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Stress-Strength Reliability Estimation for Exponentiated Generalized Inverse Weibull Distribution

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Abstract: This paper is devoted to discuss the stress-strength reliability model $R = Pr(Y < X)$ when X and Y have an exponentiated generalized inverse Weibull distribution (EGIW) with different parameters. The problem of stress-strength reliability is studied to obtain estimates of a component reliability function of EGIW distribution. Reliability for multi-component stress-strength model for EGIW distribution is also studied. Maximum likelihood estimation for stress-strength reliability of underlying distribution is performed. Bayesian estimator of R is obtained using importance sampling technique. A simulation study to investigate and compare the performance of each method of estimation is performed. Finally analysis of a real data set has also been presented for illustrative purposes.

Keywords: Exponentiated generalized inverse Weibull distribution, Stress-strength reliability, Maximum likelihood estimation, Bayesian estimation, Importance sampling technique

1 Introduction

In reliability studies, the stress-strength model is often used to describe the life of a component which has a random strength X and is subject to a random stress Y. The component fails if the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $Y < X$ thus $R = Pr(Y < X)$ is a measure of a component reliability which has many applications in physics,engineering, genetics, psychology and economics.
The term stress-strength was first introduced by Church and Harris [1] which introduced the estimation of R when X and Y are normally distributed. Since then several studies has been done both from parametric and non-parametric point of view. A good application on the different stress-strength models can be found in the monograph by Kotz et al. [2]. Some of studies on the stress-strength model can be obtained in [3,4], [5] which considered this problem when X and Y are generalized exponential, Weibull and Burr type X distributions respectively. Stress strength Reliability for three-parameter Weibull distribution has been discussed by Kundu and Raqab [6]. Krishnamoorthy et al. [7] introduced an inference on reliability in two-parameter exponential stress-strength model. Stress-strength reliability for Lindely and weighted Lindely distributions considered by Al-Mutairi et al. [8,9] respectively. Recently Hanagal and Bhalerao [10] discussed generalized inverse Weibull software reliability growth model.
In this paper we study the stress strength reliability for the Exponentiated Generalized Inverse Weibull Distribution (EGIW) which introduced in [11] as extension of exponentiated generalized family. The EGIW distribution has a p.d.f $f(x)$ and c.d.f $F(x)$:

$$f(x) = \alpha\beta\theta\lambda^{\theta}x^{-\theta-1}e^{-(\frac{\lambda}{x})^{\theta}}[1-e^{-(\frac{\lambda}{x})^{\theta}}]^{\alpha-1}[1-(1-e^{-(\frac{\lambda}{x})^{\theta}})^{\alpha}]^{\beta-1}, \quad (1)$$

$$F(x) = [1-(1-e^{-(\frac{\lambda}{x})^{\theta}})^{\alpha}]^{\beta}, \quad (2)$$

where

$$x > 0, \lambda, \theta, \alpha, \beta > 0.$$

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The EGIW distribution is very flexible model that approaches to different distributions when its parameters are changed. Its flexibility is explained in the following,if X is a random variable with pdf in Eq.(1), then we have the following special cases:
1- If $\alpha = \beta = 1$, then Eq. (1) reduces to the inverse Weibull distribution.
2- If $\alpha = 1$, then we get the generalized inverse Weibull distribution.
3- If $\theta = 1$, then we get the exponentiated generalized inverse exponential.
4- If $\alpha = \beta = \theta = 1$, then we get the inverse exponential distribution.

The rest of the article is organized as follows. In Section 2, the problem of stress-strength reliability is studied to obtain estimates of a component reliability function of EGIW distribution. Reliability for multi-component stress-strength model for EGIW distribution is also studied in Section 3. Maximum likelihood estimation for stress-strength reliability of underlying distribution is performed in Section 4. In Section 5, a general procedure of deriving the Bayesian estimator of reliability using squared error loss function is presented, wherein we adopt the importance sampling technique to compute the approximation of this estimator. Section 6 presented simulation study to investigate and compare the performance of each method of estimation. Also a real data set analysis has been presented, in section 7, for illustrating all the inferential methods developed here. Finally, conclusions appear in Section 8.

