Fuzzy Soft Semi Connected Properties in Fuzzy Soft Topological Spaces

A. Kandil\textsuperscript{1}, O. A. E. Tantawy\textsuperscript{2}, S. A. El-Sheikh\textsuperscript{3} and A. M. Abd El-latif\textsuperscript{3,*}

\textsuperscript{1} Mathematics Department, Faculty of Science, Helwan University, Helwan, Egypt.
\textsuperscript{2} Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt.
\textsuperscript{3} Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt.

Received: 30 Aug. 2014, Revised: 6 Oct. 2014, Accepted: 10 Oct. 2014
Published online: 1 May 2015

Abstract: In the present paper, we continue the study on fuzzy soft topological spaces and investigate the properties of fuzzy soft semi connected sets, fuzzy soft semi separated sets and fuzzy soft semi s-connected sets and have established several interesting properties supported by examples. Moreover, we show that a fuzzy soft semi disconnectedness property is not a hereditary property in general. Finally, we show that the fuzzy irresolute surjective soft image of fuzzy soft semi connected (resp. fuzzy soft semi s-connected) is also a fuzzy soft semi s-connected. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

Keywords: Soft set, Fuzzy soft set, Fuzzy soft topological space, Fuzzy semi soft interior, Fuzzy semi soft closure, Fuzzy semi open soft, Fuzzy semi closed soft, Fuzzy semi continuous soft functions, Fuzzy soft connected, Fuzzy soft semi connected, Fuzzy soft semi s-connected.

1 Introduction

The concept of soft sets was first introduced by Molodtov\textsuperscript{2} in 1999 as a general mathematical tool for dealing with uncertain objects. In\textsuperscript{29,30}, Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets\textsuperscript{27}, the properties and applications of soft set theory have been studied increasingly\textsuperscript{4,22,30}. Xiao et al.\textsuperscript{39} and Pei and Miao\textsuperscript{33} discussed the relationship between soft sets and information systems. They showed that soft sets are a class of special information systems. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets\textsuperscript{2,3,5,8,25,26,27,28,30,31,42}. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations\textsuperscript{9}.

Recently, in 2011, Shabir and Naz\textsuperscript{36} initiated the study of soft topological spaces. They defined soft topology on the collection \(\tau\) of soft sets over \(X\). Consequently, they defined basic notions of soft topological spaces such as open soft and closed soft sets, soft subspace, soft closure, soft nbd of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. Min in\textsuperscript{38} investigate some properties of these soft separation axioms. Kandil et al.\textsuperscript{19} introduce the notion of soft semi separation axioms. In particular they studied the properties of the soft regular spaces and soft semi normal spaces. Maji et. al.\textsuperscript{25} initiated the study involving both fuzzy sets and soft sets. The notion of soft ideal was initiated for the first time by Kandil et al.\textsuperscript{15}. They also introduced the concept of soft local function. These concepts are discussed with a view to find new soft topologies from the original one, called soft topological spaces with soft ideal \((X,\tau, E, \tilde{I})\). Applications to various fields were further investigated by Kandil et al.\textsuperscript{13,14,16,17,18}.

In\textsuperscript{6} the notion of fuzzy set soft set was introduced as a fuzzy generalization of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Then many scientists such as X. Yang et. al.\textsuperscript{40} improved the concept of fuzziness of soft sets. In\textsuperscript{1}, Karal and Ahmed

* Corresponding author e-mail: Alaa_8560@yahoo.com, dr.Alaa_daby@yahoo.com
defined the notion of a mapping on classes of fuzzy soft sets, which is fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Tanay et al. [37] introduced the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [35] gave the definition of fuzzy soft topology over the initial universe set. Chang [10] introduced the concept of fuzzy topology on a set by axiomatizing a collection $\mathcal{T}$ of fuzzy subsets of $X$.

In the present paper, we continue the study on fuzzy soft topological spaces and investigate the properties of fuzzy soft semi connected sets, fuzzy soft semi separated sets and fuzzy soft semi s-connected sets and have established several interesting properties supported by examples. Moreover, we show that a fuzzy soft semi disconnection property is not hereditary in general. Finally, we show that the fuzzy irresolute surjective soft image of fuzzy soft semi connected (resp. fuzzy soft semi s-connected) is also a fuzzy soft semi s-connected. Since the authors introduced topological structures on fuzzy soft sets [6,11,37], so the semi topological properties, which introduced by Mahanta et al. [24], is generalized here to the fuzzy soft sets which will be useful in the fuzzy systems. Because there exists compact connections between soft sets and information systems [33,39], we can use the results deduced from the studies on fuzzy soft topological space to improve these kinds of connections.

2 Preliminaries

In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel.

**Definition 2.1.**[41] A fuzzy set $A$ of a non-empty set $X$ is characterized by a membership function $\mu_A : X \rightarrow [0,1]$ whose value $\mu_A(x)$ represents the "degree of membership" of $x$ in $A$ for $x \in X$.

Let $I^X$ denotes the family of all fuzzy sets on $X$. If $A, B \in I^X$, then some basic set operations for fuzzy sets are given by Zadeh [1], as follows:

1. $A \subseteq B \iff \mu_A(x) \leq \mu_B(x) \forall x \in X$.
2. $A = B \iff \mu_A(x) = \mu_B(x) \forall x \in X$.
3. $C = A \cup B \iff \mu_C(x) = \mu_A(x) \lor \mu_B(x) \forall x \in X$.
4. $D = A \cap B \iff \mu_D(x) = \mu_A(x) \land \mu_B(x) \forall x \in X$.
5. $M = A^c \iff \mu_M(x) = 1 - \mu_A(x) \forall x \in X$.

**Definition 2.2.**[25] Let $A \subseteq E$. A pair $(f,A)$, denoted by $f_A$, is called fuzzy soft set over $X$, where $f$ is a mapping given by $f : A \rightarrow I^X$, defined by $f_A(e) = \mu_{f_A}^e$, where $\mu_{f_A}^e = 0$ if $e \notin A$ and $\mu_{f_A}^e \neq 0$ if $e \in A$, where $0 \leq x < 1$.

