Ideal Theory of BCK/BCI-algebras Based on Double-framed Soft Sets

Young Bae Jun\(^1\), G. Muhiuddin\(^2\)\(^,*\) and Abdullah M. Al-roqi\(^3\)

\(^1\) Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Korea
\(^2\) Department of Mathematics, University of Tabuk, P. O. Box 741, Tabuk 71491, Saudi Arabia
\(^3\) Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Received: 19 Jan. 2013, Revised: 22 May. 2013, Accepted: 23 May. 2013
Published online: 1 Sep. 2013

Abstract: The notion of a (closed) double-framed soft ideal (briefly, a (closed) DFS-ideal) in BCK/BCI-algebras is introduced, and related properties are investigated. Several examples are provided. The relation between a DFS-algebra and a DFS-ideal is considered, and characterizations of a (closed) DFS-ideal are established. A new DFS-ideal from old one is constructed, and we show that the int-uni DFS-set of two DFS-ideals is a DFS-ideal. Conditions for a DFS-ideal to be closed are discussed.

Keywords: Inclusive (resp. Exclusive) set, Int-uni DFS-set, DFS-algebra, (Closed) DFS-ideal

1 Introduction

Molodtsov [19] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [16] described the application of soft set theory to a decision making problem. Maji et al. [15] also studied several operations on the theory of soft sets. Jun and Park [14] studied applications of soft sets in ideal theory of BCK/BCI-algebras.

We refer the reader to the papers [1], [2], [3], [4], [5], [8], [10], [11], [12], [13], [20] and [21] for further information regarding algebraic structures/properties of soft set theory. In [9], Jun et al. introduced the notion of double-framed soft sets (briefly, DFS-sets), and applied it to BCK/BCI-algebras. They discussed double-framed soft algebras (briefly, DFS-algebras) and investigated related properties.

In this paper, we introduce the notion of a (closed) double-framed soft ideal (briefly, a (closed) DFS-ideal) in BCK/BCI-algebras. We discuss the relation between a DFS-algebra and a DFS-ideal. We establish characterizations of a (closed) DFS-ideal, and make a new DFS-ideal from old one. We show that the int-uni DFS-set of two DFS-ideals is a DFS-ideal. Conditions for a DFS-ideal to be closed are discussed.

2 Preliminaries

2.1 Basic results on BCK/BCI-algebras

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers. An algebra \((X;*,0)\) of type \((2,0)\) is called a BCI-algebra if it satisfies the following conditions:

\[(I) \forall x,y,z \in X \ ((x*y)*(x*z)) = (z*y) = 0,\]
\[(II) \forall x,y \in X \ ((x*(x*y)) = y = 0,\]
\[(III) \forall x \in X \ ((x*x) = 0,\]
\[(IV) \forall x,y \in X \ (x*y = 0, y*x = 0 \Rightarrow x = y).\]

If a BCI-algebra \(X\) satisfies the following identity:

\[(V) \forall x \in X \ (0*x = 0),\]

* Corresponding author e-mail: chishtygm@gmail.com
then $X$ is called a BCK-algebra. Any BCK/BCI-algebra $X$ satisfies the following conditions:

\[(\forall x \in X)\ (x \ast 0 = x), \quad (1)\]
\[(\forall x,y,z \in X)\ (x \leq y \Rightarrow x \ast z \leq y \ast z, z \ast y \leq z \ast x), \quad (2)\]
\[(\forall x,y,z \in X)\ ((x \ast y) \ast z = (x \ast z) \ast y), \quad (3)\]
\[(\forall x,y,z \in X)\ ((x \ast z) \ast (y \ast z) \leq x \ast y). \quad (4)\]

where $x \leq y$ if and only if $x \ast y = 0$. Note that $(X, \leq)$ is a partially ordered set (see [17]).

