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Some Subclasses of p-Valent Functions Defined by Generalized Fractional Differintegral Operator -II

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Abstract: In this paper, by applying a generalized extended fractional differintegral operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}(z\in\triangle;p\in\mathbb{N};\mu,\eta\in\mathbb{R};\ \mu<p+1;-\infty<\lambda<\eta+p+1)$ we define a new class convex functions $\mathscr{CV}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ and several sharp inclusion relationships and other interesting properties were discussed by using the techniques of differential subordination.

Keywords: Analytic function; Multivalent function; Differential subordination; Generalized fractional differintegral operator; Generalized hypergeometric function; Hadamard product (or convolution).
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1 Introduction and definitions

Let $\mathscr{A}_p$ denote the class of functions normalized by

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\qquad(p\in\mathbb{N}=\{1,2,3,\ldots\}),\ (1)$$

which are analytic and p-valent in the open disk $\triangle=\{z: z\in\mathbb{C}$ and $|z|<1\}$.

A function $f(z)\in\mathscr{A}_p$ is said to be in the class $\mathscr{S}_p^*(\alpha)$ of p-valently starlike functions of order α in $\triangle$, if $\Re\left(\frac{z\,f'(z)}{f(z)}\right)>\alpha(0\leqq\alpha<p;\ z\in\triangle)$. Furthermore, a function $f(z)\in\mathscr{A}_p$ is said to be in the class $\mathscr{K}_p(\alpha)$ of p-valently convex functions of order α in $\triangle$, if $\Re\left(1+\frac{z\,f''(z)}{f'(z)}\right)>\alpha\ (0\leqq\alpha<p;\ z\in\triangle)$. Indeed, it follows that $f(z)\in\mathscr{K}_p(\alpha)\Leftrightarrow\frac{zf'(z)}{p}\in\mathscr{S}_p^*(\alpha)(0\leqq\alpha<p;z\in\triangle)$.

We note that $\mathscr{S}_p^*(\alpha)\subseteq\mathscr{S}_p^*(0)\equiv\mathscr{S}_p^*$ and $\mathscr{K}_p(\alpha)\subseteq\mathscr{K}_p(0)\equiv\mathscr{K}_p(0\leqq\alpha<p)$, where $\mathscr{S}_p^*$ and $\mathscr{K}_p$ denote the subclass of $\mathscr{A}_p$ consisting of functions which are p-valently starlike in $\triangle$ and p-valently convex in $\triangle$, respectively (see, for details, [3]; see also [15] and [1]).

If $f(z)$ and $g(z)$ are analytic in $\triangle$, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as

$$f\prec g \text{ in } \triangle \text{ or } f(z)\prec g(z)\qquad(z\in\triangle),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\triangle$ with $w(0)=0$ and $|w(z)|<1$ in $\triangle$ such that $f(z)=g(w(z)),z\in\triangle$. It is known that

$$f(z)\prec g(z)\qquad(z\in\triangle)\quad\Rightarrow\quad f(0)=g(0)\quad\text{and}\quad f(\triangle)\subset g(\triangle).$$

In particular, if the function $g(z)$ is univalent in $\triangle$, then we have the following equivalence (cf., e.g., [9]):

$$f(z)\prec g(z)\qquad(z\in\triangle)\quad\Leftrightarrow\quad f(0)=g(0)\quad\text{and}\quad f(\triangle)\subset g(\triangle).$$

Furthermore, $f(z)$ is said to be subordinate to $g(z)$ in the disk $\triangle_r=\{z\in\mathbb{C}:\ |z|<r\}$ if the function $f_r(z)=f(rz)$ is subordinate to the function $g_r(z)=g(rz)$ in $\triangle$. It follows from the Schwarz lemma that if $f\prec g$ in $\triangle$, then $f\prec g$ in $\triangle_r$ for every $r(0<r<1)$.

The general theory of differential subordination introduced by Miller and Mocanu is given in [8]. Namely, if $\Psi:\Omega\to\mathbb{C}$ (where $\Omega\subseteq\mathbb{C}^2$) is an analytic function, h is analytic and univalent in $\triangle$, and if ϕ is analytic in $\triangle$ with $(\phi(z),z\phi'(z))\in\Omega$ when $z\in\triangle$, then we say that ϕ

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satisfies a first-order differential subordination provided that

$$\Psi(\phi(z),z\phi'(z)) \prec h(z) \quad (z\in\triangle) \quad \text{and} \quad \Psi(\phi(0),0)=h(0). \tag{2}$$

We say that a univalent function $q(z)$ is a dominant of the differential subordination (2) if $\phi(0)=q(0)$ and $\phi(z)\prec q(z)$ for all analytic functions $\phi(z)$ that satisfy the differential subordination (2). A dominant $\overline{q}(z)$ is called as the best dominant of (2), if $\overline{q}(z)\prec q(z)$ for all dominans $q(z)$ of (2)[8,9].

