducible Matrices

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In our previous study, focusing on a ranking determination, we developed two ranking models. The foundation of these ranking models is derived from either one of the two ranking methods, denoted by Ranking (I) and Ranking (II), that were proposed in our previous papers. The purpose of this paper is to analyze the mathematical structure in the process of generating Ranking (I) and Ranking (II) in detail and to study the properties of the two ranking methods.

Keywords: Ranking, Perron-Frobenius theorem, irreducible matrix.

1 Introduction

In general, a ranking is obtained through either competition or trial for a certain set of elements. Such sets, which are referred to herein as *constructed sets* and are denoted by C, include baseball teams and students in a class. The process of determining the ranking usually takes the results of data for either competition or trial into account. Authors have developed various application models to generate the ranking for a couple of different types of data [3–5]. The foundation of these ranking models is derived from either one of the two ranking methods, denoted by Ranking (I) and Ranking (II), that were proposed in our previous paper. These two ranking methods are applied to the cases that teams or individuals have same ranking in results of a sports or in score of an examination, and can determine the clear ranking even under such cases without play-off or re-examination. In the present paper, the mathematical structure in the process of generating Ranking (I) and

Ranking (II) is analyzed in detail. Through this analysis, the properties of the two ranking methods are studied.

In Section 2, a few theorems which are very important to these ranking methods are reviewed, and in Sections 3 and 4, the mathematical structures of each ranking method are given. In Section 5, the two ranking methods are compared.

2 Mathematical Foundations

In this section, two theorems and a remark that are important for analyzing the mathematical structure in the process of generating rankings are reviewed.

Theorem 2.1 (Power method $[2,7]$)**.** *Let* $\mathbf{A}$ *be a matrix and let* $\boldsymbol{x}_i$ *be an eigenvector corresponding to an eigenvalue* λ_i*. Suppose that each* λ_i $(i = 1, 2, \ldots, n)$ *satisfies following condition:*

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots |\lambda_n|.$$

Then, for any initial vector $\boldsymbol{u}_0 (\neq \mathbf{0})$*, the following iteration with respect to* k *leads to a convergence of* $\boldsymbol{u}_k$ *to* $\boldsymbol{x}_1$ *as* $k \to \infty$*:*

$$\boldsymbol{u}_k = \mathbf{A}\boldsymbol{u}_{k-1}, \quad k = 1, 2, \ldots \tag{2.1}$$

where $\boldsymbol{x}_1$ *is an eigenvector of* $\mathbf{A}$ *corresponding to the value* λ_1.

Theorem 2.2 (Perron-Frobenius Theorem $[1,9]$)**.** *Let* $\mathbf{B}$ *be a nonnegative irreducible matrix. Then, there exists a unique eigenvector* $\boldsymbol{x}_{\mathrm{B}}$ *that has all positive elements corresponding to the positive eigenvalue* λ_{B}*, where* λ_{B} *is equal to the spectral radius of* $\mathbf{B}$ *and has algebraic multiplicity* 1.

Remark 2.1. In the case of generating an eigenvector using the power method of Eq.(2.1), the elements in $\boldsymbol{u}_k$ may overflow, so such the modified iteration between $\boldsymbol{u}_k$ and $\boldsymbol{v}_k$, see Eq.(2.2), is necessary in order to avoid the overflow $\boldsymbol{u}_k$ as follows:

$$\begin{aligned} \boldsymbol{v}_k &= \mathbf{A}\boldsymbol{u}_{k-1}, \\ \boldsymbol{u}_k &= \frac{\boldsymbol{v}_k}{||\boldsymbol{v}_k||}. \end{aligned} \qquad k = 1, 2, \ldots \tag{2.2}$$

3 Ranking (I)

This section presents a detailed explanation of Ranking (I). Let

$$C = \{c(1), c(2), \ldots, c(n)\}$$

be a constructed set and let $\mathbf{M}_{(\mathrm{I})} = \{m_{(\mathrm{I})}(i,j)\}_{1 \leq i,j \leq n}$ be a matrix generated by comparing two elements in C through either competition or trial. Each element in $\mathbf{M}_{(\mathrm{I})}$ is determined in accordance with the following conditions:

Condition 1.

(3a) A matrix $\mathbf{M}_{(\mathrm{I})}$ is irreducible and primitive.
(3b) The value of $m_{(\mathrm{I})}(i,j)$ represents the nonnegative ratio of superiority of $c(i)$ over $c(j)$.
(3c) The ratio of superiority is determined depending on a common rule through either competition or trial among elements in C

From (3b) of Condition 1, no element of matrix $\mathbf{M}_{(\mathrm{I})}$ is negative, so a matrix $\mathbf{M}_{(\mathrm{I})}$ is nonnegative. A matrix $\mathbf{M}_{(\mathrm{I})}$ that satisfies Condition 1 is called *evaluation matrix (I) corresponding to* C. Then, we have the following remark and definition.

Remark 3.1. From Theorem 2.2 and Condition 1-(3a) and (3b), there exists an eigenvector $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})}} = {}^{\mathrm{T}}(x_1, x_2, \ldots, x_n)$, the elements of which are all positive, corresponding to the largest positive eigenvalue $\lambda_{\mathrm{M}_{(\mathrm{I})}}$ of $\mathbf{M}_{(\mathrm{I})}$.

Definition 3.1. The eigenvector $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})}}$, denoted in Remark 3.1, is referred to as the *ranking vector corresponding to matrix* $\mathbf{M}_{(\mathrm{I})}$ and is normalized with respect to $l_2 - norm$.

In the present study, each element in the initial vector is equal to 1 in the application of the power method. Next, the properties of each element in the ranking vector are given.

3.1 Process of generating the ranking vector for $\mathbf{M}_{(\mathrm{I})}$

In this subsection, the mathematical meaning of each element in the ranking vector is reviewed during the process of generating the ranking vector. From (3a) of Condition 1, we can generate the ranking vector for $\mathbf{M}_{(\mathrm{I})}$ by using the power method. Then, the initial vector is given as $\boldsymbol{r}_0 = {}^{\mathrm{T}}(1, 1, \ldots, 1)$ and

$$\mathbf{M}_{(\mathrm{I})}\boldsymbol{r}_0 \equiv \boldsymbol{r}_1 = {}^{\mathrm{T}}\big(r_1(1),\ r_1(2),\ \ldots,\ r_1(n)\big). \tag{3.1}$$

In Eq. (3.1), the vector $\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})}}$ is calculated as follows:

$$\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})}} = \frac{\boldsymbol{r}_1}{||\boldsymbol{r}_1||_2} = {}^{\mathrm{T}}\big(p_{[1]\mathrm{M}_{(\mathrm{I})}}(1),\ p_{[1]\mathrm{M}_{(\mathrm{I})}}(2), \ldots,\ p_{[1]\mathrm{M}_{(\mathrm{I})}}(n)\big).$$

An entry $p_{[1]\mathrm{M}_{(\mathrm{I})}}(i)$ in $\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})}}$ is referred to as the *first potential for* $c(i)$ *in constructed set C*, and $\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})}}$ is referred to as the *first potential for* $\mathbf{M}_{(\mathrm{I})}$.