The family of all these fuzzy soft sets over $X$ denoted by $FSS(X)_A$.

**Proposition 2.1.**[3] Every fuzzy set may be considered a soft set.

**Definition 2.3.**[34] The complement of fuzzy soft set $(f,A)$, denoted by $(f,A)'$, is defined by $(f,A)' = (f',A)$, $f'_A : E \rightarrow I^X$ is a mapping given by $\mu_{f'_A} = \overline{1} - \mu_{f_A} \forall e \in A$, where $\overline{1} = 1 \forall x \in X$. Clearly, $(f_A)' = f_A$.

**Definition 2.4.**[27] A fuzzy soft set $f_A$ over $X$ is said to be an absolute fuzzy soft set, denoted by $\hat{1}_A$, if for all $e \in A$, $f_A(e) = \overline{1}$.

**Definition 2.5.**[27] A fuzzy soft set $f_A$ over $X$ is said to be a NULL fuzzy soft set, denoted by $\hat{0}_A$, if for all $e \in A$, $f_A(e) = \overline{0}$.

**Definition 2.6.**[27] Let $f_A$, $g_B \in FSS(X)_E$. Then $f_A$ is fuzzy soft subset of $g_B$, denoted by $f_A \subseteq g_B$, if $A \subseteq B$ and $\mu_{f_A}^e \leq \mu_{g_B}^e \forall e \in A$, i.e., $\mu_{f_A}^e(x) \leq \mu_{g_B}^e(x) \forall x \in X$ and $\forall e \in A$.

**Definition 2.7.**[34] The union of two fuzzy soft sets $f_A$ and $g_B$ over the common universe $X$ is also a fuzzy soft set $h_C$, where $h_C(e) = \mu_{h_C}^e = \mu_{f_A}^e \lor \mu_{g_B}^e \forall e \in E$. Here, we write $h_C = f_A \lor g_B$.

**Definition 2.8.**[34] The intersection of two fuzzy soft sets $f_A$ and $g_B$ over the common universe $X$ is also a fuzzy soft set $h_C$, where $h_C(e) = \mu_{h_C}^e = \mu_{f_A}^e \land \mu_{g_B}^e \forall e \in E$. Here, we write $h_C = f_A \land g_B$.

**Theorem 2.1.**[2] Let $\{(f,A)_j : j \in J\} \subseteq FSS(X)_E$. Then the following statements hold,

1. $\bigcup_{j \in J}(f,A)_j' = \bigcap_{j \in J}(f,A)_j'$.
2. $\bigcap_{j \in J}(f,A)_j' = \bigcup_{j \in J}(f,A)_j$.

**Definition 2.9.**[34] Let $\mathcal{T}$ be a collection of fuzzy soft sets over a universe $X$ with a fixed set of parameters $E$, then $\mathcal{T} \subseteq FSS(X)_E$ is called fuzzy soft topology on $X$ if

1. $\hat{1}_E, \hat{0}_E \in \mathcal{T}$, where $\hat{0}_E(e) = \overline{0}$ and $\hat{1}_E(e) = \overline{1}$, $\forall e \in E$.
2. the union of any members of $\mathcal{T}$ belongs to $\mathcal{T}$.
3. the intersection of any two members of $\mathcal{T}$ belongs to $\mathcal{T}$.

The triplet $(X, \mathcal{T}, E)$ is called fuzzy soft topological space over $X$. Also, each member of $\mathcal{T}$ is called fuzzy open soft set in $(X, \mathcal{T}, E)$. We denote the set of all open soft sets by $FOS(X, \mathcal{T}, E)$, or $FOS(X)$.

**Definition 2.10.**[34] Let $(X, \mathcal{T}, E)$ be a fuzzy soft topological space. A fuzzy soft set $f_A$ over $X$ is said to be fuzzy closed soft set in $X$, if its relative complement $f_A'$ is fuzzy open soft set. We denote the set of all fuzzy closed soft sets by $FCS(X, \mathcal{T}, E)$, or $FCS(X)$.

**Definition 2.11.**[32] Let $(X, \mathcal{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. The fuzzy soft closure of $f_A$, denoted by $Cl(f_A)$ is the intersection of all fuzzy soft super sets of $f_A$, i.e.,

$Cl(f_A) = \cap \{h_D : h_D$ is fuzzy closed soft set and $f_A \subseteq h_D\}$.

The fuzzy soft interior of $g_B$, denoted by $Int(g_B)$ is the fuzzy soft union of all fuzzy open soft subsets of $f_A$, i.e.,

$Int(g_B) = \cup \{h_D : h_D$ is fuzzy open soft set and $h_D \subseteq g_B\}$.

**Definition 2.12.**[24] The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such
that \( \mu_{x_{\alpha}}^c(x) = \alpha (0 < \alpha \leq 1) \) and \( \mu_{f_{\alpha}}^c(y) = \emptyset \) for each \( y \in X \setminus \{x\} \), and this fuzzy soft point is denoted by \( x_{\alpha}^c \) or \( f_{\alpha} \).

**Definition 2.13.**[24] The fuzzy soft point \( x_{\alpha}^c \) is said to be belonging to the fuzzy soft set \( (g, A) \), denoted by \( x_{\alpha}^c \in (g, A) \), if for the element \( e \in A \), \( \alpha \leq \mu_{x_{\alpha}}^c(e) \).

**Theorem 2.2.**[24] Let \((X, \mathcal{T}, E)\) be a fuzzy soft topological space and \( f_e \) be a fuzzy soft point. Then the following properties hold:

1. If \( f_e \in gA \), then \( f_e \notin gA' \);
2. \( f_e \notin gA \iff f_e \in gA' \);
3. Every non-null fuzzy soft set \( f_A \) can be expressed as the union of all the fuzzy soft points belonging to \( f_A \).