2 Stress-Strength Reliability

In this section, we derive the reliability R when $X \sim EGIW(\alpha_1, \beta_1, \lambda_1, \theta_2)$ and $Y \sim EGIW(\alpha_2, \beta_2, \lambda_2, \theta_2)$ are independent random variables with pdf $f(x)$ and $w(y)$, respectively. We have

$$R = Pr(Y < X) = \int_0^\infty \int_0^x f(x)w(y)\ dydx.$$

The formula of Eqs. (1) and (2) will complicate the integration, so we write f(x) and F(x) in an expansion form, using fractional binomial theorem (See [12]), as follows:

$$\begin{aligned} f(x) &= \alpha\beta\theta\lambda^\theta x^{-\theta-1} e^{-(\frac{\lambda}{x})^\theta}[1-e^{-(\frac{\lambda}{x})^\theta}]^{\alpha-1}[1-(1-e^{-(\frac{\lambda}{x})^\theta})^\alpha]^{\beta-1} \\ &= \alpha\beta\theta\lambda^\theta x^{-\theta-1} e^{-(\frac{\lambda}{x})^\theta}[1-e^{-(\frac{\lambda}{x})^\theta}]^{\alpha-1}\sum_{j_1=0}^{\infty}(-1)^{j_1}\binom{\beta-1}{j_1}[1-e^{-(\frac{\lambda}{x})^\theta}]^{\alpha j_1} \\ &= \alpha\beta\theta\lambda^\theta x^{-\theta-1} e^{-(\frac{\lambda}{x})^\theta}\sum_{j_1=0}^{\infty}(-1)^{j}_1\binom{\beta-1}{j_1}[1-e^{-(\frac{\lambda}{x})^\theta}]^{\alpha(j_1+1)-1} \\ &= \alpha\beta\theta\lambda^\theta x^{-\theta-1} e^{-(\frac{\lambda}{x})^\theta}\sum_{j_1=0}^{\infty}(-1)^{j}_1\binom{\beta-1}{j_1}\sum_{j_2=0}^{\infty}(-1)^{j_2}\binom{\alpha(j_1+1)-1}{j_2}[e^{-(\frac{\lambda}{x})^\theta}]^{j_2} \\ &= \alpha\beta\theta\lambda^\theta x^{-\theta-1}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}(-1)^{(j_1+j_2)}\binom{\beta-1}{j_1}\binom{\alpha(j_1+1)-1}{j_2}e^{-(j_2+1)(\frac{\lambda}{x})^\theta}. \end{aligned} \tag{3}$$

Likelly for F(x),

$$\begin{aligned} F(x) &= [1-(1-e^{-(\frac{\lambda}{x})^\theta})^\alpha]^\beta \\ &= \sum_{j_3=0}^{\infty}(-1)^{j_3}\binom{\beta}{j_3}[1-e^{-(\frac{\lambda}{x})^\theta}]^{\alpha j_3} \\ &= \sum_{j_3=0}^{\infty}(-1)^{j_3}\binom{\beta}{j_3}\sum_{j_4=0}^{\infty}(-1)^{j_4}\binom{\alpha j_3}{j_4}[e^{-(\frac{\lambda}{x})^\theta}]^{j_4} \\ &= \sum_{j_3=0}^{\infty}\sum_{j_4=0}^{\infty}(-1)^{(j_3+j_4)}\binom{\beta}{j_3}\binom{\alpha j_3}{j_4}e^{-j_4(\frac{\lambda}{x})^\theta}. \end{aligned} \tag{4}$$

Now we can derive the stress strength reliability using Eqs.(3) and (4) as following:

$$\begin{aligned}
R &= Pr(Y < X) \\
&= \int_0^\infty \int_0^x f(x)w(y)\ dydx \\
&= \int_0^\infty f(x)W_y(x)\ dx \\
&= \int_0^\infty \alpha_1\beta_1\theta\lambda^\theta x^{-\theta-1} \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty (-1)^{(j_1+j_2+j_3+j_4)} \binom{\beta_1-1}{j_1} \binom{(\alpha_1(j_1+1))-1}{j_2} e^{-(j_2+1)(\frac{\lambda}{x})^\theta} \\
&\quad \times \sum_{j_3=0}^\infty \sum_{j_4=0}^\infty \binom{\beta_2}{j_3} \binom{\alpha_2 j_3}{j_4} e^{-j_4(\frac{\lambda}{x})^\theta}\ dx \\
&= \alpha_1\beta_1 \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \sum_{j_3=0}^\infty \sum_{j_4=0}^\infty (-1)^{(j_1+j_2+j_3+j_4)} \binom{\beta_1-1}{j_1} \binom{\alpha_1(j_1+1)-1}{j_2} \binom{\beta_2}{j_3} \binom{\alpha_2 j_3}{j_4} \\
&\quad \times \int_0^\infty \theta\lambda^\theta x^{-\theta-1} e^{-(j_2+j_4+1)(\frac{\lambda}{x})^\theta}\ dx \\
&= \alpha_1\beta_1 \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \sum_{j_3=0}^\infty \sum_{j_4=0}^\infty (-1)^{(j_1+j_2+j_3+j_4)} \binom{\beta_1-1}{j_1} \binom{\alpha_1(j_1+1)-1}{j_2} \binom{\beta_2}{j_3} \binom{\alpha_2 j_3}{j_4} \frac{1}{(j_2+j_4+1)}.
\end{aligned}$$

Note that when the exponents in Eqs. (1) and (2) are integers, the expansions in Eqs.(3), (4) and (5) become finite and this is a special case from fractional binomial theorem.