For functions $f_j(z)\in\mathscr{A}_p$, given by

$$f_j(z)=z^p+\sum_{n=1}^{\infty}a_{p+n,j}z^{p+n} \qquad (j\in 1,2;\ p\in\mathbb{N}),$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1\star f_2)(z)=z^p+\sum_{n=1}^{\infty}a_{p+n,1}a_{p+n,2}z^{p+n}=(f_2\star f_1)(z) \qquad (p\in\mathbb{N}\ z\in\triangle).$$

In our present investigation, we shall also make use of the Guassian hypergeometric function functions ${}_2F_1$, ${}_3F_2$ defined by

$$2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\ (a,b,c\in\mathbb{C},\ c\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}), \tag{3}$$

$$3F_2(a,b,c;d,e;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{z^n}{n!}\ (a,b,c,d,e\in\mathbb{C},\ d,e\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}) \tag{4}$$

where $(\kappa)_n$ denote the Pochhammer symbol (or the shifted factorial) given in terms of Gamma function $(\kappa)_n=\frac{\Gamma(\kappa+n)}{\Gamma(\kappa)}$ by We note that the series defined by (3) and (4) converges absolutely for $z\in\triangle$ and hence ${}_2F_1$ and ${}_3F_2$ represents analytic functions in the open unit disk $\triangle$.

We recall here the following generalized fractional integral and generalized fractional derivative operators due to Srivastava et al. [20] (see also [5,6,14]).

Definition 11[20] *For real numbers $\lambda>0,\mu$ and η, Saigo hypergeometric fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ is defined by*

$$I_{0,z}^{\lambda,\mu,\eta}f(z)=\frac{z^{-\lambda-\mu}}{\Gamma(\lambda)}\int_0^z(z-t)^{\lambda-1}\,{}_2F_1\left(\lambda+\mu,-\eta;\lambda;1-\frac{t}{z}\right)f(t)dt,$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z-$plane containing the origin, with the order

$$f(z)=O(|z|^{\varepsilon}) \qquad (z\longrightarrow 0;\varepsilon>\max\{0,\mu-\eta\}-1),$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ *to be real when* $(z-t)>0$.

Definition 12[20] *Under the hypotheses of Definition 11, Saigo hypergeometric fractional derivative operator $\mathfrak{S}_{0,z}^{\lambda,\mu,\eta}$ is defined by*

$$\mathfrak{S}_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\dfrac{1}{\Gamma(1-\lambda)}\dfrac{d}{dz}\left\{z^{\lambda-\mu}\int_0^z(z-t)^{-\lambda}\,{}_2F_1\left(\mu-\lambda,1-\eta;1-\lambda;1-\dfrac{t}{z}\right)f(t)dt\right\} & (0\le\lambda<1);\\ \dfrac{d^n}{dz^n}\mathfrak{S}_{0,z}^{\lambda-n,\mu,\eta}f(z) & (n\le\lambda<n+1;n\in\mathbb{N}),\end{cases}$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 11.

It may be remarked that

$$I_{0,z}^{\lambda,-\lambda,\eta}f(z)=D_z^{-\lambda}f(z) \quad (\lambda>0) \quad \text{and} \quad \mathfrak{S}_{0,z}^{\lambda,\lambda,\eta}f(z)=D_z^{\lambda}f(z) \quad (0\le\lambda<1),$$

where $D_z^{-\lambda}$ denotes fractional integral operator and D_z^{λ} denotes fractional derivative operator considered by Owa [11].