A nonempty subset $S$ of a BCK/BCI-algebra $X$ is called a subalgebra of $X$ if $x \ast y \in S$ for all $x,y \in S$. A subset $I$ of a BCK/BCI-algebra $X$ is called an ideal of $X$ if it satisfies:

\[0 \in I, \quad (5)\]
\[(\forall x \in X)\ (\forall y \in I) (x \ast y \in I \Rightarrow x \in I). \quad (6)\]

We refer the reader to the books [6] [7] and [17] for further information regarding BCK/BCI-algebras.

### 2.2 Basic results on soft sets

Molodtsov [19] defined the soft set in the following way: Let $U$ be an initial universe set and $E$ be a set of parameters. We say that the pair $(U,E)$ is a soft universe. Let $\mathcal{P}(U)$ denotes the power set of $U$ and $A,B,C,\ldots \subseteq E$.

**Definition 1([19]).** A pair $(\alpha, A)$ is called a soft set over $U$, where $\alpha$ is a mapping given by

\[\alpha : A \to \mathcal{P}(U).\]

In other words, a soft set over $U$ is a parameterized family of subsets of the universe $U$. For $e \in A$, $\alpha(e)$ may be considered as the set of $e$-approximate elements of the soft set $(\alpha, A)$. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [19].

In what follows, we take $E = X$, as a set of parameters, which is a BCK/BCI-algebra and $A,B,C,\ldots$ be subalgebras of $E$ unless otherwise specified.

**Definition 2([9]).** A double-framed soft pair $((\alpha, \beta); A)$ is called a double-framed soft set of $A$ over $U$ (briefly, DFS-set of $A$), where $\alpha$ and $\beta$ are mappings from $A$ to $\mathcal{P}(U)$.

**Definition 3([9]).** A DFS-set $((\alpha, \beta); A)$ of $A$ is called a double-framed soft algebra of $A$ over $U$ (briefly, DFS-algebra of $A$) if it satisfies:

\[(\forall x,y \in A)\left(\alpha(x \ast y) \supseteq \alpha(x) \cap \alpha(y), \quad (7)\right)\]

\[\beta(x \ast y) \subseteq \beta(x) \cup \beta(y). \quad (7)\]

### 3 Double-framed soft ideals

**Definition 4.** A DFS-set $((\alpha, \beta); A)$ of $A$ is called a double-framed soft ideal of $A$ over $U$ (briefly, DFS-ideal of $A$) if it satisfies:

\[(\forall x \in A)\left(\alpha(0) \supseteq \alpha(x), \quad (8)\right)\]

\[\beta(0) \subseteq \beta(x). \quad (9)\]

**Example 1.** Let $(U,E) = (U,X)$ where $X = \{0,1,2,a,b\}$ is a BCI-algebra with the following Cayley table:

\begin{array}{c|cccc}
  & 0 & 1 & 2 & a & b \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & b & a \\
a & 2 & 0 & 0 & a & a \\
b & a & a & 0 & 0 & 0 \\
\end{array}

Let $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ be classes of subsets of $U$ which are posets with the following Hasse diagrams:

\[
\begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\gamma_5 \\
\end{array}
\]

\[
\begin{array}{c}
\tau_1 \\
\tau_2 \\
\tau_3 \\
\tau_4 \\
\tau_5 \\
\end{array}
\]

Define a DFS-set $((\alpha, \beta); E)$ of $E$ as follows:

\[\alpha : E \to \mathcal{P}(U), \ x \mapsto \begin{cases} 
\gamma_1 & \text{if } x = 0, \\
\gamma_2 & \text{if } x = 1, \\
\gamma_3 & \text{if } x = 2, \\
\gamma_4 & \text{if } x = a, \\
\gamma_5 & \text{if } x = b
\end{cases} \quad (10)\]

\[\beta : E \to \mathcal{P}(U), \ x \mapsto \begin{cases} 
\tau_1 & \text{if } x = 0, \\
\tau_2 & \text{if } x = 1, \\
\tau_3 & \text{if } x = 2, \\
\tau_4 & \text{if } x = a, \\
\tau_5 & \text{if } x = b
\end{cases} \quad (11)\]

Routine calculations show that $((\alpha, \beta); E)$ is a DFS-ideal of $E$.