Elements $p_{[1]\mathrm{M}_{(\mathrm{I})}}(i)$ $(i = 1, \ldots, n)$ in $\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})}}$ represent the total degree of superiority of $c(i)$ to other elements $c(j)$ (also involving the superiority of $c(i)$ to $c(i)$). By calculating $\mathbf{M}_{(\mathrm{I})}\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})}}$,

$$\mathbf{M}_{(\mathrm{I})}\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})}} = \begin{pmatrix} m_{(\mathrm{I})}(1,1) & m_{(\mathrm{I})}(1,2) & \cdots & m_{(\mathrm{I})}(1,n) \\ m_{(\mathrm{I})}(2,1) & m_{(\mathrm{I})}(2,2) & \cdots & m_{(\mathrm{I})}(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ m_{(\mathrm{I})}(n,1) & m_{(\mathrm{I})}(n,2) & \cdots & m_{(\mathrm{I})}(n,n) \end{pmatrix} \begin{pmatrix} p_{[1]\mathrm{M}_{(\mathrm{I})}}(1) \\ p_{[1]\mathrm{M}_{(\mathrm{I})}}(2) \\ \cdots \\ p_{[1]\mathrm{M}_{(\mathrm{I})}}(n) \end{pmatrix}$$

$$= {}^{\mathrm{T}}\left(\sum_{k=1}^{n} m_{(\mathrm{I})}(1,k)\, p_{[1]\mathrm{M}_{(\mathrm{I})}}(k), \ldots,\ \sum_{k=1}^{n} m_{(\mathrm{I})}(n,k)\, p_{[1]\mathrm{M}_{(\mathrm{I})}}(k)\right)$$

$$\equiv \boldsymbol{r}_2 = {}^{\mathrm{T}}\big(r_2(1),\ r_2(2),\ \ldots,\ r_2(n)\big). \tag{3.2}$$

In a similar manner, $\boldsymbol{p}_{[2]\mathrm{M}_{(\mathrm{I})}}$ is lead as follows:

$$\boldsymbol{p}_{[2]\mathrm{M}_{(\mathrm{I})}} = \frac{\boldsymbol{r}_2}{||\boldsymbol{r}_2||_2} = {}^{\mathrm{T}}\big(p_{[2]\mathrm{M}_{(\mathrm{I})}}(1),\ p_{[2]\mathrm{M}_{(\mathrm{I})}}(2),\ \ldots,\ p_{[2]\mathrm{M}_{(\mathrm{I})}}(n)\big), \tag{3.3}$$

and

$$\begin{aligned}
\mathbf{M}_{(\mathrm{I})}\boldsymbol{p}_{[2]\mathrm{M}_{(\mathrm{I})}} &\equiv \boldsymbol{r}_3 = {}^{\mathrm{T}}\big(r_3(1),\ r_3(2),\ \ldots,\ r_3(n)\big),\\
&\ \vdots\\
\mathbf{M}_{(\mathrm{I})}\boldsymbol{p}_{[\mathrm{n}-1]\mathrm{M}_{(\mathrm{I})}} &\equiv \boldsymbol{r}_n,\\
\boldsymbol{p}_{[\mathrm{n}]\mathrm{M}_{(\mathrm{I})}} &= \frac{\boldsymbol{r}_n}{||\boldsymbol{r}_n||_2}.
\end{aligned} \tag{3.4}$$

The value $r_2(i)$ in Eq. (3.2) is taken by the following equation

$$r_2(i) = m_{(\mathrm{I})}(i,1)p_{[1]\mathrm{M}_{(\mathrm{I})}}(1) + m_{(\mathrm{I})}(i,2)p_{[1]\mathrm{M}_{(\mathrm{I})}}(2) + \cdots + m_{(\mathrm{I})}(i,n)p_{[1]\mathrm{M}_{(\mathrm{I})}}(n), \tag{3.5}$$

for $i = 1, \ldots, n$. Thus, the value of $r_2(i)$, which has a high rate of superiority compared to $\{p_{[1]\mathrm{M}_{(\mathrm{I})}}(t)\}$ with high potentials, becomes characteristically larger than that of $r_2(j)$, which has a high rate of superiority compared to $\{p_{[1]\mathrm{M}_{(\mathrm{I})}}(s)\}$ with low potentials. Since the vector $\boldsymbol{p}_{[2]\mathrm{M}_{(\mathrm{I})}}$ in Eq.(3.3) is the second potential for $\mathbf{M}_{(\mathrm{I})}$ and is lead by normalizing $\boldsymbol{r}_2$, the characteristic for $\{r_2(i)\}$ $(i = 1, \ldots, n)$ mentioned above is similar to the characteristic for $\{p_{[2]\mathrm{M}_{(\mathrm{I})}}(i)\},\ (i = 1, \ldots, n)$. This characteristic is commonly satisfied in each stage of $\boldsymbol{p}_{[3]\mathrm{M}_{(\mathrm{I})}},\ \boldsymbol{p}_{[4]\mathrm{M}_{(\mathrm{I})}}, \cdots$. Therefore, for the vector

$$\boldsymbol{p}_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}} = {}^{\mathrm{T}}\big(p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(1),\ p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(2),\ \ldots,\ p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(n)\big),$$

we have the following property:

Property 1. The value of element $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(i)$ in $\boldsymbol{p}_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}$, which has a high rate of superiority compared to $\{p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{I})}}(t)\}$ with high potentials, becomes larger than that of element $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(j)$, which has a high rate of superiority compared to $\{p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{I})}}(s)\}$ with low potentials.