3 Reliability For Multi-Component Stress-Strength Model

Let the random samples $Y, X_1, X_2, \cdots, X_k$ be independent, $G(y)$ be the cumulative distribution function of Y and $F(x)$ be the common cumulative distribution function of $X_1, X_2, \cdots, X_k$. The reliability for a multi-component stress-strength model has developed by Bhattacharyya and Johnson [13] is:

$$\begin{aligned}
R_{s,k} &= Pr[at\ least\ s\ of\ the\ (X_1, X_2, \cdots, X_k)\ exceed\ Y] \\
&= \sum_{i=s}^k \binom{k}{i} \int_0^\infty [1-F(y)]^i [F(y)]^{k-i} dG(y).
\end{aligned} \quad (5)$$

The reliability for multi-component stress-strength of the exponentiated generalized inverse Weibull distribution is:

$$\begin{aligned}
R_{s,k} &= \sum_{i=s}^k \binom{k}{i} \int_0^\infty \left[1-\left(1-(1-e^{-(\frac{\lambda}{y})^\theta})^{\alpha_1}\right)^{\beta_1}\right]^i \left[(1-(1-e^{-(\frac{\lambda}{y})^\theta})^{\alpha_1})^{\beta_1}\right]^{k-i} \\
&\quad \times \alpha_2\beta_2\theta\lambda^\theta y^{-\theta-1} e^{-(\frac{\lambda}{y})^\theta} \left[1-e^{-(\frac{\lambda}{y})^\theta}\right]^{\alpha_2-1} \left[1-(1-e^{-(\frac{\lambda}{y})^\theta})^{\alpha_2}\right]^{\beta_2-1} dy \\
&= \sum_{i=s}^k \binom{k}{i} \alpha_2\beta_2 \int_0^\infty \left[1-(1-t^{\alpha_1})^{\beta_1}\right]^i [1-t^{\alpha_1}]^{\beta_1(k-i)} \\
&\quad \times \alpha_2\beta_2\theta\lambda^\theta y^{-\theta-1} e^{-(\frac{\lambda}{y})^\theta} t^{\alpha_2-1} [1-t^{\alpha_2}]^{\beta_2-1}\ dy \\
&= \sum_{i=s}^k \binom{k}{i} \sum_{j_1=0}^i \binom{i}{j_1} (-1)^{j_1} \alpha_2\beta_2 \int_0^\infty [1-t^{\alpha_1}]^{\beta_1(j_1+k-i)} \\
&\quad \times \alpha_2\beta_2\theta\lambda^\theta y^{-\theta-1} e^{-(\frac{\lambda}{y})^\theta} t^{\alpha_2-1} [1-t^{\alpha_2}]^{\beta_2-1}\ dy
\end{aligned}$$

$$= \sum_{i=s}^{k}\sum_{j_1=0}^{i}\sum_{j_2=0}^{\beta_1(j_1+k-i)}\sum_{j_3=0}^{\beta_2-1}\binom{k}{i}\binom{i}{j_1}\binom{\beta_1(j_1+k-i)}{j_2}\binom{\beta_2-1}{j_3}(-1)^{j_1+j_2+j_3}\alpha_2\beta_2 \times \int_0^\infty \theta\lambda^\theta y^{-\theta-1}e^{-(\frac{\lambda}{y})^\theta} t^{\alpha_1 j_2+\alpha_2 j_3+\alpha_2-1}\,dy$$

$$= \sum_{i=s}^{k}\sum_{j_1=0}^{i}\sum_{j_2=0}^{\beta_1(j_1+k-i)}\sum_{j_3=0}^{\beta_2-1}\binom{k}{i}\binom{i}{j_1}\binom{\beta_1(j_1+k-i)}{j_2}\binom{\beta_2-1}{j_3}(-1)^{j_1+j_2+j_3}\alpha_2\beta_2 \times \int_0^1 t^{\alpha_1 j_2+\alpha_2(j_3+1)-1}\,dt$$