Recently Goyal and Prajapat [4] introduced generalized fractional differintegral operator $S_{0,z}^{\lambda,\mu,\eta}:\mathscr{A}_p\longrightarrow\mathscr{A}_p$, by

$$S_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\dfrac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}z^{\mu}\mathfrak{S}_{0,z}^{\lambda,\mu,\eta}f(z) & (0\le\lambda<\eta+p+1,z\in\triangle);\\ \dfrac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}z^{\mu}I_{0,z}^{-\lambda,\mu,\eta}f(z) & (-\infty<\lambda<0,z\in\triangle).\end{cases} \tag{5}$$

It is easily seen from (5) that for a function $f\in\mathscr{A}_p$ we have

$$\begin{aligned}S_{0,z}^{\lambda,\mu,\eta}f(z) &= z^p+\sum_{n=1}^{\infty}\frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n}a_{p+n}z^{p+n}\\ &= z^p\ {}_3F_2(1,1+p,1+p+\eta-\mu;1+p-\mu,1+p+\eta-\lambda;z)*f(z)\\ &\quad (z\in\triangle;p\in\mathbb{N};\mu,\eta\in\mathbb{R};\mu<p+1;-\infty<\lambda<\eta+p+1).\end{aligned} \tag{6}$$

We note that

$$S_{0,z,0,b}^{p,0,0,0}f(z)=f(z)$$

$$S_{0,z,0,b}^{p,1,1,1}f(z)=S_{0,z,0,b}^{p,1,0,0}f(z)=\frac{zf'(z)}{p}$$

$$S_{0,z,0,b}^{p,2,1,1}f(z)=\frac{zf'(z)+z^2f''(z)}{p^2}$$

and

$$S_{0,z,0,b}^{p,\lambda,\lambda,\eta}f(z)=S_{0,z,0,b}^{p,\lambda,\mu,0}f(z)=\Omega_z^{\lambda,p}f(z),$$

where $\Omega_z^{\lambda,p}$ is an extended fractional differintegral operator studied very recently by [13].

On the other hand, if we set $\lambda=-\alpha$, $\mu=0$ and $\eta=\beta-1$, in (5) and using

$$I_{0,z}^{\alpha,0,\beta-1}f(z)=\frac{1}{z^{\beta}\Gamma(\alpha)}\int_0^z t^{\beta-1}\left(1-\frac{t}{z}\right)^{\alpha-1}f(t)dt,$$

we obtain following $p-$valent generalization of multiplier transformation operator [7]

$$\begin{aligned}\mathscr{Q}_{\beta,p}^{\alpha}f(z) &= \binom{p+\alpha+\beta-1}{p+\beta-1}\frac{\alpha}{z^{\beta}}\int_0^z t^{\beta-1}\left(1-\frac{t}{z}\right)^{\alpha-1}f(t)dt\\ &= z^p+\sum_{n=1}^{\infty}\frac{\Gamma(p+\beta+n)\Gamma(p+\alpha+\beta)}{\Gamma(p+\alpha+\beta+n)\Gamma(p+\beta)}a_{p+n}z^{p+n} \quad (\beta>-p;\alpha+\beta>-p)\end{aligned} \tag{7}$$

On the other hand, if we set $\lambda=-1$, $\mu=0$, and $\eta=\beta-1$ in (6), we obtain the generalized Bernardi-Libera integral operator $\mathscr{F}_{\beta,p}:\mathscr{A}_p\longrightarrow\mathscr{A}_p(\beta>-p)$ defined by

$$\begin{aligned}S_{0,z}^{-1,0,\beta-1}f(z) &= \mathscr{F}_{\beta,p}f(z)=\frac{p+\beta}{z^{\beta}}\int_0^z t^{\beta-1}f(t)dt\\ &= z^p+\sum_{n=1}^{\infty}\frac{p+\beta}{p+\beta+n}a_{p+n}z^{p+n}\\ &= z^p\ {}_2F_1(1,p+\beta;p+\beta+1;z)(\beta>-p;z\in\triangle).\end{aligned} \tag{8}$$

For the choice $p=1$, where $\beta\in\mathbb{N}$, the operator defined by (8) reduces to the well known Bernardi integral operator [2].