**Proposition 1.** Every DFS-ideal $((\alpha, \beta); A)$ of $A$ satisfies the following conditions:

\[(1)(\forall x,y \in A)\ (x \leq y \Rightarrow \alpha(x) \supseteq \alpha(y), \quad (12)\]

\[\beta(x) \subseteq \beta(y). \quad (13)\]

\[(2)(\forall x,y,z \in A)\ (x \ast y \leq z \Rightarrow \alpha(x \ast y) \supseteq \alpha(x) \cap \alpha(y), \quad (14)\]

\[\beta(x \ast y) \subseteq \beta(x) \cup \beta(y). \quad (15)\]
Proof. (1) Let $x, y \in A$ be such that $x \leq y$. Then $x * y = 0 \in A$, and so
\[
\alpha(y) = \alpha(0) \cap \alpha(y) = \alpha(x * y) \cap \alpha(y) \subseteq \alpha(x),
\]
\[
\beta(y) = \beta(0) \cup \beta(y) = \beta(x * y) \cup \beta(y) \supseteq \beta(x)
\]
by (8) and (9).
(2) Let $x, y, z \in A$ be such that $x * y \leq z$. Then
\[
\alpha(z) = \alpha(0) \cap \alpha(z) = \alpha((x * y) * z) \cap \alpha(z) \subseteq \alpha(x * y),
\]
\[
\beta(z) = \beta(0) \cup \beta(z) = \beta((x * y) * z) \cup \beta(z) \supseteq \beta(x * y).
\]
It follows from (9) that
\[
\alpha(y) \cap \alpha(z) \subseteq \alpha(y) \cap \alpha(x * y) \subseteq \alpha(x),
\]
\[
\beta(y) \cup \beta(z) \supseteq \beta(y) \cup \beta(x * y) \supseteq \beta(x).
\]
This completes the proof.

Corollary 1. Every DFS-ideal $\langle (\alpha, \beta); E \rangle$ of $E$ satisfies the following conditions:

(1) $\forall x, y \in E \ (x \leq y \Rightarrow \alpha(x) \supseteq \alpha(y), \ \beta(x) \subseteq \beta(y))$.

(2) $\forall x, y, z \in E \ \left( \ x * y \leq z \Rightarrow \left\{ \begin{array}{l}
\alpha(x) \supseteq \alpha(x * y) \cap \alpha(y), \\
\beta(x) \subseteq \beta(y) \cup \beta(z) \end{array} \right. \right)$. 

Proposition 2. Every DFS-ideal $\langle (\alpha, \beta); E \rangle$ of $E$ satisfies the following conditions:

(1) $\forall x, y, z \in E \ \left( \alpha(x * y) \supseteq \alpha(x) \cap \alpha(z) \cap \alpha(y) \right)$. 

(2) $\forall x, y \in E \ \left( \alpha(x * y) = \alpha(0) \Rightarrow \alpha(x) \supseteq \alpha(y) \right)$.

Proof. (1) Since $(x * y) * (x * z) \leq z * y$ for all $x, y, z \in E$, it follows from Corollary 1(1) that
\[
\alpha(z * y) \subseteq \alpha((x * y) * (x * z)),
\]
\[
\beta((x * y) * (x * z)) \subseteq \beta(z * y).
\]
Using (9) we have
\[
\alpha(x * y) \supseteq \alpha((x * y) * (x * z)) \cap \alpha(x * z)
\]
\[
\supseteq \alpha(x) \cap \alpha(z * y),
\]
\[
\beta(x * y) \subseteq \beta((x * y) * (x * z)) \cup \beta(x * z)
\]
\[
\subseteq \beta(z * y) \cup \beta(z * y).
\]
(2) Let $x, y \in E$ be such that $\alpha(x * y) = \alpha(0)$ and $\beta(x * y) = \beta(0)$. Then
\[
\alpha(x) \supseteq \alpha(x * y) \cap \alpha(y) = \alpha(0) \cap \alpha(y) = \alpha(y)
\]
\[
\beta(x) \subseteq \beta(x * y) \cup \beta(y) = \beta(0) \cup \beta(y) = \beta(y)
\]
by (8) and (9).