$$= \sum_{i=s}^{k}\sum_{j_1=0}^{i}\sum_{j_2=0}^{\beta_1(j_1+k-i)}\sum_{j_3=0}^{\beta_2-1}\binom{k}{i}\binom{i}{j_1}\binom{\beta_1(j_1+k-i)}{j_2}\binom{\beta_2-1}{j_3}(-1)^{j_1+j_2+j_3+1}\frac{\alpha_2\beta_2}{\alpha_1 j_2+\alpha_2(j_3+1)}, \tag{6}$$

where

$$t = (1-e^{-(\frac{\lambda}{y})^\theta})$$

4 Maximum Likelihood Estimation for Reliability

Suppose

$$X \sim EGIW(\alpha_1,\beta_1,\lambda,\theta)$$

and

$$Y \sim EGIW(\alpha_2,\beta_2,\lambda,\theta)$$

and they are independent random variables. We need to compute the MLE of the vector of parameters $\underline{\phi} = (\alpha_1,\beta_1,\beta_2,\alpha_2,\lambda,\theta)$ to compute the MLE of R.

Suppose $x_1,x_2,\cdots,x_n$ is random sample from $EGIW(\alpha_1,\beta_1,\lambda,\theta)$, and $y_1,y_2,\cdots,y_m$ is random sample from $EGIW(\alpha_2,\beta_2,\lambda,\theta)$. The log likelihood function can be written as :

$$\begin{aligned} logL(x,y;\underline{\phi}) &= n\log\alpha_1 + n\log\beta_1 + mlog\alpha_2 + mlog\beta_2 + (n+m)\log\theta + \theta(n+m)\log\lambda \\ &\quad -(\theta+1)[\sum_{i=1}^{n}\log x_i + \sum_{j=1}^{m}\log y_j] - \sum_{i=1}^{n}(\frac{\lambda}{x_i})^\theta - \sum_{j=1}^{m}(\frac{\lambda}{y_j})^\theta \\ &\quad +(\alpha_1-1)\sum_{i=1}^{n}\log(1-e^{-(\frac{\lambda}{x_i})^\theta} + (\alpha_2-1)\sum_{j=1}^{m}\log(1-e^{-(\frac{\lambda}{y_j})^\theta} \\ &\quad +(\beta_1-1)\sum_{i=1}^{n}\log[1-(1-e^{-(\frac{\lambda}{x_i})^\theta})^{\alpha_1}] + (\beta_2-1)\sum_{j=1}^{m}\log[1-(1-e^{-(\frac{\lambda}{y_j})^\theta})^{\alpha_2}]. \end{aligned} \tag{7}$$

The MLE of $\alpha_1,\beta_1,\beta_2,\alpha_2,\lambda,\theta$ can be obtained as a solution of the following equations:

$$\frac{\partial L}{\partial\beta_1} = \frac{n}{\beta_1} + \sum_{i=1}^{n}\log[1-(1-e^{-(\frac{\lambda}{x_i})^\theta})^{\alpha_1}] = 0, \tag{8}$$

$$\frac{\partial L}{\partial\beta_2} = \frac{m}{\beta_2} + \sum_{j=1}^{m}\log[1-(1-e^{-(\frac{\lambda}{y_j})^\theta})^{\alpha_2}] = 0, \tag{9}$$

$$\frac{\partial L}{\partial\alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n}\log[1-e^{-(\frac{\lambda}{x_i})^\theta}] + (\beta_1-1)\sum_{i=1}^{n}\frac{-(1-e^{-(\frac{\lambda}{x_i})^\theta})^{\alpha_1}\times\log(1-e^{-(\frac{\lambda}{x_i})^\theta})}{[1-(1-e^{-(\frac{\lambda}{x_i})^\theta})^{\alpha_1}]} = 0, \tag{10}$$

$$\frac{\partial L}{\partial \alpha_2} = \frac{m}{\alpha_2} + \sum_{j=1}^{m} \log[1-e^{-(\frac{\lambda}{y_j})^{\theta}}] + (\beta_2-1)\sum_{j=1}^{m} \frac{-(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2} \times \log(1-e^{-(\frac{\lambda}{y_j})^{\theta}})}{[1-(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}]} = 0, \tag{11}$$