It is easily seen from (6) that

$$z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))'=(p+\eta-\lambda)(\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta}f(z))-(\eta-\lambda)(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z)). \tag{9}$$

On differentiating (9), we get

$$z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))''=(p+\eta-\lambda)(\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta}f(z))'-(\eta+1-\lambda)(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))'. \tag{10}$$

It follows from (6) and (8) that

$$z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z))'=(p+\eta+\beta)(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))-(\eta+\beta)(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)). \tag{11}$$

On differentiating (11), we get

$$z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z))''=(p+\eta+\beta)(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))'-(\eta+1+\beta)(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z))'. \tag{12}$$

Making use of an extended fractional differintegral operator various mapping properties and inclusion relationships between certain subclasses of multivalently starlike functions are investigated by applying the techniques of differential subordination by Patel and Misra [13] also by Selvaraj et al.[15,16]. Using the generalized Saigo fractional differintegral operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}$, we now introduce the following subclass of $\mathscr{A}_p$:

Definition 13*For fixed parameters A,B with $-1\leq B<A\leq 1, 0\leqq\alpha<p, f(z)\in\mathscr{A}_p$ is in the class $\mathscr{CV}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ if*

$$\frac{1}{p-\alpha}\left(1+\frac{z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))''}{(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))'}-\alpha\right)\prec\frac{1+Az}{1+Bz}(z\in\triangle;p\in\mathbb{N};\mu,\eta\in\mathbb{R};\mu<p+1;-\infty<\lambda<\eta+p+1) \tag{13}$$

For $A=1, B=-1$, we have

$$\frac{1}{p-\alpha}\left(1+\frac{z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))''}{(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))'}-\alpha\right)\prec\frac{1+z}{1-z}.$$

For convenience, we write

$$C\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;1,-1)=C\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha)$$
$$=\left\{f(z)\in\mathscr{A}_p:\Re\left(1+\frac{z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))''}{(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z))'}\right)>\alpha,\ 0\leqq\alpha<p,z\in\triangle\right\}.$$

We further observe that

$$\mathscr{CV}_p^{\lambda,\mu,\eta}(\alpha;A,B)=\mathscr{CV}_p^{\lambda,\mu,\eta}(0;A+\tfrac{\alpha}{p}(B-A),B);\qquad \mathscr{CV}_p^{0,0,0}(\alpha)=\mathscr{K}_p(\alpha)$$

also note that for $(0\leq\lambda<1;\ 0\leq\alpha<1)$ we have $\mathscr{CV}_1^{\lambda,\mu,0}(\alpha)=\mathscr{K}_\lambda(\alpha)$ due to Srivastava and Owa [18].

In the present paper we obtain several sharp inclusion relationships and other interesting properties of the class $C\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ for $\eta\in\mathbb{R},\mu<p+1$ and for all admissible non-negative values of λ and also for negative values of λ by using the techniques of differential subordination. Further we determine mapping properties of a variety of operators involving the operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}$.

2 A set of preliminary lemmas

We denote by $\mathscr{P}(\gamma)$ the class of functions $\varphi(z)$ given by

$$\varphi(z)=1+b_1z+b_2z^2+\cdots \tag{14}$$

which are analytic in $\triangle$ and satisfy the following inequality:

$$\Re(\varphi(z))>\gamma \qquad (0\leqq\gamma<1;\ z\in\triangle).$$

In order to prove our main results, we recall the following lemmas.

Lemma 21*[8,10] Let the function $h(z)$ be analytic and convex (univalent) in $\triangle$ with $h(0)=1$. Suppose also that the function $\phi(z)$ given by*

$$\phi(z)=1+c_1z+c_2z^2+\cdots$$

is analytic in $\triangle$. If

$$\phi(z)+\frac{z\phi'(z)}{\gamma}\prec h(z) \qquad (z\in\triangle;\Re(\gamma)\geqq 0;\gamma\neq 0), \tag{15}$$

then

$$\phi(z)\prec\psi(z)=\frac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1}h(t)dt\prec h(z) \qquad (z\in\triangle)$$

and $\psi(z)$ is the best dominant of (15).

Lemma 22*[9] If $-1\leqq B<A\leqq 1,\beta>0$, and the complex number γ is constrained by*

$$\Re(\gamma)\geqq-\beta(1-A)/(1-B),$$

then the following differential equation:

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz} \qquad (z\in\triangle)$$

has a univalent solution in $\triangle$ given by

$$q(z)=\begin{cases}\dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int\limits_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}dt}-\dfrac{\gamma}{\beta}, & (B\neq 0)\\[2ex] \dfrac{z^{\beta+\gamma}\ exp\ (\beta Az)}{\beta\int\limits_0^z t^{\beta+\gamma-1}\ exp\ (\beta At)dt}-\dfrac{\gamma}{\beta}, & (B=0)\end{cases} \tag{16}$$

If the function $\phi(z)$ given by

$$\phi(z)=1+c_1z+c_2z^2+\cdots$$

is analytic in $\triangle$ and satisfies the following subordination:

$$\phi(z)+\frac{z\phi'(z)}{\beta\phi(z)+\gamma}\prec\frac{1+Az}{1+Bz} \qquad (z\in\triangle), \tag{17}$$

then

$$\phi(z)\prec q(z)\prec\frac{1+Az}{1+Bz} \qquad (z\in\triangle)$$

and $q(z)$ is the best dominant of (17).