**Definition 2.14.**[24] A fuzzy soft set \( gB \) in a fuzzy soft topological space \((X, \mathcal{T}, E)\) is said to be a soft neighborhood of the fuzzy soft point \( x_{\alpha}^c \) if there exists a fuzzy open soft set \( h_C \) such that \( x_{\alpha}^c \in h_C \subseteq gB \). A fuzzy soft set \( gB \) in a fuzzy soft topological space \((X, \mathcal{T}, E)\) is called fuzzy soft neighborhood of the soft set \( f_A \) if there exists a fuzzy open soft set \( h_C \) such that \( f_A \subseteq h_C \subseteq gB \). Let \( N_{x_{\alpha}^c} \) denote the family of all its fuzzy soft neighborhoods.

**Definition 2.15.**[24] Let \((X, \mathcal{T}, E)\) be a fuzzy soft topological space and \( Y \subseteq X \). Let \( h_E^c \) be a fuzzy soft set over \((Y, E)\) such that \( h_E^c : E \rightarrow Y \) and \( h_E^c(e) = \mu_{x_{\alpha}}^c \).

Let \( \mathcal{T}_Y = \{h_E^c \cap gB : gB \in \mathcal{T}\} \), then the fuzzy soft topology \( \mathcal{T}_Y \) on \((Y, E)\) is called fuzzy soft subspace topology for \((Y, E)\) and \((X, \mathcal{T}, E)\) is called fuzzy soft subspace of \((X, \mathcal{T}, E)\). If \( h_E^c \in \mathcal{T} \), then \((Y, \mathcal{T}_Y, E)\) is called fuzzy open (resp. closed) soft subspace of \((X, \mathcal{T}, E)\).

**Definition 2.16.**[32] Let \( FSS(X)_E \) and \( FSS(Y)_K \) be families of fuzzy soft sets over \( X \) and \( Y \), respectively. Let \( u : X \rightarrow Y \) and \( p : E \rightarrow K \) be mappings. Then the map \( f_{pu} \) is called fuzzy soft mapping from \( X \) to \( Y \) and denoted by \( f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K \) such that:

1. If \( f_A \in FSS(X)_E \), then the image of \( f_A \) under the fuzzy soft mapping \( f_{pu} \) is the fuzzy soft set over \( Y \) defined by \( f_{pu}(f_A)(K)(y) = \begin{cases} \bigvee_{u(x) = y} [\bigvee_{p(e) = k} (f_A(e))](x) & \text{if } x \in u^{-1}(y), \\ 0 & \text{otherwise}. \end{cases} \)

2. If \( gB \in FSS(Y)_K \), then the pre-image of \( gB \) under the fuzzy soft mapping \( f_{pu} \) is the fuzzy soft set over \( X \) defined by \( f_{pu}^{-1}(gB)(e)(x) = \begin{cases} gB(p(e))(u(x)) & \text{for } p(e) \in B, \\ 0 & \text{otherwise}. \end{cases} \)

The fuzzy soft mapping \( f_{pu} \) is called surjective (resp. injective) if \( p \) and \( u \) are surjective (resp. injective), also it is said to be constant if \( p \) and \( u \) are constant.

**Definition 2.17.**[32] Let \((X, \mathcal{T}_1, E)\) and \((Y, \mathcal{T}_2, K)\) be two fuzzy soft topological spaces and \( f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K \) be a fuzzy soft mapping. Then \( f_{pu} \) is called:

1. Fuzzy continuous soft if \( f_{pu}^{-1}(gB) \in \mathcal{T}_1 \forall (gB) \in \mathcal{T}_2 \).
2. Fuzzy open soft if \( f_{pu}(gA) \in \mathcal{T}_2 \forall (gA) \in \mathcal{T}_1 \).

**Theorem 2.3.**[1] Let \( FSS(X)_E \) and \( FSS(Y)_K \) be two families of fuzzy soft sets. For the fuzzy soft function \( f_{pu} : FSS(X)_E \rightarrow FSS(Y)_K \), the following statements hold:

1. If \( f_{pu} \) is surjective, then the equality holds.
2. If \( f_{pu} \) is injective, then the equality holds.
3. If \( f_{pu} \) is surjective (resp. injective), also it is said to be semi open (resp. closed) soft.

**Definition 2.18.**[24] Let \((X, \mathcal{T}, E)\) be a fuzzy soft topological space. A fuzzy soft separation of \( \mathcal{I}_E \) is a pair of non null proper fuzzy open soft sets \( gB, hC \) such that \( gB \cap hC = 0 \) and \( \mathcal{I}_E = gB \cup hC \).

**Definition 2.19.**[24] A fuzzy soft topological space \((X, \mathcal{T}, E)\) is said to be fuzzy soft connected if and only if there is no fuzzy soft separations of \( X \). Otherwise, \((X, \mathcal{T}, E)\) is said to be fuzzy soft disconnected space.

**Definition 2.20.**[12] Let \((X, \tau, E)\) be a soft topological space and \( F_A \in SS(X)_E \). If \( F_A \subseteq (\text{int}(F_A)) \), then \( F_A \) is called semi open soft set. We denote the set of all semi open soft sets by \( SOS(X, \tau, E) \), or \( SOS(X) \) and the set of all semi closed soft sets by \( SCS(X, \tau, E) \), or \( SCS(X) \).

**Definition 2.21.**[7] Let \((X, \tau, E)\) be a soft topological space. A soft semi separation on \( X \) is a pair of non null semi proper open soft sets \( F_A, G_B \) such that \( F_A \cap G_B = \emptyset \) and \( X = F_A \cup G_B \).

**Definition 2.22.**[7] A soft topological space \((X, \tau, E)\) is said to be semi soft connected if and only if there is no soft semi separations of \( X \). Otherwise, \((X, \tau, E)\) is said to be soft semi disconnected space.

**Definition 2.23.**[20] Let \((X, \mathcal{T}, E)\) be a fuzzy soft topological space and \( f_A \in FSS(X)_E \). If \( f_A \subseteq Fcl(Fint(f_A)) \), then \( f_A \) is called fuzzy semi open soft set. We denote the set of all fuzzy semi open soft sets.
by $FSOS(X, \mathcal{I}, E)$, or $FSOS(X)$ and the set of all fuzzy semi closed soft sets by $FSCS(X, \mathcal{I}, E)$, or $FSCS(X)$.