$$\begin{aligned}\frac{\partial L}{\partial \lambda} &= \frac{\theta(n+m)}{\lambda} - \sum_{i=1}^{n} \theta(\frac{1}{x_i})(\frac{\lambda}{x_i})^{\theta-1} - \sum_{j=1}^{m} \theta(\frac{1}{y_j})(\frac{\lambda}{y_j})^{\theta-1} \\ &+\theta(\alpha_1-1)\sum_{i=1}^{n} \frac{e^{-(\frac{\lambda}{x_i})^{\theta}}(\frac{1}{x_i})(\frac{\lambda}{x_i})^{\theta-1}}{[1-e^{-(\frac{\lambda}{x_i})^{\theta}}]} + \theta(\alpha_2-1)\sum_{j=1}^{m} \frac{e^{-(\frac{\lambda}{y_j})^{\theta}}(\frac{1}{y_j})(\frac{\lambda}{y_j})^{\theta-1}}{[1-e^{-(\frac{\lambda}{y_j})^{\theta}}]} \\ &+\theta\alpha_1(\beta_1-1)\sum_{i=1}^{n} \frac{e^{-(\frac{\lambda}{x_i})^{\theta}}(\frac{1}{x_i})(\frac{\lambda}{x_i})^{\theta-1}}{[1-(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}]} + \theta\alpha_2(\beta_2-1)\sum_{j=1}^{m} \frac{e^{-(\frac{\lambda}{y_j})^{\theta}}(\frac{1}{y_j})(\frac{\lambda}{y_j})^{\theta-1}}{[1-(1-e^{-(\frac{\lambda}{y_j})^{\theta}})^{\alpha_2}]} = 0,\end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{\partial L}{\partial \theta} &= \frac{(n+m)}{\theta} + (n+m)\log\lambda - \sum_{i=1}^{n}\log x_i - \sum_{j=1}^{m}\log y_j - \sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\theta}\log(\frac{\lambda}{x_i}) \\ &-\sum_{j=1}^{m}(\frac{\lambda}{y_j})^{\theta}\log(\frac{\lambda}{y_j}) + (\alpha_1-1)\sum_{i=1}^{n} \frac{e^{-(\frac{\lambda}{x_i})^{\theta}}(\frac{\lambda}{x_i})^{\theta}\log(\frac{\lambda}{x_i})}{(1-e^{-(\frac{\lambda}{x_i})^{\theta}})} \\ &+(\alpha_2-1)\sum_{j=1}^{m} \frac{e^{-(\frac{\lambda}{y_j})^{\theta}}(\frac{\lambda}{y_j})^{\theta}\log(\frac{\lambda}{y_j})}{(1-e^{-(\frac{\lambda}{y_j})^{\theta}})} \\ &+\alpha_1(\beta_1-1)\sum_{i=1}^{n} \frac{[1-e^{-(\frac{\lambda}{x_i})^{\theta}}]^{\alpha_1-1}e^{-(\frac{\lambda}{x_i})^{\theta}}(\frac{\lambda}{x_i})^{\theta}\log(\frac{\lambda}{x_i})}{[1-(1-e^{-(\frac{\lambda}{x_i})^{\theta}})^{\alpha_1}]} = 0.\end{aligned} \tag{13}$$

These nonlinear equations are solved numerically using iterative process as Newton Raphson to get

$$\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\theta}, \hat{\lambda},$$

then we can get the MLE of R as follows

$$\hat{R} = \hat{\alpha}_1\hat{\beta}_1 \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{\infty}(-1)^{(j_1+j_2+j_3+j_4)}\binom{\hat{\beta}_1-1}{j_1}\binom{\hat{\alpha}_1(j_1+1)-1}{j_2}\binom{\hat{\beta}_2}{j_3}\binom{\hat{\alpha}_2 j_3}{j_4}\frac{1}{(j_2+j_4+1)}. \tag{14}$$

Similarly, We can calculate the MLE of reliability for multi-component stress-strength model from Eq. (6).

5 Bayesian Estimation

In this section we provide the Bayes estimate of R where $\underline{\phi} = (\lambda, \theta, \alpha_1, \alpha_2, \beta_1, \beta_2)$ are unknown parameters and all of these parameters having independent gamma prior distribution as following:

$$\pi(\lambda) \sim Gamma(b_1, a_1),$$

$$\pi(\theta) \sim Gamma(b_2, a_2),$$

$$\pi(\alpha_1) \sim Gamma(b_3, a_3),$$

$$\pi(\alpha_2) \sim Gamma(b_4, a_4),$$

$$\pi(\beta_1) \sim Gamma(b_5, a_5),$$

and

$$\pi(\beta_2) \sim Gamma(b_6, a_6).$$

The joint posterior PDF is defined as

$$g(\underline{\phi}/data) = \frac{L(x,y/\alpha_1,\beta_1,\lambda,\theta,\alpha_2,\beta_2)\pi(\lambda)\pi(\theta)\pi(\alpha_1)\pi(\alpha_2)\pi(\beta_1)\pi(\beta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(x,y/\alpha_1,\beta_1,\lambda,\theta,\alpha_2,\beta_2)\pi(\lambda)\pi(\theta)\pi(\alpha_1)\pi(\alpha_2)\pi(\beta_1)\pi(\beta_2)d\underline{\phi}}.$$