Lemma 23[21] *For real or complex numbers a,b and c $(c \neq 0,-1,-2,\ldots)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z) \qquad (Re(c) > Re(b) > 0); \tag{18}$$

$$ {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z); \tag{19}$$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1(a,c-b;c;\frac{z}{z-1}); \tag{20}$$

$$ (a+1)\,{}_2F_1(1,a;a+1;z) = (a+1)+az\,{}_2F_1(1,a+1;a+2;z) \tag{21}$$

and

$$ {}_2F_1(a,b;\frac{a+b+1}{2};\frac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}. \tag{22}$$

3 Inclusion relationships for function class $\mathscr{C}\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$

Unless otherwise mentioned, we assume throughout the sequel that

$$-1 \le B < A \le 1,\ p \in \mathbb{N};\ 0 \le \alpha < p; \mu,\eta \in \mathbb{R}; \mu < p+1; -\infty < \lambda < \eta+p+1$$

Theorem 31*Let $f(z) \in C\mathscr{V}_p^{\lambda+1,\mu,\eta}(\alpha;A,B)$,*

$$(p-\alpha)(1-A)+(\alpha+\eta-\lambda)(1-B) \ge 0 \tag{23}$$

and the function $Q(z)$ be defined on $\triangle$ by

$$Q(z) = \begin{cases} \int_0^1 t^{p+\eta-\lambda-1}(\frac{1+Btz}{1+\beta z})^{(p-\alpha)(A-B)/B}dt & (B \neq 0), \\ \int_0^1 t^{p+\eta-\lambda-1}\ exp\ (A(p-\alpha)(t-1)z)dt & (B=0). \end{cases} \tag{24}$$

Then

$$\frac{1}{p-\alpha}\left(1+\frac{z\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)''}{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}-\alpha\right) \prec \frac{1}{p-\alpha}\left(\frac{1}{Q(z)}-\alpha-\eta+\lambda\right) = q(z) \prec \frac{1+Az}{1+Bz} \ (z \in \triangle), \tag{25}$$

$$C\mathscr{V}_p^{\lambda+1,\mu,\eta}(\alpha;A,B) \subset C\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B),$$

and $q(z)$ is the best dominant of (25).

If, in addition to (23) one has $A \le -\frac{\alpha+\eta-\lambda+1}{p-\alpha}$ with $-1 \le B < 0$, then

$$C\mathscr{V}_p^{\lambda+1,\mu,\eta}(\alpha;A,B) \subset C\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;1-2\rho,-1), \tag{26}$$

where

$$\rho = \frac{1}{p-\alpha}\left[(p+\eta-\lambda)\left\{{}_2F_1(1,\frac{(p-\alpha)(B-A)}{B};p+\eta-\lambda+1;\frac{B}{B-1})\right\}^{-1}-\alpha-\eta+\lambda\right].$$

The bound in (26) is the best possible.

*Proof.*Let $f(z) \in C\mathscr{V}_p^{\lambda+1,\mu,\eta}(\alpha;A,B)$, and $g(z)$ be defined by

$$g(z) = z\left(\frac{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}} \qquad (z \in \triangle). \tag{27}$$

Write $r_1 = \sup\{r : g(z) \neq 0, 0 < |z| < r < 1\}$. Then $g(z)$ is single-valued and analytic in $|z| < r_1$. Taking logarithmic differentiation in (27), it follows that the function

$$\phi(z) = \frac{zg'(z)}{g(z)} = \frac{1}{p-\alpha}\left(1+\frac{z\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)''}{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}-\alpha\right) \tag{28}$$

is of the form (14) and is analytic in $|z| < r_1$. Using the identity (10) in (28) and again carrying out logarithmic differentiation in the resulting equation, we get

$$\phi(z)+\frac{z\phi'(z)}{(p-\alpha)\phi(z)+\alpha+\eta-\lambda} = \frac{1}{p-\alpha}\left(1+\frac{z\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)''}{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}-\alpha\right) \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1). \tag{29}$$

Hence, by using Lemma 21 we find that

$$\phi(z) \prec \frac{1}{p-\alpha}\left(\frac{1}{Q(z)}-\alpha-\eta+\lambda\right) = q(z) \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1), \tag{30}$$

where $q(z)$ is the best dominant of (25) and $Q(z)$ is given by (24). The remaining part of the proof can now be deduced on the same lines as in [[12], Theorem 1]. This evidently completes the proof.