The matrix $\mathbf{M}_{(\mathrm{I})}$ is assumed to be irreducible and primitive. Then, we can generate the ranking vector $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})}}$, defined in Definition 3.1, corresponding to the largest positive eigenvalue $\lambda_{\mathrm{M}_{(\mathrm{I})}}$. The iteration process is represented by Eqs. (3.3) through (3.4) and is identical to generating the process of $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})}}$ by the power method. Therefore, we have

$$\lim_{k\to\infty} \boldsymbol{p}_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}} = \boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})}}.$$

We refer to

$$\boldsymbol{p}_{[\infty]\mathrm{M}_{(\mathrm{I})}} = \lim_{k\to\infty} \boldsymbol{p}_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}$$

as the *final potential* for $\mathbf{M}_{(\mathrm{I})}$. A vector $\boldsymbol{p}_{[\infty]\mathrm{M}_{(\mathrm{I})}}$ is generated through the successive transition of each step's potentials for all elements in C. Thus, we obtain another property for $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})}}$, as follows:

Property 2. The value of $c(i)$ in $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})}}$ is determined based on its superiority compared to other elements $\{c(j)\}$ that have relatively high potentials.

In the present paper, a ranking that is ordered according to the highest-value element in $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})}}$ is referred to as Ranking (I) for $\mathbf{M}_{(\mathrm{I})}$ in C.

3.2 Properties of the ranking vector for $\mathrm{M}_{(\mathrm{I})}$

In this subsection, we introduce the property of the ranking vector for evaluation matrix $\mathbf{M}_{(\mathrm{I})}$. First, two corollaries and a property are given. Here, elements of $\mathbf{M}_{(\mathrm{I})}$ are assumed to satisfy (3a), (3b) and (3c) of Condition 1.

Corollary 3.1. *If the first potential of each element in C is equivalent, then the elements in the ranking vector are identical.*

Proof. The proof of this corollary follows from Theorem 2.1. □

Property 3. A family of sets $C_x = \{C_x(1), C_x(2), \ldots\}$ can be constructed from elements that are divided according to potential, where each set $\{C_x(i)\}$ $(i = 1, 2, \ldots)$ is arranged in order of potential from lowest to highest. Then, for two elements $c(\alpha),\ c(\beta)$, $(\alpha \neq \beta)$, where $c(\alpha),\ c(\beta) \in C_x(i)$, if there exists at least one element $c(k) \notin C_x(i)$ such that

$$\begin{array}{c}\text{the superiority of } c(\alpha) \\ \text{compared to } c(k)\end{array} \neq \begin{array}{c}\text{the superiority of } c(\beta) \\ \text{compared to } c(k)\end{array},$$

then $c(\alpha)$ and $c(\beta)$ have different rankings (see Examples 3.1 and 3.2).

Example 3.1. The superiority relation among $C = \{c(1), c(2), c(3)\}$ is given in evaluation matrix $\mathbf{M}_{(\mathrm{I})1}$ as follows:

$$\mathbf{M}_{(\mathrm{I})1} = \begin{pmatrix} 9/10 & 3/10 & 9/10 \\ 8/10 & 5/10 & 7/10 \\ 7/10 & 5/10 & 8/10 \end{pmatrix}.$$

From simple calculus, the first potential $\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})1}}$ is

$$\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})1}} = {}^{\mathrm{T}}(0.59612, 0.567733, 0.567733),$$

so we can divide the set C into $C_x = C_x(1) \cup C_x(2)$, where $C_x(1) \ni c(1)$ and $C_x(2) \ni c(2), c(3)$. Potentials $\boldsymbol{p}_{[2]\mathrm{M}_{(\mathrm{I})1}}$ and $\boldsymbol{p}_{[3]\mathrm{M}_{(\mathrm{I})1}}$ and ranking vector $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})}}$ are

$$\begin{aligned} \boldsymbol{p}_{[2]\mathrm{M}_{(\mathrm{I})1}} &= {}^{\mathrm{T}}(0.597128, 0.567898, 0.566506), \\ \boldsymbol{p}_{[3]\mathrm{M}_{(\mathrm{I})1}} &= {}^{\mathrm{T}}(0.597109, 0.567963, 0.566462), \\ \boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})1}} &= {}^{\mathrm{T}}(0.597102, 0.567967, 0.566465). \end{aligned}$$

The values of superiority of $c(2)$ and $c(3)$ compared to $c(1)$ are different, and the first potential of $c(1)$ is the highest. In this case, the value of second potential of $c(2)$ is higher than that of $c(3)$ because the ratio of superiority of $c(2)$ is higher than that of $c(3)$ compared to $c(1)$. Then, the ranking does not change in the process of the subsequent potential transition. Finally, from elements in $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})1}}$, Ranking (I) is

$$\text{First} \cdots c(1), \quad \text{Second} \cdots c(2), \quad \text{Third} \cdots c(3).$$

Example 3.2. The superiority relation among $C = \{c(1), c(2), c(3)\}$ is assumed to be given in evaluation matrix $\mathbf{M}_{(\mathrm{I})2}$ as follows:

$$\mathbf{M}_{(\mathrm{I})2} = \begin{pmatrix} 9/10 & 3/10 & 9/10 \\ 7/10 & 6/10 & 7/10 \\ 7/10 & 5/10 & 8/10 \end{pmatrix}.$$

The first potential $\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})2}}$ is

$$\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})2}} = {}^{\mathrm{T}}(0.59612, 0.567733, 0.567733).$$

Similar to Example 3.1, set C is divided into $C_x = C_x(1) \cup C_x(2)$, where $C_x(1) \ni c(1)$, $C_x(2) \ni c(2)$, $c(3)$. However, the rate of superiority of both $c(2)$ and $c(3)$ compared to $c(1)$ is the same $7/10$, and the second and third elements are equal in $\{\boldsymbol{p}_{[\mathrm{i}]\mathrm{M}_{(\mathrm{I})2}}\}$ $(i = 1, 2, \ldots)$. As a result, a clear ranking between $c(2)$ and $c(3)$ cannot be determined in Ranking (I).

$$\begin{aligned} \boldsymbol{p}_{[2]\mathrm{M}_{(\mathrm{I})2}} &= {}^{\mathrm{T}}(0.5976, 0.566954, 0.566954), \\ \boldsymbol{p}_{[3]\mathrm{M}_{(\mathrm{I})2}} &= {}^{\mathrm{T}}(0.597718, 0.566892, 0.566892), \\ \boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})2}} &= {}^{\mathrm{T}}(0.597728, 0.566887, 0.566887). \end{aligned}$$

Property 4. Such a ranking, which does not depend on the order of highest first potentials, may be generated (see Example 3.3).