Then

$$\begin{aligned} g(\underline{\phi}/data) &\propto \alpha_1{}^n\alpha_1{}^m\beta_1{}^n\beta_2{}^m\theta^{n+m}\lambda^{(n+m)\theta}\prod_{i=1}^{n}x_i{}^{-\theta-1}(1-e^{-(\frac{\lambda}{x_i})^\theta})^{\alpha_1-1}[1-(1-e^{-(\frac{\lambda}{x_i})^\theta})^{\alpha_1}]^{\beta_1-1} \\ &\times e^{-\sum_{i=1}^n(\frac{\lambda}{x_i})^\theta-\sum_{j=1}^m(\frac{\lambda}{y_j})^\theta}\prod_{j=1}^{m}y_j{}^{-\theta-1}(1-e^{-(\frac{\lambda}{y_j})^\theta})^{\alpha_2-1}[1-(1-e^{-(\frac{\lambda}{y_j})^\theta})^{\alpha_2}]^{\beta_2-1} \\ &\times \lambda^{a_1-1}e^{-b_1\lambda}\theta^{a_2-1}e^{-b_2\theta}\alpha_1{}^{a_3-1}e^{-b_3\alpha_1}\alpha_2{}^{a_4-1}e^{-b_4\alpha_2}\beta_1{}^{a_5-1}e^{-b_5\beta_1}\beta_2{}^{a_6-1}e^{-b_6\beta_2} \\ &\propto g_1(\theta/data)g_2(\lambda/\theta,data)g_3(\alpha_1/\lambda,\theta,data)g_4(\alpha_2/\lambda,\theta,data)g_5(\beta_1/data)g_6(\beta_2/data)h(\underline{\phi}/data), \end{aligned} \tag{15}$$

where

$$g_1(\theta/data) \propto Gamma\left(b_2+\sum_{i=1}^{n}lnx_i+\sum_{j=1}^{m}lny_j, a_2+n+m\right), \tag{16}$$

$$g_2(\lambda/\theta,data) \propto Gamma\left(b_1, a_1+(n+m)\theta\right), \tag{17}$$

$$g_3(\alpha_1/\lambda,\theta,data) \propto Gamma\left(b_3-\sum_{i=1}^{n}ln(1-e^{-(\frac{\lambda}{x_i})^\theta}), a_3+n\right), \tag{18}$$

$$g_4(\alpha_2/\lambda,\theta,data) \propto Gamma\left(b_4-\sum_{j=1}^{m}ln(1-e^{-(\frac{\lambda}{y_j})^\theta}), a_4+m\right), \tag{19}$$

$$g_5(\beta_1/data) \propto Gamma\left(b_5, a_5+n\right), \tag{20}$$

$$g_6(\beta_2/data) \propto Gamma\left(b_6, a_6+m\right), \tag{21}$$

and

$$\begin{aligned} h(\underline{\phi}/data) = \; & e^{-\sum_{i=1}^n(\frac{\lambda}{x_i})^\theta-\sum_{j=1}^m(\frac{\lambda}{y_j})^\theta}\prod_{i=1}^{n}\left[1-(1-e^{-(\frac{\lambda}{x_i})^\theta})^{\alpha_1}\right]^{\beta_1-1}\prod_{j=1}^{m}\left[1-(1-e^{-(\frac{\lambda}{y_j})^\theta})^{\alpha_2}\right]^{\beta_2-1} \\ & \times\frac{\Gamma(a_1+(n+m)\theta)e^{-\sum_{i=1}^n ln(1-e^{-(\frac{\lambda}{x_i})^\theta})}e^{-\sum_{j=1}^m ln(1-e^{-(\frac{\lambda}{y_j})^\theta})}}{b_1{}^{(n+m)\theta}[b_3-\sum_{i=1}^n ln(1-e^{-(\frac{\lambda}{x_i})^\theta})]^{a_3+n}[b_4-\sum_{j=1}^m ln(1-e^{-(\frac{\lambda}{y_j})^\theta})]^{a_4+m}}. \end{aligned} \tag{22}$$