**Definition 2.24.** Let $(X, \mathcal{I}, E)$ be a fuzzy soft topological space, $f_A \in FSS(X)_E$ and $f_e \in FSS(X)_E$. Then

1. $f_e$ is called fuzzy semi interior soft point of $f_A$ if $\exists g_B \in FSOS(X)$ such that $f_e \subseteq g_B \subseteq f_A$. The set of all fuzzy semi interior soft points of $f_A$ is called the fuzzy semi interior of $f_A$ and is denoted by $FSint(f_A)$ consequently.

2. $f_e$ is called fuzzy semi closure soft point of $f_A$ if $f_e \cap h_C \neq \emptyset \forall h_D \in FSOS(X)$. The set of all fuzzy semi closure soft points of $f_A$ is called fuzzy semi soft closure of $f_A$ and denoted by $FScl(f_A)$. Consequently, $FScl(f_A) = \bigcap \{h_D : h_D \in FSCS(X), f_A \subseteq h_D\}$.

**Definition 2.25.** Let $(X, \mathcal{I}_1, E)$, $(X, \mathcal{I}_2, K)$ be fuzzy soft topological spaces and $f_{p_u} : FSS(X)_E \rightarrow FSS(Y)_K$ be a soft function. Then $f_{p_u}$ is called:

1. Fuzzy semi continuous soft function if $f_{p_u}(g_B) \in FSOS(Y) \forall g_B \in \mathcal{I}_2$.
2. Fuzzy semi open soft if $f_{p_u}(g_A) \in FSOS(Y) \forall g_A \in \mathcal{I}_1$.
3. Fuzzy semi closed soft if $f_{p_u}(g_A) \in FSCS(Y) \forall g_A \in \mathcal{I}_1$.
4. Fuzzy irresolute soft if $f_{p_u}(g_B) \subseteq FSOS(Y) \forall g_B \in FSOS(X)$.
5. Fuzzy irresolute open soft if $f_{p_u}(g_A) \subseteq FSOS(Y) \forall g_A \in FSOS(X)$.
6. Fuzzy irresolute closed soft if $f_{p_u}(f_A) \subseteq FSCS(Y) \forall f_A \in FSCS(Y)$.

### 3 Fuzzy soft semi connectedness

Connectedness is one of the important notions of topology. F. Lin [23] introduced the notions of soft connectedness in soft topological spaces. Mahanta and Das [24] introduce the notions of fuzzy soft connectedness in fuzzy soft topological spaces. In this section, we introduce the notions of fuzzy soft semi connectedness in fuzzy soft topological space and examine its basic properties.

**Definition 3.1.** Two fuzzy soft sets $f_A$ and $g_B$ are said to be disjoint, denoted by $f_A \cap g_B = \emptyset_E$, if $A \cap B = \emptyset$ and $\mu_{f_A} \cap \mu_{g_B} = 0 \forall e \in E$.

**Definition 3.2.** Let $(X, \mathcal{I}, E)$ be a fuzzy soft topological space. A fuzzy soft semi separation on $I_E$ is a pair of non null proper fuzzy semi open soft sets $f_A, g_B$ such that $f_A \cap g_B = \emptyset_E$ and $I_E = f_A \cup g_B$.

**Definition 3.3.** A fuzzy soft topological space $(X, \mathcal{I}, E)$ is said to be fuzzy soft semi connected if and only if there is no fuzzy soft semi separation of $I_E$. Otherwise, $(X, \mathcal{I}, E)$ is said to be fuzzy soft semi disconnected space.

**Examples 3.1.**

(1) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$ and $\mathcal{I}$ be the discrete fuzzy soft topology on $X$. Then $(X, \mathcal{I}, E)$ is not fuzzy soft semi connected.

(2) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$ and $\mathcal{I}$ be the indiscrete fuzzy soft topology on $X$. Then $\mathcal{I}$ is always fuzzy soft semi connected.

**Definition 3.4.** A fuzzy soft subspace $(Y, \mathcal{I}_Y, E)$ of fuzzy soft topological space $(X, \mathcal{I}, E)$ is said to be fuzzy semi open soft (resp. semi closed soft, soft semi connected) subspace if $h^1_Y \subseteq FSOS(X)$ (resp. $h^2_Y \subseteq FSCS(X)$, $h^3_Y$ is fuzzy soft semi connected).

**Theorem 3.1.** Let $(Y, \mathcal{I}_Y, E)$ be a fuzzy soft semi connected subspace of fuzzy soft topological space $(X, \mathcal{I}, E)$ such that $h^1_Y \subseteq FSOS(X)$ $g_A \subseteq FSOS(X)$. If $I_E$ has a fuzzy soft semi separations $f_A, g_B$, then either $h^1_Y \subseteq f_A$, or $h^1_Y \subseteq g_B$.

**Proof.** Let $f_A, g_B$ be fuzzy soft semi separation on $I_E$. By hypothesis, $f_A \cap h^1_Y \subseteq FSOS(X)$, $g_B \cap h^1_Y \subseteq FSOS(X)$ and $g_B \cap h^1_Y \subseteq f_A \cap h^1_Y = h^1_Y$. Therefore, either $h^1_Y \subseteq f_A$, or $h^1_Y \subseteq g_B$.

**Theorem 3.2.** If $(X, \mathcal{I}_2, E)$ is a fuzzy soft semi connected space and $\mathcal{I}_1$ is fuzzy soft coarser than $\mathcal{I}_2$, then $(X, \mathcal{I}_1, E)$ is also a fuzzy soft semi connected.

**Proof.** Let $f_A, g_B$ be fuzzy soft semi separation on $(X, \mathcal{I}_1, E)$. Then $f_A, g_B \subseteq \mathcal{I}_1$. Since $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $f_A, g_B \subseteq \mathcal{I}_2$ such that $f_A, g_B$ is fuzzy soft semi separation on $(X, \mathcal{I}_2, E)$, which is a contradiction with the fuzzy soft semi connectedness of $(X, \mathcal{I}_2, E)$. Hence, $(X, \mathcal{I}_1, E)$ is fuzzy soft semi connected.