Example 3.3. The superiority relation among $C = \{c(1), c(2), c(3)\}$ is assumed to be given in evaluation matrix $\mathbf{M}_{(\mathrm{I})3}$ as follows:

$$\mathbf{M}_{(\mathrm{I})3} = \begin{pmatrix} 9/10 & 5/10 & 7/10 \\ 10/10 & 8/10 & 1/10 \\ 7/10 & 4/10 & 2/10 \end{pmatrix}.$$

The first potential $\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})3}}$ is

$$\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})3}} = {}^{\mathrm{T}}(0.673922, 0.609739, 0.41719),$$

the order of highest first potentials in C is $c(1)$, $c(2)$, and $c(3)$. Potentials $\boldsymbol{p}_{[2]\mathrm{M}_{(\mathrm{I})3}}$ and $\boldsymbol{p}_{[3]\mathrm{M}_{(\mathrm{I})3}}$ and ranking vector $\boldsymbol{r}_{\mathbf{M}_{3(\mathrm{I})}}$ are as follows:

$$\begin{aligned}
\boldsymbol{p}_{[2]\mathrm{M}_{(\mathrm{I})3}} &= {}^{\mathrm{T}}(0.640067, 0.640067, 0.425004),\\
\boldsymbol{p}_{[3]\mathrm{M}_{(\mathrm{I})3}} &= {}^{\mathrm{T}}(0.640348, 0.640897, 0.423327),\\
\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})3}} &= {}^{\mathrm{T}}(0.63998, 0.641245, 0.423356).
\end{aligned}$$

The order of $c(1)$ and $c(2)$ for the second potential is even, while the order of $c(1)$ and $c(2)$ for the third potential reverses the order of that for the first potential. The final ranking is determined from $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{I})3}}$ as follows:

$$\text{First} \cdots c(2), \quad \text{Second} \cdots c(1), \quad \text{Third} \cdots c(3).$$

Sporting events produce results that are easy to understand in terms of the superiority of $c(i)$ to $c(j)$. In applying evaluation matrix (I) to sporting events, the rank of element $c(i)$, which has a high superiority compared to high potential elements, is higher than that of element $c(j)$, which has low superiority compared to high potential elements. Based on this property, Keener presented the concept of rank determination using the Perron-Frobenius theorem [8].

4 Ranking (II)

In this section, Ranking (II) is described in detail. Unlike Ranking (I), two constructed sets, $C = \{c(1), c(2), \ldots, c(n)\}$ and $Q = \{q(1), q(2), \ldots, q(m)\}$, are needed in order to determine Ranking (II). A matrix $\mathbf{M}_{(\mathrm{II})}$ is generated by evaluating the superiority of $c(i),\ (i = 1, \ldots, n)$ to $q(j),\ (j = 1, \ldots, m)$, and then applying the Perron-Frobenius theorem. The conditions for generating $\mathbf{M}_{(\mathrm{II})}$ are as follows:

Condition 2.

(4a) The matrix $\mathbf{M}_{(\mathrm{II})}$ is irreducible.

(4b) The value of $m_{(\mathrm{II})}(i, n+j)$ is the nonnegative ratio of superiority of $c(i)$ to $q(j)$, and the value of $m_{(\mathrm{II})}(n+j, i)$ is the nonnegative ratio of superiority of $q(j)$ to $c(i)$.

(4c) The superiority is determined by maintaining the conditions such that

$$m_{(\mathrm{II})}(i, n+j) + m_{(\mathrm{II})}(n+j, i) = h(\text{const}) > 0.$$

(4d) The ratio of superiority is determined based on a common rule either competition or trial among all 1-paired elements that do not belong to the same set.

(4e) The ratio of superiority is assumed to be zero among all 1-paired elements that belong to the same set.

From (4b) and (4e) of Condition 2, no element of a matrix $\mathbf{M}_{(\mathrm{II})}$ is negative, so a matirx $\mathbf{M}_{(\mathrm{II})}$ is nonnegative. A matrix $\mathbf{M}_{(\mathrm{II})} = \{m_{(\mathrm{II})}(i,j)\}_{1 \le i,j \le n+m}$, which satisfies Condition 2 is referred to as *evaluation matrix (II) for constructed set C and Q*. As for the case of evaluation matrix (I), we have the following remark and definition:

Remark 4.1. From Theorem 2.2 and (4a), (4b) and (4e) of Condition 2, there exists an eigenvector $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{II})}} = {}^{\mathrm{T}}(x_1, x_2, \ldots, x_{n+m})$, the elements of which are all positive, corresponding to the largest positive eigenvalue $\lambda_{\mathrm{M}_{(\mathrm{II})}}$ of $\mathbf{M}_{(\mathrm{II})}$.

Definition 4.1. The vector $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{II})}}$, denoted in Remark 4.1, is referred to as the ranking vector corresponding to the matrix $\mathbf{M}_{(\mathrm{II})}$.

Evaluation matrix $\mathbf{M}_{(\mathrm{II})}$ has the following form

$$\mathbf{M}_{(\mathrm{II})} = \begin{pmatrix} \mathbf{0} & \mathbf{V} \\ \mathbf{W} & \mathbf{0} \end{pmatrix},$$

where

$$\mathbf{V} = \begin{pmatrix} m_{(\mathrm{II})}(1, n+1) & \cdots & m_{(\mathrm{II})}(1, n+m) \\ \vdots & \ddots & \vdots \\ m_{(\mathrm{II})}(n, n+1) & \cdots & m_{(\mathrm{II})}(n, n+m) \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} m_{(\mathrm{II})}(n+1, 1) & \cdots & m_{(\mathrm{II})}(n+1, n) \\ \vdots & \ddots & \vdots \\ m_{(\mathrm{II})}(n+m, 1) & \cdots & m_{(\mathrm{II})}(n+m, n) \end{pmatrix}.$$

The sizes of $\mathbf{V}$ and $\mathbf{W}$ are $n \times m$ and $m \times n$, respectively. Here, a very simple example to generate $\mathbf{M}_{(\mathrm{II})}$ and a property for $\mathbf{M}_{(\mathrm{II})}$ are given.

Example 4.1. Let $C = \{c(1), c(2), c(3)\}$ be a set of students, and let $Q = \{q(1), q(2)\}$ be the set of questions. Table 4.1 lists the distributions between students and questions. A maximum of 10 points may be received for each question. From Table 4.1, for the value of $h = 10$ in (4c) of Condition 2, the following evaluation matrix $\mathbf{M}_{(\mathrm{II})1}$ is obtained:

$$\mathbf{M}_{(\mathrm{II})1} = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 5 & 4 \\ \hline 9 & 7 & 5 & 0 & 0 \\ 1 & 3 & 6 & 0 & 0 \end{array}\right).$$

Property 5. A matrix $\mathbf{M}_{(\mathrm{II})}$ is not primitive and has a period 2.