Taking $A = 1, B = -1, \eta = 0$ and $p = 1$ in Theorem 31 we get the following result which both extends and sharpens the work of Srivastava et al. [17].

Corollary 32*If $-\infty < \max\{\lambda, \frac{\lambda}{2}\} \le \alpha < 1$, then*

$$\mathscr{K}_{\lambda+1}(\alpha) \subseteq \mathscr{K}_{\lambda}(\gamma) \subseteq \mathscr{K}_{\lambda}(\alpha),$$

where $\gamma = (1-\lambda)\left[{}_2F_1(1,2(1-\alpha);2-\lambda;\frac{1}{2})\right]^{-1}+\lambda$. The value of γ is the best possible.

Theorem 33*Let β be a real number satisfying*

$$(p-\alpha)(1-A)+(\eta+\beta+\alpha)(1-B) \ge 0.$$

(i)If $f(z) \in C\mathscr{V}_p^{\lambda}(\alpha;A,B)$, then

$$\frac{1}{p-\alpha}\left(1+\frac{z\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)''}{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)'}-\alpha\right) \prec \frac{1}{p-\alpha}\left(\frac{1}{Q(z)}-\eta-\beta-\alpha\right) = \tilde{q}(z) \prec \frac{1+Az}{1+Bz} \ (z \in \triangle), \tag{31}$$

where

$$Q(z) = \begin{cases} \int_0^1 t^{p+\eta+\beta-1}(\frac{1+Btz}{1+\beta z})^{(p-\alpha)(A-B)/B}dt & (B \neq 0) \\ \int_0^1 t^{p+\eta+\beta-1}\ exp\ (A(p-\alpha)(t-1)z)dt & (B=0) \end{cases}$$

and $\tilde{q}(z)$ is the best dominant of (31). Consequently, the operator $\mathscr{F}_{\beta,p}$ maps the class $C\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ into itself.

(ii)If $-1 \le B < 0$ and

$$\beta \ge \max\left\{\frac{(p-\alpha)(B-A)}{B} - p - \eta - 1, -\frac{(p-\alpha)(1-A)}{1-B} - \alpha - \eta\right\}, \tag{32}$$

then the operator $\mathscr{F}_{\beta,p}$ maps the class $C\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ into the class $C\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;1-2\rho,-1)$, where

$$\rho = \frac{1}{p-\alpha}\left[(\eta+\beta+p)\left\{{}_2F_1(1,\frac{(p-\alpha)(B-A)}{B};\eta+\beta+p+1;\frac{B}{B-1})\right\}^{-1} - \eta - \beta - \alpha\right].$$

The bound ρ is the best possible.

*Proof.*Upon replacing

$$g(z) \text{ by } z\left(\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)'}{pz^{p-1}}\right)^{\frac{1}{p-\alpha}}, (z \in \triangle)$$

in (27) and carrying out logarithmic differentiation it follows that the function $\phi(z)$ given by

$$\phi(z) = \frac{zg'(z)}{g(z)} = \frac{1}{p-\alpha}\left(1 + \frac{z\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)''}{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)'} - \alpha\right) \tag{33}$$

is of the form (14) and is analytic in $|z| < r_1$. Using the identity (12) in (33) and the fact that $\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) \neq 0$ in $0 < |z| < 1$, we get

$$\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)'}{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'} = \frac{\eta+\beta+p}{(p-\alpha)\phi(z)+\eta+\beta+\alpha} \qquad (|z| < r_1). \tag{34}$$

Again, by taking logarithmic differentiation in (34) and using (33) in the resulting equation, we deduce that

$$\frac{1}{p-\alpha}\left(1 + \frac{z\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)''}{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'} - \alpha\right) = \phi(z) + \frac{z\phi'(z)}{(p-\alpha)\phi(z)+\eta+\alpha} \prec \frac{1+Az}{1+Bz} \quad (|z| < r_1).$$

The remaining part of the proof is similar to that of [[12], Theorem 1] and we choose to omit the details.