**Remark 3.1.** The converse of Theorem 3.2 is true in general, as shown in the following example.

**Example 3.1.** Let $X = \{a, b, c\}$, $E = \{e_1, e_2, e_3, e_4\}$ and $A, B \subseteq E$ where $A = \{e_1, e_2\}$ and $B = \{e_3, e_4\}$. Let $\mathcal{I}_1$ be the indiscrete fuzzy soft semi topology, then $\mathcal{I}_1$ is fuzzy soft semi connected, on the other hand, let $\mathcal{I}_2 = \{\emptyset_E, \{0_E, f_A, g_A, k_B, h_B, s_E, v_E\}\}$ where $f_A, g_A, k_B, h_B, s_E, v_E$ are fuzzy soft sets over $X$ defined as follows:

- $f_A^c = \{a_1, b_1, c_1\}$, $g_A^c = \{a_1, b_1, c_1\}$,
- $k_B^c = \{a_0, b_0, c_0\}$, $h_B^c = \{a_0, b_0, c_0\}$,
- $s_E^c = \{a_1, b_1, c_1\}$, $v_E^c = \{a_1, b_1, c_1\}$,
- $\mu_{f_A}^c = \{a_1, b_1, c_1\}$, $\mu_{g_A}^c = \{a_1, b_1, c_1\}$,
- $\mu_{k_B}^c = \{a_0, b_0, c_0\}$, $\mu_{h_B}^c = \{a_0, b_0, c_0\}$,
- $\mu_{s_E}^c = \{a_1, b_1, c_1\}$, $\mu_{v_E}^c = \{a_1, b_1, c_1\}$.

Then $\mathcal{I}_2$ defines a fuzzy soft topology on $X$ such that $\mathcal{I}_1 \subseteq \mathcal{I}_2$. Now, $f_A$ and $k_B$ are fuzzy semi open soft sets in which form a fuzzy soft semi separation of $(X, \mathcal{I}_2, E)$ where $f_A \cap k_B = \emptyset_E$ and $I_E = f_A \cup k_B$. Hence, $(X, \mathcal{I}_2, E)$ is fuzzy soft semi disconnected.

**Theorem 3.3.** A fuzzy soft subspace $(Y, \mathcal{I}_Y, E)$ of fuzzy soft semi disconnected space $(X, \mathcal{I}, E)$ is fuzzy soft semi disconnected if $h^1_Y \subseteq FSOS(X)$ $g_A \subseteq FSOS(X)$. 

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Proof. Let \((Y, \Sigma_Y, E)\) be fuzzy soft semi connected space. Since \((X, \Sigma, E)\) is fuzzy soft semi disconnected. Then there exist fuzzy soft semi separation \(f_A, g_B\) on \((X, \Sigma, E)\).

By hypothesis, \(h_E \cap h_E ^{\pi} \in F SOS(X), g_B \cap h_E ^{\pi} \in F SOS(X)\) and \([g_B \cap h_E ^{\pi}] \cup [f_A \cap h_E ^{\pi}] = h_E ^{\pi}\), which is a contradiction with the fuzzy soft semi connectedness of \((Y, \Sigma_Y, E)\). Therefore, \((Y, \Sigma_Y, E)\) is fuzzy soft semi disconnected.

Remark 3.2 A fuzzy soft semi disconnectedness property is not hereditary property in general, as in the following example.

Example 3.2. In Example 3.1, let \(Y = \{a, b\} \subseteq X\). We consider the fuzzy soft set \(h_Y ^{\pi}\) over \((Y, E)\) defined as follows:

\[
\mu ^{c_e}_{h_E} = \{a, b, 1, 0\}, \quad \mu ^{c_e}_{h_E ^{\pi}} = \{a, b, 1, 0\}, \quad \mu ^{e_{c_e}}_{h_E} = \{a, b, 1, 0\},
\]

Then we find \(\Sigma_Y\) as follows, \(\Sigma_Y = \{h_Y \cap z_E : z_E \in \Sigma\}\) where

\[h_Y ^{\pi} \cap \emptyset = \emptyset, \quad h_Y ^{\pi} \cap \tilde{I}_E = h_Y ^{\pi}, \quad h_Y ^{\pi} \cap f_A = h_C, \quad \mu ^{h_C}_{\Sigma_Y} = \{a, b, 1, 0\}, \quad \mu ^{h_C}_{\Sigma_Y ^{\pi}} = \{a, b, 1, 0\}, \quad \mu ^{h_C}_{\Sigma_Y ^{\pi}} = \{a, b, 1, 0\}, \quad \mu ^{h_C}_{\Sigma_Y ^{\pi}} = \{a, b, 1, 0\}.
\]

Thus, the collection \(\Sigma_Y = \{h_Y \cap z_E : z_E \in \Sigma\}\) is a fuzzy soft topology on \((Y, E)\). Therefore, \((Y, \Sigma_Y, E)\) is fuzzy soft semi connected at the time that \((X, \Sigma, E)\) is fuzzy soft semi disconnected as shown in Example 3.1.

Theorem 4.3. Let \((X_1, \Sigma_1, E)\) and \((X_2, \Sigma_2, E)\) be fuzzy soft topological spaces and \(f_{\mu} : (X_1, \Sigma_1, E) \rightarrow (X_2, \Sigma_2, E)\) be a fuzzy irresolute surjective soft function. If \((X_1, \Sigma_1, E)\) is fuzzy soft semi connected, then \((X_2, \Sigma_2, E)\) is also a fuzzy soft semi connected.