Table 4.1: Scores received by students $c(i)$ in C.

	$q(1)$	$q(2)$	Total
$c(1)$	1	9	10
$c(2)$	3	7	10
$c(3)$	5	4	9
Mean	3.0	6.7	

4.1 Characteristics of ranking vector for $\mathrm{M}_{(\mathrm{II})}$

In this subsection, the mathematical properties of each element in ranking vector $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{II})}}$ are given. As is mentioned in Property 5, the matrix $\mathbf{M}_{(\mathrm{II})}$ has a period 2. Therefore, we cannot have the characteristics of each element in $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{II})}}$ by the transition of successive potential in applying the power method for the case of $\mathbf{M}_{(\mathrm{I})}$. Therefore, we present the following theorems concerning $\mathbf{M}_{(\mathrm{II})}$.

Theorem 4.1. *For a matrix* $\mathrm{M}_{(\mathrm{II})} = \begin{pmatrix} \mathbf{0} & \mathbf{V} \\ \mathbf{W} & \mathbf{0} \end{pmatrix}$*, matrices* $\mathbf{VW}$ *and* $\mathbf{WV}$ *are irreducible in* $\mathrm{M}^2_{(\mathrm{II})} = \begin{pmatrix} \mathbf{VW} & \mathbf{0} \\ \mathbf{0} & \mathbf{WV} \end{pmatrix}$.

Proof. Let the diagonal elements $m_{(\mathrm{II})}(i,i)$ $(1 \leq i \leq n)$ and $m_{(\mathrm{II})}(j,j)$ $(n+1 \leq j \leq n+m)$ in $\mathbf{M}_{(\mathrm{II})} = \{m[i,j]_{(\mathrm{II})}\}_{1 \leq i,j \leq n+m}$ be P_i and Q_j, respectively, and let P_i and Q_j be referred to as nodes corresponding to each element in $\mathbf{M}_{(\mathrm{II})}$. If $m(i,j)_{(\mathrm{II})} \neq 0$, then $m(i,j)_{(\mathrm{II})}$ represents $P_i \longrightarrow Q_j$. Since the matrix $\mathbf{M}_{(\mathrm{II})}$ is irreducible, the structure of the directed graph of each node in $\mathbf{M}_{(\mathrm{II})}$ generates a bipartite graph between $\{P_i\}$ and $\{Q_j\}$, and all nodes in each node $\{P_i\} \cup \{Q_i\}$ are strongly connected. Similarly, denote the diagonal elements $m'_{(\mathrm{II})}(i,i)$ $(1 \leq i \leq n)$ and $m'_{(\mathrm{II})}(j,j)$ $(n+1 \leq j \leq n+m)$ in $\mathbf{M}^2_{(\mathrm{II})} = \{m'[i,j]_{(\mathrm{II})}\}_{1 \leq i,j \leq n+m}$ as P'_i and Q'_j, respectively.

To prove the irreducibility of matrix $\mathbf{VW}$, we apply the reduction theory. If it is assumed that a matrix $\mathbf{VW}$ is not irreducible, then there exists at least one pair nodes $P'_{\alpha 1}$ and $P'_{\beta 1}$ which are not strongly connected each other. If nodes P'_i, P'_j in $\mathbf{VW}$ are strongly connected, then there is at least one set of nodes $\mathcal{Q}_\alpha$ that are strongly connected to P_i and P_j. Here, let $\mathcal{P}'_\alpha$ be a set of nodes $\{p'_{\alpha 1}, p'_{\alpha 2}, \ldots, p'_{\alpha n}\}$ that are strongly connected to $\{p'_{\alpha 1}\}$, and let $\mathcal{P}'_\beta$ be a set of node $\{p'_{\beta 1}, p'_{\beta 2}, \ldots, p'_{\beta n}\}$ that are strongly connected to $\{p'_{\beta 1}\}$. Under this the assumption, $P'_{\alpha 1}$ and $P'_{\beta 1}$ are not strongly connected, and there exist two sets of nodes, namely, $\mathcal{Q}_\alpha$, which are strongly connected to $\{P_{\alpha 1}, P_{\alpha 2}, \ldots, P_{\alpha n}\}$, and $\mathcal{Q}_\beta$, which are strongly connected to $\{P_{\beta 1}, P_{\beta 2}, \ldots, P_{\beta n}\}$, that satisfy $\mathcal{Q}_\alpha \cap \mathcal{Q}_\beta = \phi$. Thus, the relations between the nodes in $\mathcal{Q}_\alpha$ and the nodes in $\mathcal{Q}_\beta$ are not strongly connected, which contradicts the assertion that $\mathbf{VW}$ is irreducible. Therefore, $\mathbf{VW}$ is irreducible. Similarly, it is proven that $\mathbf{WV}$ is irreducible. □

Theorem 4.2. *If* $\boldsymbol{s}_1 = {}^{\mathrm{T}}(x_1, \ldots, x_n, y_1, \ldots, y_m),\ ||\boldsymbol{s}_1||_2 = 1$ *is the ranking vector for a matrix* $\mathbf{M}_{(\mathrm{II})} = \begin{pmatrix} \mathbf{0} & \mathbf{V} \\ \mathbf{W} & \mathbf{0} \end{pmatrix}$ *corresponding to the largest positive eigenvalue* α*, then*

$$\boldsymbol{s}_2 = {}^{\mathrm{T}}(x_1, \ldots, x_n, -y_1, \ldots, -y_m)$$

is also an eigenvector for $\mathbf{M}_{(\mathrm{II})}$ *corresponding to an eigenvalue* $-\alpha$*.*