Proposition 3. If $\langle (\alpha, \beta); E \rangle$ is a DFS-ideal of $E$, then the following are equivalent:

(1) $\forall x, y \in E \ \left( \alpha(x * y) \supseteq \alpha((x * y) * y) \right)$.

(2) $\forall x, y, z \in E \ \left( \begin{array}{l}
\alpha((x * z) * (y * z)) \supseteq \alpha((x * y) * z) \\
\beta((x * z) * (y * z)) \subseteq \beta((x * y) * z)
\end{array} \right)$. 

Proof. Assume that (1) is valid and let $x, y, z \in E$. Since
\[
((x * (y * z)) * z) = ((x * z) * (y * z)) * z \leq (x * y) * z,
\]
and
\[
\frac{\alpha((x * y) * z)}{\beta((x * y) * z)} = \frac{\alpha((x * y) * z) \supseteq \alpha((x * y) * z)}{\beta((x * y) * z) \subseteq \beta((x * y) * z)}.
\]
Therefore $\langle (\alpha, \beta); E \rangle$ is a DFS-algebra of $E$. 

The converse of Theorem 1 is not true as seen in the following example.
Example 2. Let $U = \mathbb{N}$ be the initial universe set and let $E = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

\[
\begin{array}{c|cccc}
+ & 0 & a & b & c & d \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 & 0 \\
b & b & b & 0 & 0 & 0 \\
c & c & c & c & 0 & 0 \\
d & d & c & c & c & 0 \\
\end{array}
\]

Define a DFS-set $\langle (\alpha, \beta); E \rangle$ of $E$ as follows:

\[
\alpha : E \to \mathcal{P}(U), \quad x \mapsto \begin{cases}
N & \text{if } x = 0, \\
4N & \text{if } x = a, \\
2N & \text{if } x = b, \\
3N & \text{if } x = c, \\
8N & \text{if } x = d
\end{cases}
\]

and

\[
\beta : E \to \mathcal{P}(U), \quad x \mapsto \begin{cases}
12N & \text{if } x = 0, \\
3N & \text{if } x = a, \\
6N & \text{if } x = b, \\
5N & \text{if } x = c, \\
N & \text{if } x = d
\end{cases}
\]

Then $\langle (\alpha, \beta); E \rangle$ is a DFS-ideal of $E$, but it is not a DFS-ideal of $E$ since

\[
\alpha(a \ast b) \cap \alpha(b) = 3N \cap 2N = 6N \not\subseteq \alpha(d)
\]

and/or

\[
\beta(d \ast b) \cup \beta(b) = 5N \cup 6N \not\supseteq \beta(d).
\]

The following example shows that Theorem 1 is not true in BCK-algebras.

Example 3. Consider the BCK-algebra $(\mathbb{Z}, +, 0)$ as the initial universe set $U$, where $a + b = a - b$ for all $a, b \in \mathbb{Z}$. Let $E = X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:

\[
\begin{array}{c|cccc}
+ & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 \\
b & b & b & 0 & 0 \\
c & c & c & c & 0 \\
\end{array}
\]

Define a DFS-set $\langle (\alpha, \beta); E \rangle$ of $E$ as follows:

\[
\alpha : E \to \mathcal{P}(U), \quad x \mapsto \begin{cases}
Z & \text{if } x = 0, \\
2Z & \text{if } x = a, \\
3Z & \text{if } x = b, \\
8Z & \text{if } x = c
\end{cases}
\]

and

\[
\beta : E \to \mathcal{P}(U), \quad x \mapsto \begin{cases}
12Z & \text{if } x = 0, \\
8Z & \text{if } x = a, \\
3Z & \text{if } x = b, \\
2Z & \text{if } x = c
\end{cases}
\]