Putting $A = 1$ and $B = -1$ in Theorem 33, we get

Corollary 34*If β is a real number satisfying $\beta \ge \max\{p - 2\alpha - \eta - 1, -\alpha - \eta\}$, then*

$$\mathscr{F}_{\beta,p}(C\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha)) \subset C\mathscr{V}_p^{\lambda,\mu,\eta}(\sigma),$$

where $\sigma = (\eta + \beta + p)\left[{}_2F_1(1,2(p-\alpha);\eta+\beta+p+1;\frac{1}{2})\right]^{-1} - \eta - \beta$. The result is the best possible.

In particular, when $\eta = 0$, Corollary 3.4 gives [[15] , corollary 3.4]. Further, for $\eta = 0$ and $\lambda = 0$, corollary 34 gives the following result which, in turn, the second half of Remark 2 [[12],p.330].

Corollary 35*If β is a real number satisfy $\beta \ge \max\{p - 2\alpha - 1, -\alpha\}$, then*

$$\mathscr{F}_{\beta,p}(\mathscr{K}_p(\alpha)) \subset \mathscr{K}_p(\sigma),$$

where $\sigma = (\beta+p)[\,{}_2F_1(1,2(p-\alpha);\beta+p+1;\frac{1}{2})]^{-1} - \beta$. The value of σ is the best possible.

It is interest to note that, by setting $\beta = 0$ in corollary 35, we have the further consequence [[19], Corollary 7].

4 Some properties of the operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}$

Now we discuss some properties of the operator$\mathcal{S}_{0,z}^{\lambda,\mu,\eta}$.

Theorem 41*Let $\delta > 0, \eta \in \mathbb{R}, \mu < p+1, -\infty < \lambda < p, p \neq 1$ and the function $f(z) \in \mathscr{A}_p$ satisfies the following subordination:*

$$(1-\delta)\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{pz^{p-1}} + \delta\frac{\left(\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz} \quad (z \in \triangle). \tag{35}$$

Then

$$\Re\left[\left(\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{pz^{p-1}}\right)^{\frac{1}{m}}\right] > \chi_1^{\frac{1}{m}} \quad (m \in \mathbb{N};\ z \in \triangle), \tag{36}$$

where

$$\chi_1 = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1(1,1;\frac{p+\eta-\lambda}{\delta}+1;\frac{B}{B-1}) & (B \neq 0), \\ 1 - \frac{(p+\eta-\lambda)A}{p+\eta-\lambda+\delta}, & (B = 0). \end{cases}$$

The result is the best possible.

*Proof.*For $f(z) \in \mathscr{A}_p$, consider the function given by

$$\phi(z) = \frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{pz^{p-1}} \qquad (z \in \triangle). \tag{37}$$

Then $\phi(z)$ is of the form (14) and analytic in $\triangle$. By differentiating (37) and making use of (10), we obtain

$$\phi(z) + \frac{\delta}{p+\eta-\lambda}z\phi'(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \triangle).$$

Now, by applying Lemma 21 we get

$$\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{pz^{p-1}} \prec Q(z) = \frac{p+\eta-\lambda}{\delta}z^{-\frac{p+\eta-\lambda}{\delta}}\int_0^z t^{\frac{p+\eta-\lambda}{\delta}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1(1,1;\frac{p+\eta-\lambda}{\delta}+1;\frac{Bz}{1+Bz}) & (B \neq 0), \\ 1 + \frac{(p+\eta-\lambda)A}{p+\eta-\lambda+\delta} & (B = 0), \end{cases}$$

where we have also made a change of variable followed by the use of identities (18) and (20). The remaining part of the proof can be deduced on the same lines as in [[12], Theorem 4]. The proof of Theorem 41 is thus completed.