Proof. Since the matrix $\mathbf{M}_{(\mathrm{II})}$ has a period 2, if the largest positive eigenvalue is α, then $-\alpha$ is also an eigenvalue of $\mathbf{M}_{(\mathrm{II})}$. From $\mathbf{M}_{(\mathrm{II})}\boldsymbol{s}_1 = \alpha \boldsymbol{s}_1$, and by setting $\boldsymbol{x}$ and $\boldsymbol{y}$ as

$$\boldsymbol{x} = {}^{\mathrm{T}}(x_1, x_2, \ldots, x_n), \quad \boldsymbol{y} = {}^{\mathrm{T}}(y_1, y_2, \ldots, y_m),$$

$$\mathbf{V}\boldsymbol{y} = \alpha \boldsymbol{x}, \quad \mathbf{W}\boldsymbol{x} = \alpha \boldsymbol{y},$$

we have

$$\mathbf{V}\frac{1}{\alpha}\mathbf{W}\boldsymbol{x} = \alpha \boldsymbol{x}, \tag{4.1}$$

$$\mathbf{V}\mathbf{W}\boldsymbol{x} = \alpha^2 \boldsymbol{x}. \tag{4.2}$$

If it is assumed that $\boldsymbol{v} = {}^{\mathrm{T}}(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+m}),\ ||\boldsymbol{v}||_2 = 1$ is an eigenvector for an eigenvalue $-\alpha$, then $\mathbf{M}_{(\mathrm{II})}\boldsymbol{v} = -\alpha \boldsymbol{v}$. Setting $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as

$$\boldsymbol{v}_1 = {}^{\mathrm{T}}(v_1, v_2, \ldots, v_n),\ \boldsymbol{v}_2 = {}^{\mathrm{T}}(v_{n+1}, \ldots, v_{n+m}),$$

$$\mathbf{V}\boldsymbol{v}_2 = -\alpha \boldsymbol{v}_1, \quad \mathbf{W}\boldsymbol{v}_1 = -\alpha \boldsymbol{v}_2,$$

we have

$$\mathbf{V}\mathbf{W}\boldsymbol{v}_1 = \alpha^2 \boldsymbol{v}_1. \tag{4.3}$$

The matrix $\mathbf{M}^2_{(\mathrm{II})}$ is not irreducible, but the eigenvector α^2 is the largest positive eigenvalue for $\mathbf{M}^2_{(\mathrm{II})}$. On the other hand, the matrix $\mathbf{V}\mathbf{W}$ is irreducible, and α^2 is the largest positive eigenvalue for $\mathbf{V}\mathbf{W}$. Therefore, $\boldsymbol{v}_1$ is a unique positive eigenvector corresponding to an eigenvalue α^2. Therefore, from Eqs. (4.2) through (4.3), $\boldsymbol{x} = \boldsymbol{v}_1$. In addition, we have

$$\mathbf{W}\boldsymbol{x} = \mathbf{W}\boldsymbol{v}_1 = -\alpha \boldsymbol{v}_2 = \alpha \boldsymbol{y}.$$

Therefore, the eigenvector $\boldsymbol{s}$ for the matrix $\mathbf{M}_{(\mathrm{II})}$ corresponding to $-\alpha$ is given

$$\boldsymbol{s}_2 = {}^{\mathrm{T}}(x_1, \ldots, x_n, -y_1, \ldots, -y_m). \qquad \square$$

The matrix $\mathbf{M}_{(\mathrm{II})}$ has a period 2 and does not converge to the eigenvector corresponding to the largest positive eigenvalue α in the application of the power method. However, the form of the eigenvector $\boldsymbol{s}_2$ corresponding to the eigenvalue -α was determined by Theorem 4.2. Then, the following corollary for $\mathbf{M}_{(\mathrm{II})}$ is taken in the process of applying the power method.

Corollary 4.1. *Let* $\boldsymbol{r}_{\mathrm{M}_{(\mathrm{II})}} = {}^{\mathrm{T}}(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m),\ ||\boldsymbol{r}_{\mathrm{M}_{(\mathrm{II})}}||_2 = 1$ *be the ranking vector corresponding to the largest positive eigenvalue* α *for a matrix* $\mathbf{M}_{(\mathrm{II})} = \begin{pmatrix} \mathbf{0} & \mathbf{V} \\ \mathbf{W} & \mathbf{0} \end{pmatrix}$*. Then, if the power method is applied to* $\mathbf{M}_{(\mathrm{II})}$*, the following two vectors,* $\boldsymbol{w}_1$ *and* $\boldsymbol{w}_2$*, are obtained:*

$$\begin{cases} \boldsymbol{w}_1 = \frac{1}{w_{f1}} {}^{\mathrm{T}}\big((c_1 + c_2)x_1, \ldots, (c_1 + c_2)x_n, (c_1 - c_2)y_1, \ldots, (c_1 - c_2)y_m\big), \\ \boldsymbol{w}_2 = \frac{1}{w_{f2}} {}^{\mathrm{T}}\big((c_1 - c_2)x_1, \ldots, (c_1 - c_2)x_n, (c_1 + c_2)y_1, \ldots, (c_1 + c_2)y_m\big), \end{cases} \tag{4.4}$$

where w_{f1} *and* w_{f2} *are constants to normalize the vectors* $\boldsymbol{w}_1$ *and* $\boldsymbol{w}_2$*, respectively, with respect to* $l_2 - norm$*.*

Proof. It is assumed that the eigenvalues for $\mathbf{M}_{(\mathrm{II})}$ are $\alpha, -\alpha, \alpha_3, \ldots$ and that the following is satisfied:

$$\alpha = |-\alpha| > |\alpha_3| \geq |\alpha_4| \geq \cdots .$$

From Theorem 4.2, eigenvectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ corresponding to α and $-\alpha_2$, respectively, are given as

$$\begin{aligned} \boldsymbol{s}_1 &= {}^{\mathrm{T}}(x_1, \ldots, x_n, y_1, \ldots, y_m), \\ \boldsymbol{s}_2 &= {}^{\mathrm{T}}(x_1, \ldots, x_n, -y_1, \ldots, -y_m). \end{aligned}$$

Since the two vectors, $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, are linearly independent, the initial vector $\boldsymbol{u}_0$ can be represented by following equation while assuming that $c_1,\ c_2 \neq 0$:

$$\boldsymbol{u}_0 = {}^{\mathrm{T}}(1, 1, \ldots, 1) = c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2 + \sum_{i=1} c_3\boldsymbol{x}_i. \tag{4.5}$$

Thus,

$$\begin{aligned} \mathbf{A}\boldsymbol{u}_0 &= \mathbf{A}\big(c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2 + \sum_{i=1} c_3\boldsymbol{x}_i\big) \\ &= c_1\alpha\boldsymbol{s}_1 + c_2(-\alpha)\boldsymbol{s}_2 + c_3\alpha_3\boldsymbol{x}_3 + \cdots \equiv \boldsymbol{u}_1. \end{aligned} \tag{4.6}$$