Therefore, the Bayes estimate of reliability, say$\hat{R_B}$ under the squared error loss function

$$\hat{R_B} = \frac{\int_0^\infty R*g_1(\theta/data)g_2(\lambda/\theta,data)g_3(\alpha_1/\lambda,\theta,data)g_4(\alpha_2/\lambda,\theta,data)g_5(\beta_1/data)g_6(\beta_2/data)h(\underline{\phi}/data)d\underline{\phi}}{\int_0^\infty g_1(\theta/data)g_2(\lambda/\theta,data)g_3(\alpha_1/\lambda,\theta,data)g_4(\alpha_2/\lambda,\theta,data)g_5(\beta_1/data)g_6(\beta_2/data)h(\underline{\phi}/data)d\underline{\phi}}. \tag{23}$$

It is impossible to compute Eq.(23) analytically, therefore instead, we propose to approximate it by using importance sampling technique as suggested by Chen and Shao [14].

5.1 Importance Sampling Technique

In statistics, importance sampling is the name for the general technique of determining the properties of a distribution by drawing samples from another distribution. The focus of importance sampling here is to determine as easily and accurately as possible the properties of the posterior from a representative sample from the second distribution.

Since $g_1(\theta/data)$, $g_2(\lambda/\theta,data)$, $g_3(\alpha_1/\lambda,\theta,data)$, $g_4(\alpha_2/\lambda,\theta,data)$, $g_5(\beta_1/data)$, $g_6(\beta_2/data)$ follow gamma, it is quite simple to generate from them. Now we use the following algorithm assuming that $a_1,\cdots,a_6$ and $b_1,\cdots,b_6$ are known a prior, and assuming initial values for λ, θ, α_1, α_2, β_1, β_2.

Importance Sampling Algorithm:

–Step 1: Generate θ_1 from $g_1(./data)$.
–Step 2: Generate λ_1 from $g_2(./\theta,data)$.
–Step 3: Generate

$$\alpha_{11}\ from\ g_3(./\lambda,\theta,data),$$

and

$$\alpha_{21}\ from\ g_4(./\lambda,\theta,data).$$

–Step 4: Generate

$$\beta_{11}\ from\ g_5(./data),$$

and

$$\beta_{21}\ from\ g_6(./data).$$

–Step 5: Repeat this procedure N times to obtain $(\theta_1,\lambda_1,\alpha_{11},\alpha_{21},\beta_{11},\beta_{21}),\cdots,(\theta_N,\lambda_N,\alpha_{1N},\alpha_{2N},\beta_{1N},\beta_{2N})$.
–Step 6: An approximate Bayes estimate of R under a squared error loss function can be obtained as

$$\hat{R_B}=\frac{\frac{1}{N}\sum_{i=1}^{N}R_ih(\theta_i,\lambda_i,\alpha_1i,\alpha_2i,\beta_1i,\beta_2i/data)}{\frac{1}{N}\sum_{i=1}^{N}h(\theta_i,\lambda_i,\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i}/data)},$$

where

$$R_i=R(\theta_i,\lambda_i,\alpha_{1i},\alpha_{2i},\beta_{1i},\beta_{2i}),$$

as defined in Eq(5), for $i=1,\cdots,N$.

Using the same technique, We can obtained the Bayesian estimation of reliability for multi-component stress-strength model.

6 Simulation Study

In this section, we mainly present some simulation experiments to see the behavior of the proposed methods for various sample sizes and for parameter values $\alpha_1=0.75$, $\alpha_2=1.5$, $\beta_1=3.5$, $\beta_2=2.2$, $\lambda=1.008$, $\theta=0.61$, so that the true reliability value is 0.847751. We compared the performances of the MLEs and the Bayes estimates with respect to the squared error loss function in terms of biases and mean squares errors (MSEs). We have taken sample sizes namely $(n,m)=(5,5),(10,10),(20,20),(30,30)$.

For Bayesian estimation, we used importance sampling method under the informative gamma priors. For choosing a suitable hyper-parameters, the experimenters can incorporate their prior guess in terms of location and precision for the parameter of interest. Such that
$mean=a/b$, and $varience=a/b^2$.
We assume a small value of prior $varience(0.005)$, and taken the mean equal to the parameter of interest. For each parameter priors we solve the two equations of the mean and the varience, we obtain the following values of hyper-parameters :
$a_1=201.6$, $a_2=76.25$, $a_3=107.14$, $a_4=500$, $a_5=2500$, $a_6=956.5$ and $b_1=200$, $b_2=125$, $b_3=142.857$, $b_4=333.333$, $b_5=714.286$, $b_6=434.783$.