Proof. Let \((X_2, \Sigma_2, E)\) be a fuzzy soft semi disconnected space. Then there exist \(f_A, g_B\) pair of non null proper fuzzy semi open soft subsets of \(I_E\) such that \(f_A \cap g_B = 0\) and \(\tilde{I}_E = f_A \cup g_B\). Since \(f_{\mu}^{-1}(f_A, g_B)\) is a pair of non null proper fuzzy soft semi open soft subsets of \(I_E\) such that \(f_{\mu}^{-1}(f_A) \cap f_{\mu}^{-1}(g_B) = f_{\mu}^{-1}(f_A) \cap g_B = f_{\mu}^{-1}(f_A) \cap g_B = f_{\mu}^{-1}(I_E) = I_E\) from Theorem 2.3. This means that, \(f_{\mu}^{-1}(f_A, g_B)\) forms a fuzzy soft semi separation of \(I_E\), which is a contradiction with the fuzzy soft semi connectedness of \((X_1, \Sigma_1, E)\). Therefore, \((X_2, \Sigma_2, E)\) is fuzzy soft semi connected.

4 Fuzzy soft semi s-connected spaces

In this section, we introduce the notions of fuzzy soft semi separated sets and use it to introduce the notions of fuzzy soft semi connectedness in fuzzy soft topological spaces and study its basic properties. Definition 4.1.

A non null fuzzy soft subsets \(f_A, g_B\) of fuzzy soft topological space \((X, \Sigma, E)\) are said to be fuzzy soft semi separated sets if \(F Sc l(f_A) \cap g_B = F Sc l(g_B) \cap f_A = 0\).

Theorem 4.1. Let \(f_A \cap g_B\) be fuzzy soft semi separated sets. Then \(f_A \cap g_B\) are fuzzy soft semi separated subsets of fuzzy soft topological space \((X, \Sigma, E)\). Then \(f_A, g_B\) are fuzzy soft semi separated sets.

Proof. Let \(f_A \cap g_B\). Then \(F Sc l(f_A) \cap g_B = F Sc l(g_B) \cap f_A = 0\). It follows that, \(F Sc l(f_A) \cap g_B = F Sc l(g_B) \cap f_A = 0\).

Theorem 4.2. Two fuzzy semi closed subsets of fuzzy soft topological space \((X, \Sigma, E)\) are fuzzy soft semi separated sets if and only if they are disjoint.

Proof. Let \(f_A, g_B\) be fuzzy soft semi separated sets. Then \(F Sc l(g_B) \cap f_A = g_B \cap F Sc l(f_A) = 0\). Since \(f_A, g_B\) are fuzzy soft semi closed sets. Then \(f_A \cap g_B = 0\).

Theorem 4.3. Let \((Z, \Sigma_Z, E)\) be a fuzzy soft subspace of fuzzy soft topological space \((X, \Sigma, E)\) and \(f_A, g_B\) are fuzzy soft semi separated sets in \((X, \Sigma, E)\). Then \(f_A, g_B\) are fuzzy soft semi separated sets.

Definition 4.2. A fuzzy soft topological space \((X, \Sigma, E)\) is said to be fuzzy soft semi s-connected if and only if \(I_E\) can not be expressed as the fuzzy soft union of two fuzzy soft semi separated sets in \((X, \Sigma, E)\).

Theorem 4.4. Let \((Z, \Sigma_Z, E)\) be a fuzzy soft subspace of fuzzy soft topological space \((X, \Sigma, E)\) and \(f_A, g_B\) are fuzzy soft semi separated sets on \(\Sigma_Z\) if and only if \(f_A, g_B\) are fuzzy soft semi connected on \(\Sigma\) where \(\Sigma_Z\) is the fuzzy soft subspace for \(\Sigma\).

Proof. Suppose that \(f_A, g_B\) are fuzzy soft semi separated on \(\Sigma_Z\) if and only if \(f_A, g_B\) are fuzzy soft semi connected w.r.t \((Z, \Sigma_Z, E)\).

Theorem 4.5. Let \(\Sigma_Z\) be a fuzzy soft subset of fuzzy soft topological space \((X, \Sigma, E)\). Then \(\Sigma_Z\) is fuzzy soft semi s-connected w.r.t \((X, \Sigma, E)\) if and only if \(\Sigma_Z\) is fuzzy soft semi s-connected w.r.t \((Z, \Sigma_Z, E)\).

Proof. Suppose that \(\Sigma_Z\) is not fuzzy soft semi s-connected w.r.t \((Z, \Sigma_Z, E)\). Then \(\Sigma_Z = f_{1A} \cup f_{2B}\), where \(f_{1A}\) and \(f_{2B}\) are fuzzy soft semi separated sets on \(\Sigma_Z\).
Theorem 4.5. Let \((Z, \mathcal{Z}_N, E)\) be a fuzzy soft semi s-connected subspace of fuzzy soft topological space \((X, \mathcal{X}, E)\) and \(f_A, g_B\) be fuzzy soft semi separated of \(I_E\) with \(z_E \subseteq f_A \cup g_B\), then either \(z_E \subseteq f_A\) or \(z_E \subseteq g_B\).

Proof. Let \(z_E \subseteq f_A \cup g_B\) for some fuzzy soft semi separated subsets \(f_A, g_B\) of \(I_E\). Since \(z_E = (z_E \cap f_A) \cup (z_E \cap g_B)\). \(z_E \subseteq f_A \cap g_B\). Then \((z_E \cap f_A) \cap \text{FScI}(z_E \cap g_B) \subseteq (f_A \cap \text{FScI}(g_B) = \emptyset_E\). \(\text{FScI}(z_E \cap f_A) \cap (z_E \cap g_B) = \emptyset_E\). \(\text{FScI}(z_E)\) is fuzzy soft semi connected. Thus, either \(z_E \subseteq f_A\) or \(z_E \subseteq g_B\). This implies that, \(z_E \subseteq f_A\) or \(z_E \subseteq g_B\).

Theorem 4.6. Let \((Z, \mathcal{Z}_N, N)\) and \((Y, \mathcal{Y}_M, M)\) be fuzzy soft semi s-connected subspaces of fuzzy soft topological space \((X, \mathcal{X}, E)\) such that none of them is fuzzy soft semi connected. Then \(\mathcal{Z}_N \cup \mathcal{Y}_M\) is fuzzy soft semi s-connected.