To normalize the vector $\boldsymbol{u}_1$, operating w_1 on Eq. (4.6) yields

$$\boldsymbol{v}_1 \equiv \frac{1}{w_1}A\boldsymbol{u}_0 = \frac{1}{w_1}\big(c_1\alpha\boldsymbol{s}_1 + c_2(-\alpha)\boldsymbol{s}_2 + \cdots\big).$$

By repeating this procedure, we obtain

$$\boldsymbol{v}_{\boldsymbol{k}} \equiv \frac{1}{w_k}\mathbf{A}^{k-1}\boldsymbol{u}_0 = \frac{\alpha^k}{w_k}\Big(c_1\boldsymbol{s}_1 + c_2(-1)^k\boldsymbol{s}_2 + \Big(\frac{\alpha_3}{\alpha}\Big)^k\boldsymbol{x}_3 + \cdots\Big).$$

Therefore, the value of $\lim_{k\to\infty}\boldsymbol{v}_k$ as $k \to \infty$ is

$$\lim_{k\to\infty}\boldsymbol{v}_k$$

$$= \begin{cases} \boldsymbol{w}_1 = \frac{1}{w_{f1}}(c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2) \\ \boldsymbol{w}_2 = \frac{1}{w_{f_2}}(c_1\boldsymbol{s}_1 - c_2\boldsymbol{s}_2), \end{cases} \tag{4.7}$$

$$= \begin{cases} \boldsymbol{w}_1 = \frac{1}{w_{f1}}{}^{\mathrm{T}}\big((c_1+c_2)x_1,\ldots,(c_1+c_2)x_n,(c_1-c_2)y_1,\ldots,(c_1-c_2)y_m\big) \\ \boldsymbol{w}_2 = \frac{1}{w_{f2}}{}^{\mathrm{T}}\big((c_1-c_2)x_1,\ldots,(c_1-c_2)x_n,(c_1+c_2)y_1,\ldots,(c_1+c_2)y_m\big). \end{cases}$$

where w_{f1} and w_{f2} are constants to normalize the vectors $c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$ and $c_1\boldsymbol{s}_1 - c_2\boldsymbol{s}_2$ with respect to $l_2 - norm$, respectively. □

If the power method is applied to evaluation matrix $\mathbf{M}_{(\mathrm{II})}$ with initial vector $\boldsymbol{u_0} = {}^{\mathrm{T}}(1,1,\ldots,1)$, then the k-th potential $\boldsymbol{p}_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}$ for $\mathbf{M}_{(\mathrm{II})}$ can be defined in a similar manner by applying the power method to $\mathbf{M}_{(\mathrm{I})}$, and $\boldsymbol{p}_{[\infty]\mathrm{M}_{(\mathrm{II})}}$ can be also defined a final potential for $\mathbf{M}_{(\mathrm{II})}$, as follows:

$$\boldsymbol{p}_{[\infty]\mathrm{M}_{(\mathrm{II})}} = \lim_{k\to\infty} \boldsymbol{v}_k. \tag{4.8}$$

Since the final potential $\boldsymbol{p}_{[\infty]\mathrm{M}_{(\mathrm{II})}}$ is oscillated between $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ in Corollary 4.1 by applying the power method with an initial vector $\boldsymbol{u_0}$ and $\mathbf{M}_{(\mathrm{II})}$ is nonnegative, we can assume that, in Eq. (4.7), $c_1 > c_2$. Next, we have the following property.

Property 6. In Eq. (4.7) of Corollary 4.1, denoting $\boldsymbol{w}_1 = {}^{T}\big(w_1(1), w_1(2), \ldots, w_1(n+m)\big)$, $\boldsymbol{w}_2 = {}^{T}\big(w_2(1), w_2(2), \ldots, w_2(n+m)\big)$, where all $w_1(i),\ w_2(i),\ (i = 1,\ldots,n+m)$ are positive, the following equations are satisfied:

$$\begin{aligned} \frac{1}{\sum_{v=1}^{n} w_1(v)^2}{}^{\mathrm{T}}\big(w_1(1),\ldots,w_1(n)\big) &= \frac{1}{\sum_{v=1}^{n} w_2(v)^2}{}^{\mathrm{T}}\big(w_2(1),\ldots,w_2(n)\big) \\ &= \frac{1}{\sum_{v=1}^{n} x_v}{}^{\mathrm{T}}(x_1,\ldots,x_n), \end{aligned}$$

$$\begin{aligned} \frac{1}{\sum_{v=n+1}^{m} w_1(v)^2}{}^{\mathrm{T}}\big(w_1(n+1),\ldots,w_1(n+m)\big) &= \frac{1}{\sum_{v=n+1}^{m} w_2(v)^2}{}^{\mathrm{T}}\big(w_2(n+1),\ldots,w_2(n+m)\big) \\ &= \frac{1}{\sum_{v=1}^{m} y_v}{}^{\mathrm{T}}(y_1,\ldots,y_m). \end{aligned}$$

Since the k-th potential $\boldsymbol{p}_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}$ for $\mathbf{M}_{(\mathrm{II})}$ in the process of applying the power method is denoted as

$$\boldsymbol{p}_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}} = {}^{\mathrm{T}}\big(p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(1),\ldots,p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n),p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+1),\ldots,p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+m)\big),$$

each element of $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(i)$ $(1 \le i \le n)$ represents the k-th potential for $c(i)$ and is calculated by the following equation:

$$p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(i) = \sum_{v=1}^{m} m_{(\mathrm{II})}(i, n+v)\, p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{II})}}(n+v). \tag{4.9}$$

In Eq. (4.9), the value of $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(i)$ indicates that the element $c(i)$, which has a high degree of superiority compared to $\{q(j)\}$ with high potentials $\{p_{(\mathrm{II})k-1}(n+v)\}_{1\le v\le m}$, is becoming larger. Similarly, the k-th potential of $q(j)$ $(n+1 \le j \le n+m)$ is

$$p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+j) = \sum_{v=1}^{n} m_{(\mathrm{II})}(n+j,v)p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{II})}}(v). \tag{4.10}$$