Upon setting $A = 1-2\alpha, (0 \le \alpha < 1)$, $B = -1, m = 1, \eta = 0$, and $\lambda = 0$ in Theorem 41 , we state the following

Corollary 42*For $\delta > 0$, if*

$$\Re\left((1-\delta)\frac{f'(z)}{pz^{p-1}} + \delta\frac{(zf'(z))'}{p^2z^{p-1}}\right) > \alpha,$$

then

$$\Re\left(\frac{f'(z)}{pz^{p-1}}\right) > \alpha + (1-\alpha)\left[{}_2F_1\left(1,1;\frac{p}{\delta}+1;\frac{1}{2}\right)-1\right].$$

Upon setting $A = 1-2\alpha, (0 \le \alpha < 1)$, $B = -1, m = 1, \eta = 0$, and $\lambda = -1$ in Theorem 41, we state the following

Corollary 43*For $\delta > 0$, if* $\Re\left(\frac{(1-\delta)}{pz^{p-1}}\left[\frac{p+1}{z}\int_0^z f(\xi)d\xi\right]' + \delta\frac{f'(z)}{pz^{p-1}}\right) > \alpha$, *then*

$$\Re\left(\frac{1}{pz^{p-1}}\left[\frac{p+1}{z}\int_0^z f(\xi)d\xi\right]'\right) > \alpha + (1-\alpha)\left[{}_2F_1\left(1,1;\frac{p+1}{\delta}+1;\frac{1}{2}\right)-1\right].$$

Theorem 44*Let $\delta > 0, \eta \in \mathbb{R}, \mu < p+1, -\infty < \lambda < p+1, p \neq 1$ and $f(z) \in \mathscr{A}_p$. If the function $\mathscr{F}_{\mu,p}(f)(z)$ be defined by (8) satisfies*

$$(1-\delta)\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)'}{pz^{p-1}} + \delta\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz} \quad (z \in \triangle), \tag{38}$$

then

$$\Re\left[\left(\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)'}{pz^{p-1}}\right)^{\frac{1}{m}}\right] > \varsigma_1^{\frac{1}{m}} \quad (m \in \mathbb{N};\ z \in \triangle), \tag{39}$$

where

$$\varsigma_1 = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{p+\eta+\beta}{\delta}+1;\frac{B}{B-1}\right) & (B \neq 0), \\ 1 - \frac{(p+\eta+\beta)A}{p+\eta+\beta+\delta} & (B = 0). \end{cases}$$

*Proof.*For $f(z) \in \mathscr{A}_p$, consider the function given by

$$\psi(z) = \frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)'}{pz^{p-1}} \qquad (z \in \triangle). \tag{40}$$

Then $\psi(z)$ is of the form (14) and analytic in $\triangle$. By differentiating (40) and making use of the identity (12), we obtain

$$\psi(z) = \frac{\delta}{p+\eta+\beta} z\,\psi'(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \triangle).$$

The remaining part of the proof of Theorem 44 is similar to that of Theorem 41 and we omit the details.

Upon setting $A = 1-2\alpha, (0 \le \alpha < 1)$, $B = -1, m = \delta = 1, \eta = 0$ and $\lambda = 0$ in Theorem 44 we state the following

Corollary 45*If* $\Re\left(\frac{f'(z)}{pz^{p-1}}\right) > \alpha$, *then*

$$\Re\left(\frac{1}{pz^{p-1}}\left[\frac{p+\beta}{z^\beta}\int_0^z \xi^{\beta-1}f(\xi)d\xi\right]'\right) > \alpha + (1-\alpha)\left[{}_2F_1\left(1,1;p+\beta+1;\frac{1}{2}\right)-1\right].$$

Upon setting $A = 1-2\alpha, (0 \le \alpha < 1)$, $B = -1, \eta = 0$ and $m = \delta = \lambda = 1$ in Theorem 44 we state the following

Corollary 46*If* $\Re\left(\frac{(zf'(z))'}{p^2z^{p-1}}\right) > \alpha$ *then*

$$\Re\left(\frac{1}{p^2z^{p-1}}\left[z\left\{\frac{p+\beta}{z^\beta}\int_0^z \xi^{\beta-1}f(\xi)d\xi\right\}'\right]'\right) > \alpha + (1-\alpha)\left[{}_2F_1\left(1,1;p+\beta+1;\frac{1}{2}\right)-1\right].$$

In particular, for $\beta = 0$, Corollary 46 gives

Corollary 47*If* $\Re\left(\frac{f'(z)}{pz^{p-1}}\right) > \alpha$, *then*

$$\Re\left(\frac{f'(z)}{pz^{p-1}}\right) > \alpha + (1-\alpha)\left[{}_2F_1\left(1,1;p+1;\frac{1}{2}\right)-1\right].$$

Concluding remark

Taking $\eta = 0$ in Theorem 3 .1, Theorem 3 .3, Theorem 4 .1 and Theorem 4 .4, we get corresponding theorems for the operator $\Omega_z^{\lambda,p}$ (see [15]).

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