Proof. Let \((Z, \mathcal{Z}_N, N)\) and \((Y, \mathcal{Y}_M, M)\) be fuzzy soft semi s-connected subspaces of \(I_E\) such that \(\mathcal{Z}_N \subseteq k_D \cap \mathcal{X}_h\). Since \(\mathcal{Z}_N, \mathcal{Y}_M\) are fuzzy soft semi s-connected, \(\mathcal{Z}_N, \mathcal{Y}_M \subseteq z_E \cup k_D \cap \mathcal{X}_h\). By Theorem 4.5, either \(z_E \subseteq k_D \cap \mathcal{X}_h\) or \(z_E \subseteq \mathcal{X}_h\), also, either \(\mathcal{Y}_M \subseteq k_D \cap \mathcal{X}_h\) or \(\mathcal{Y}_M \subseteq \mathcal{X}_h\). Therefore, \((z_E \subseteq k_D \cap \mathcal{X}_h\) or \(\mathcal{Y}_M \subseteq \mathcal{X}_h\), we get \(\mathcal{Z}_N \cup \mathcal{Y}_M \subseteq k_D \cap \mathcal{X}_h\). Hence, \(z_E \subseteq k_D \cap \mathcal{X}_h\) is fuzzy soft semi connected, which is a contradiction. Hence, \(\mathcal{Z}_N \cup \mathcal{Y}_M\) is fuzzy soft semi s-connected.

Theorem 4.7. Let \((Z, \mathcal{Z}_N, N)\) be a fuzzy soft semi s-connected subspace of fuzzy soft topological space \((X, \mathcal{X}, E)\) and \(S_M \in SS(X, E)\). If \(\mathcal{Z}_N \subseteq S_M \subseteq \text{FScI}(\mathcal{Z}_N)\). Then \((S, \mathcal{S}_M, M)\) is fuzzy soft semi s-connected subspace of \((X, \mathcal{X}, E)\).

Proof. Suppose that \((S, \mathcal{S}_M, M)\) is not fuzzy soft semi s-connected subspace of \((X, \mathcal{X}, E)\). Then, there exist fuzzy soft semi separated sets \(f_A\) and \(g_B\) on \(\mathcal{X}\) such that \(S_M = f_A \cup g_B\). So, we have \(\mathcal{Z}_N\) is fuzzy soft semi s-connected subset of fuzzy soft semi s-disconnected space. By Theorem 4.5, either \(z_E \subseteq f_A\) or \(z_E \subseteq g_B\). If \(z_E \subseteq f_A\). Then \(\text{FScI}(z_E) \subseteq \text{FScI}(f_A)\). But \(\text{FScI}(z_E) \supseteq \text{FScI}(g_B)\). Hence, \(g_B = \emptyset_E\), which is a contradiction.

Corollary 4.1. If \((Z, \mathcal{Z}_N, N)\) is fuzzy soft semi s-connected subspace of fuzzy soft topological space \((X, \mathcal{X}, E)\). Then \(\text{FScI}(\mathcal{Z}_N)\) is fuzzy soft semi s-connected.

Proof. It obvious from Theorem 4.7.

Theorem 4.8. If for all pair of distinct fuzzy soft point \(f_e, g_e\), there exists a fuzzy soft semi s-connected set \(\mathcal{Z}_N\) \(f_e, g_e, \mathcal{Z}_N\), then \(I_E\) is fuzzy soft semi s-connected.

Proof. Suppose that \(I_E\) is fuzzy soft semi s-disconnected. Then \(I_E = f_A \cup g_B\), where \(f_A, g_B\) are fuzzy soft semi separated sets. It follows \(f_A \cap g_B = \emptyset_E\). So, \(z_E \in f_A\) and \(g_B\). Since \(f_A \cap g_B = \emptyset_E\). Then \(f_e, g_e\) are distinct fuzzy soft point in \(I_E\). By hypothesis, there exists a fuzzy soft semi s-connected set \(z_N\) such that \(f_e, g_e \subseteq \mathcal{Z}_N\). Moreover, we have \(\mathcal{Z}_N\) is fuzzy soft semi s-disconnected subset of a fuzzy soft semi s-disconnected space. Therefore by Theorem 4.5, either \(\mathcal{Z}_N \subseteq f_A\) or \(\mathcal{Z}_N \subseteq g_B\) and both cases is a contradiction with the hypothesis. Therefore, \(I_E\) is fuzzy soft semi s-connected.

Theorem 4.9. Let \(\{Z_j, \mathcal{Z}_N, J\) : \(j \in J\) \} be a non null family of fuzzy soft semi s-connected subspaces of fuzzy soft topological space \((X, \mathcal{X}, E)\). If \(\cap_{j \in J} \mathcal{Z}_N \neq \emptyset_E\), then \((\cup_{j \in J} \mathcal{Z}_N, \mathcal{N}_{\cup j \in J} \mathcal{Z}_N, N)\) is also fuzzy soft semi s-connected fuzzy subspace of \((X, \mathcal{X}, E)\).

Proof. Suppose that \((Z, \mathcal{Z}_N, N) = (\cup_{j \in J} \mathcal{Z}_N, \mathcal{N}_{\cup j \in J} \mathcal{Z}_N, N)\) is fuzzy soft semi s-disconnected. Then \(\mathcal{Z}_N = f_A \cup g_B\) for some fuzzy soft semi separated subspaces \(f_A, g_B\). Since \(\cap_{j \in J} \mathcal{Z}_N \neq \emptyset_E\). Then \(\exists f_e \in f_A, g_e \in g_B\). So, \(f_e, g_e \in \mathcal{Z}_N\). Hence, \(\mathcal{Z}_N \subseteq f_A\) or \(\mathcal{Z}_N \subseteq g_B\) since \(\mathcal{Z}_N \subseteq \mathcal{X}_h\). Therefore, \(\mathcal{Z}_N \subseteq f_A\) or \(\mathcal{Z}_N \subseteq g_B\), which is a contradiction. Also, if \((Z, \mathcal{Z}_N, N) \neq \emptyset_E\), then \(\mathcal{Z}_N \subseteq \mathcal{X}_h\). So, \(\mathcal{Z}_N \subseteq f_A\) or \(\mathcal{Z}_N \subseteq g_B\).