The maximum likelihood and Bayes estimates of the stress-strength reliability are obtained in Table 1.

Table 1: Average Bias and MSE of R using different estimators

Estimators	MLE			Bayesian		
(m,n)	$\hat{R}$	Bias	MSE	$\hat{R}_B$	Bias	MSE
(5,5)	0.90616	0.05841	0.01502	0.88337	0.0316	0.00425
(10,10)	0.89713	0.04938	0.01116	0.87597	0.02822	0.00124
(20,20)	0.89001	0.04226	0.00612	0.86825	0.0205	0.00096
(30,30)	0.87338	0.02563	0.00379	0.86002	0.01227	0.0007

7 Real data analysis

In this section, we present a data analysis of the strength data introduced in [15]. The data stand for the strength data measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50*mm*. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300*mm*. For illustrative purpose, we consider the data sets consisting the single fibers of 20 mm (Data Set 1) and 10 mm in gauge lengths (Data Set 2), with sample sizes 69 and 63 respectively. Data sets are provided below:

Data set 1:(strength measurements)
0.312, 0.314, 0.479, 0.552, 0.7, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.997, 1.006, 1.021, 1.055, 1.063, 1.098, 1.14, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585.

Data set 2:(stress measurements)
0.101, 0.332, 0.403, 0.428, 0.457, 0.550, 0.561, 0.596, 0.597, 0.645, 0.6540, 0.674, 0.718, 0.722, 0.725, 0.732, 0.775, 0.814, 0.816, 0.818, 0.824, 0.859, 0.875, 0.938, 0.940, 1.056, 1.117, 1.128, 1.137, 1.137, 1.177, 1.196, 1.230, 1.325, 1.339, 1.345, 1.420, 1.423, 1.435, 1.443, 1.464, 1.472, 1.494, 1.532, 1.546, 1.577, 1.608, 1.635, 1.693, 1.701, 1.737, 1.754, 1.762, 1.828, 2.052, 2.071, 2.086, 2.171, 2.224, 2.227, 2.425, 2.595, 3.2.

We fit the two data sets separately with the exponentiated generalized inverse Weibull distribution(EGIW) . we provide the Kolmogorov-Smirnov (K-S),Anderson-Darling(A-D) and Cramr-von Mises goodness-of-fit tests in Table 2. Obviously, the (EGIW) model fits well to Data Set 1 and Data Set 2.
The MLE and Bayesian estimates of R for the real data are provided in Table 3.

Table 2: P-value of different goodness-of-fit tests for data set 1, 2.

	K-S	A-D	Cramr-von
data set 1.	0.231248	0.143961	0.152425
data set 2.	0.192997	0.126852	0.213019

Table 3: Maximum likelihood ,Bayesian estimates of the parameters and R.

	α_1	α_2	β_1	β_2	λ	θ	R
MLE	2.7192	1.9639	4.4707	2.0057	0.9511	1.0789	0.55826
Bayes	1.1070	1.5513	3.6196	2.1963	1.5344	1.06724	0.7493

In case of multi-component stress-strength model, the maximum likelihood and Bayes estimates of the stress-strength reliability based on the real data sets, are presented in Table 4 for different values of s and k.

Table 4: The Maximum likelihood ,Bayesian estimates of $R_{s,k}$.

(s,k)	MLE	Bayes
(1,3)	0.73573	0.82293
(1,5)	0.83869	0.91084
(2,4)	0.54955	0.70667
(3,3)	0.16096	0.34609
(3,5)	0.42262	0.6608

From Table 4 we notice that: For fixed k, as s increases then the value of $R_{s,k}$ decreases, also for fixed s, as k increases then the value of $R_{s,k}$ increases.

8 Conclusion

In this paper we presented two methods for estimating $R = Pr(Y < X)$ when X and Y both follow exponentiated generalized inverse Weibull distribution with different parameters. We investigated Maximum likelihood and Bayesian estimation methods of R and their performances are examined by simulation study.
We have computed the Bayes estimate of R based on the independent gamma priors and using squared error loss function. Since the Bayes estimate cannot be obtained in explicit form, we have used he importance sampling technique to compute the Bayes estimate. Simulation results suggest that the performance of the Bayes estimator is better than maximum likelihood for all different sample sizes, also, maximum likelihood method provides very satisfactory results as sample size increased.

It is hoped that our investigation will be useful for researchers dealing with the kind of data considered in this paper.

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