This means that the element $q(j)$, which has relatively high superiority compared to $\{c(i)\}$ with high potentials $\{p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{II})}}(i)\}_{1\le i\le n}$ is becoming larger. From Corollary 4.1 and Eq. (4.8), the final potential $\boldsymbol{p}_{[\infty]\mathrm{M}_{(\mathrm{II})}}$ is oscillated between $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, and the ratios of relation among the first through n-th elements in $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are identical and the relation among the $(n+1)$-th through $(n+m)$-th elements in $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are also identical (see Property 6). Therefore, we can redefine the k-th potential for C, denoted by $\boldsymbol{p}_{[k](C)}$, and k-th potential for Q, denoted by $\boldsymbol{p}_{[k](Q)}$, as follows:

$$\begin{aligned}\boldsymbol{p}_{[k](C)} &= \frac{1}{\sum_{v=1}^{n} p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(v)^2}{}^{\mathrm{T}}\big(p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(1),\ \ldots,\ p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n)\big),\\ \boldsymbol{p}_{[k](Q)} &= \frac{1}{\sum_{v=n+1}^{n+m} p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(v)^2}{}^{\mathrm{T}}\big(p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+1),\ \ldots,\ p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+m)\big).\end{aligned} \tag{4.11}$$

Therefore, we have the final potential for C as $\boldsymbol{x}_{(C)\mathrm{M}_{(\mathrm{II})}}$ and that for Q as $\boldsymbol{y}_{(Q)\mathrm{M}_{(\mathrm{II})}}$, as follows:

$$\boldsymbol{x}_{(C)\mathrm{M}_{(\mathrm{II})}} = \frac{1}{\sum_{v=1}^{n} x_v}{}^{T}(x_1,\ldots,x_n) = \lim_{k\to\infty} \boldsymbol{p}_{[k](C)}, \tag{4.12}$$

$$\boldsymbol{y}_{(Q)\mathrm{M}_{(\mathrm{II})}} = \frac{1}{\sum_{v=1}^{m} y_v}{}^{T}(y_1,\ldots,y_m) = \lim_{k\to\infty} \boldsymbol{p}_{[k](Q)}. \tag{4.13}$$

Finally, from Eqs. (4.9)-(4.13), the following property of Ranking (II) for $\mathbf{M}_{(\mathrm{II})}$ is obtained:

Property 7. Among the elements belonging to C, the rank of element $c(i)$, which has a high superiority compared to $\{q(j)\}$ with high potential, is increasing, and among the elements belonging to Q, the rank of element $q(j)$, which has a high superiority compared to $\{c(i)\}$ with high potential, is increasing.

5 Comparison of Ranking (I) and Ranking (II)

In this section, we discuss the similarities and differences of Ranking (I) and Ranking (II). The primary difference between Ranking (I) and Ranking (II) is that, while each potential $c(i)$ in C for $\mathbf{M}_{(\mathrm{I})}$ is calculated successively considering only the relation among elements belonging to C, each potential $\mathbf{M}_{(\mathrm{II})}$ is calculated successively considering the relations between $c(i)$ in C and $q(j)$ in Q. As a result, each of the elements in the ranking vector for $\mathbf{M}_{(\mathrm{I})}$ generated by Condition 1 tends to reflect the average superiority as compared to the other elements, because the characteristics of elements in first potential vector $\boldsymbol{p}_{[1]\mathrm{M}_{(\mathrm{I})}}$ for $\mathbf{M}_{(\mathrm{I})}$ depend entirely on the average superiority of each $c(i)$ compared the other elements (involving the superiority of $c(i)$ to $c(i)$). On the other hand, each of the elements in the ranking vector for $\mathbf{M}_{(\mathrm{II})}$ generated by Condition 2 tend to reflect directly the superiority of that element as compared to the other elements. Since $\mathbf{M}_{(\mathrm{II})}$ is generated while considering both the relation $C \longrightarrow Q$ and the relation $Q \longrightarrow C$, the first through n-th elements in $\boldsymbol{p}_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}$ directly reflect the superiority among all elements $q(j)$ in Q and the $(n+1)$-th through $(n+m)$-th elements in $\boldsymbol{p}_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}$ directly reflect the superiority among all elements $c(i)$ in C in the process of applying the power method. In addition, compared to Ranking (I), ranking vector $\boldsymbol{r}_{\mathrm{M(II)}}$ incorporates two types of ranking, namely, the ranking among $c(i)$ in C and the ranking among $q(j)$ in Q. Thus, Ranking (II) has an advantage in that two distinct types of ranking can be generated using one data as the two aspects of results between C and Q according to (4b) of Condition 2.

The processes of generating ranking vectors are similar between Ranking (I) and Ranking (II) in the meaning of that, in both processes, the first potential vector is defined initially, and then the final potential vector is lead using the power method. As a result, the characteristics for Ranking (I) and Ranking (II) are taken as similar Property 2 and Property 7, respectively.

6 Conclusions

In the present study, we have proposed two types of ranking, Ranking (I) and Ranking (II). One advantage of applying these rankings is the ability to determine a clear ranking even among $c(i)$ in C that are considered to be equal. A fundamental method by which to generate these rankings was proposed as follows. First, an evaluation matrix that satisfies two conditions that correspond to either Ranking (I) or Ranking (II) is generated. Then, the first potential vector is defined. Finally, the final potential vector successively is lead using the power method. As a result, the characteristics for Ranking (I) and Ranking (II) are taken as Property 2 and Property 7, respectively. However, irregular results may occur in both Ranking (I) and Ranking (II) such that the generated rankings are not ordered according to the actual data each element took. (This is referred to as the rank inversion phenomenon.) Therefore, we are currently investigating a technique for controlling the rank inversion phenomenon of Ranking (I) and Ranking (II).

References

[1] Abraham Berman and Robert J. Plemmons, *Nonnegative Mathematical Science*, Academic Press, New York, U.S.A., 1979.

[2] Curtid F. Gerald, *Applied Numerical Analysis*, Addison Wesley, Reading, MA, U.S.A., 1990.
[3] S. Furuichi, K. Oshima and I. Hofuku, An improved ranking procedure applying a continuous weighted functions, *Int. J. Appl. Math.* **2** (2000), 553–558.
[4] I. Hofuku and K. Oshima, A total ranking based on exam scores for a small number of subjects, *Information* **5** (2002), 417–423.
[5] I. Hofuku and K. Oshima, Rankings methods for various aspects based on Perron Frobenius theorem, *Information* **9** (2006), 37–52.
[6] I. Hofuku and K. Oshima, A controlled absolute ranking method applied to an exam of multiplex choice form, *Int. J. Pure Appl. Math.* **47** (2008), 267–280.
[7] James M. Ortega, *Numerical Analysis - A Second Course*, SIAM, Philadelphia, PA, U.S.A., 1990.
[8] James P. Keener, The Perron-Frobenius theorem and the ranking of football, *SIAM Review* **35** (1993), 80–93.
[9] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Academic Press, New York, U.S.A., 1985.

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