Theorem 4.10. Let \(\{Z_j, \mathcal{Z}_N, J\) : \(j \in J\) \} be a family of fuzzy soft semi s-connected subspaces of fuzzy soft topological space \((X, \mathcal{X}, E)\) such that one of the members of the family fuzzy soft intersects every other members, then \((\cap_{j \in J} \mathcal{Z}_N, \mathcal{N}_{\cap j \in J} \mathcal{Z}_N, N)\) is fuzzy subspace of \((X, \mathcal{X}, E)\).

Proof. Let \((Z, \mathcal{Z}_N, N) = (\cap_{j \in J} \mathcal{Z}_N, \mathcal{N}_{\cap j \in J} \mathcal{Z}_N, N)\) and \((z, N)_{j_0} \in \{\{z, N\}_{j_0} : j_0 \in J\} \) such that \((z, N)_{j_0} \cup \mathcal{Z}_N \neq \emptyset_E\). Then \((z, N)_{j_0} \cup \mathcal{Z}_N\) is fuzzy soft semi s-connected \(\forall j \in J\) by Theorem 4.6. Hence, the collection \(\{z, N\}_{j_0} \cup \mathcal{Z}_N\) \(j_0 \in J\) \} is a collection of fuzzy soft semi s-connected subspaces of \((X, \mathcal{X}, E)\) by Theorem 4.6.

Theorem 4.11. Let \((X_1, \mathcal{X}_1, E)\) and \((X_2, \mathcal{X}_2, K)\) be fuzzy soft topological spaces and \(f_{pu} : (X_1, \mathcal{X}_1, E) \to (X_2, \mathcal{X}_2, K)\) be a fuzzy irresolute surjective soft function. If \((X_1, \mathcal{X}_1, E)\) is fuzzy soft semi s-connected, then \((X_2, \mathcal{X}_2, K)\) is also fuzzy soft semi s-connected.

Proof. Let \((X_2, \mathcal{X}_2, K)\) be fuzzy soft semi disconnected space. Then there exist \(f_A, g_B\) pair of non null proper
Theorem 4.2. This means that, $\bar{f}_A \sqcup \bar{g}_B = \bar{f}_A \cap \bar{g}_B = \bar{f}_A \cap \bar{g}_B = \emptyset_E$. Since $f_{pu}$ is fuzzy irresolute soft function, then $f_{pu}^{-1}(\bar{f}_A), f_{pu}^{-1}(\bar{g}_B)$ are pair of non null proper fuzzy semi open soft subsets of $\bar{I}_E$ such that

$$\bar{I}_E = \bar{f}_A \cup \bar{g}_B = \bar{f}_A \cap \bar{g}_B = \bar{f}_A \cap \bar{g}_B = \emptyset_E, \quad f_{pu}^{-1}(\bar{f}_A) \sqcup f_{pu}^{-1}(\bar{g}_B) = f_{pu}^{-1}(\bar{f}_A) \cap f_{pu}^{-1}(\bar{g}_B) = \emptyset_E.$$ 

Therefore, $(\bar{X}_2, \bar{\mathcal{T}}_2, \mathcal{K})$ is fuzzy soft semi $s$-connected.

5 Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. Recently, many scientists have studied the soft set theory, which is initiated by Molodtsvov and easily applied to many problems having uncertainties from social life. In the present work, we have continued to study the properties of fuzzy soft topological spaces. We introduce some new concepts in fuzzy soft topological spaces such as fuzzy soft semi connected sets, fuzzy soft semi separated sets and fuzzy soft semi $s$-connected sets and have established several interesting properties. Since the authors introduced topological structures on fuzzy soft sets and the semi topological properties, which introduced by Mahanta et al., is generalized here to the fuzzy soft sets which will be useful in the fuzzy systems. Because there exists compact connections between soft sets and information systems, we can use the results deduced from the studies on fuzzy soft topological space to improve these kinds of connections. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life. This paper, not only can form the theoretical basis for further applications of topology on soft sets, but also lead to the development of information systems.

6 Acknowledgements

The authors express their sincere thanks to the reviewers for their careful checking of the details and for helpful comments that improved this paper. The authors are also thankful to the editors-in-chief and managing editors for their important comments which helped to improve the presentation of the paper.

References


Dr. Kandil has published over 80 papers in refereed journals and contributed several book chapters in various types of Mathematics textbooks. He is a Fellow of the Egyptian Mathematical Society and Egyptian Physics Mathematical Society. He was the Supervisor of 20 PHD and about 30 MSC students.

Osama Abd El-Hamid El-Tantawy is a Professor of Mathematics at Zagazig University. He born in 1951. He received the Ph.D. degree in Topology from the University of Moscow in 1978. His primary research areas are General Topology, Fuzzy Topology, double sets and theory of sets.

Sobhy Ahmed Aly El-Sheikh is a Professor of pure Mathematics, Ain Shams University Faculty of Education, Mathematic Department, Cairo, Egypt. He born in 1955. He received the Ph.D. degree in Topology from the University of Zagazig. His primary research areas are General Topology, Fuzzy Topology, double sets and theory of sets. Dr. Sobhy has published over 15 papers in refereed journals. He is a Fellow of the Egyptian Mathematical Society and Egyptian Physics Mathematical Society. He was the Supervisor of many PHD and MSC Thesis.
Alaa Mohamed Abd El-Latif is a lecturer of pure Mathematics (Topology) at Ain Shams University, Faculty of Education, Mathematic Department, Cairo, Egypt. He was born in 1985. He received the MSC degree in Topology from Ain Shams University in 2012. He received the Ph.D degree in Topology from Ain Shams University in 2015. He is a Fellow of the Egyptian Mathematical Society. His primary research areas are General Topology, Fuzzy Topology, Set theory, Soft set theory and Soft topology. He is referee of several international journals in the pure mathematics. Dr. Alaa has published many papers in refereed journals.