Then $\langle (\alpha, \beta); E \rangle$ is a DFS-ideal of $E$, but it is not a DFS-ideal of $E$ since

\[
\alpha(a) \cap \alpha(b) = 2Z \cap 3Z \not\subseteq 8Z = \alpha(a \ast b)
\]

and/or

\[
\beta(a) \cup \beta(c) = 8Z \cup 3Z \not\supseteq 3Z = \beta(a \ast c).
\]

For a DFS-set $\langle (\alpha, \beta); A \rangle$ of $A$ and two subsets $\gamma$ and $\delta$ of $U$, the $\gamma$-inclusive and the $\delta$-exclusive set of $\langle (\alpha, \beta); A \rangle$, denoted by $i_{A}(\alpha; \gamma)$ and $e_{A}(\beta; \delta)$, respectively, are defined as follows:

\[
i_{A}(\alpha; \gamma) := \{x \in A \mid \gamma \subseteq \alpha(x)\}
\]

and

\[
e_{A}(\beta; \delta) := \{x \in A \mid \delta \supseteq \beta(x)\}
\]

respectively. The set

\[
D_{A}(\alpha, \beta)(\gamma, \delta) := \{x \in A \mid \gamma \subseteq \alpha(x), \delta \supseteq \beta(x)\}
\]

is called a double-framed including set (DFI) of $\langle (\alpha, \beta); A \rangle$.

It is clear that $D_{A}(\alpha, \beta)(\gamma, \delta) = i_{A}(\alpha; \gamma) \cap e_{A}(\beta; \delta)$.

Theorem 2. For a DFS-set $\langle (\alpha, \beta); A \rangle$ of $A$, the following are equivalent:

1. $\langle (\alpha, \beta); A \rangle$ is a DFS-ideal of $A$.
2. The nonempty $\gamma$-inclusive and the $\delta$-exclusive set of $\langle (\alpha, \beta); A \rangle$ are ideals of $A$ for any subsets $\gamma$, $\delta$ of $U$.

Proof. Suppose that $\langle (\alpha, \beta); A \rangle$ is a DFS-ideal of $A$. Let $\gamma$ and $\delta$ be subsets of $U$ such that $i_{A}(\alpha; \gamma) \neq \emptyset \neq e_{A}(\beta; \delta)$. Then $\gamma \subseteq \alpha(x)$ and $\delta \supseteq \beta(a)$ for some $x, a \in A$, which imply from (8) that $\gamma \subseteq \alpha(x) \subseteq \alpha(0)$ and $\delta \supseteq \beta(a) \supseteq \beta(0)$. Hence $0 \in i_{A}(\alpha; \gamma) \cap e_{A}(\beta; \delta)$.

Therefore $\gamma \subseteq \alpha(x)$ and $\delta \supseteq \beta(a)$ for some $x, a \in A$. Hence $0 \in i_{A}(\alpha; \gamma) \cap e_{A}(\beta; \delta)$. Hence $\gamma \subseteq \gamma(x)$ and $\delta \supseteq \delta(a)$. Hence $\gamma \subseteq \alpha(x) \cap e_{A}(\beta; \delta)$.

Let $x, y, a, b \in A$ be such that $x \in i_{A}(\alpha; \gamma) \cap e_{A}(\beta; \delta)$. Then $x \in i_{A}(\alpha; \gamma)$, $y \in i_{A}(\alpha; \gamma)$, $a \in e_{A}(\beta; \delta)$, and $b \in e_{A}(\beta; \delta)$. Then $\gamma \subseteq \alpha(x \ast y)$, $\gamma \subseteq \beta(a \ast b)$, and $\delta \supseteq \beta(a)$. It follows from (9) that

\[
\gamma \subseteq \alpha(x \ast y) \cap e_{A}(\beta; \delta)
\]

and

\[
\delta \supseteq \beta(a) \cap \beta(b) \supseteq \beta(a)
\]

so that $x \in i_{A}(\alpha; \gamma)$ and $a \in e_{A}(\beta; \delta)$. Hence $i_{A}(\alpha; \gamma)$ and $e_{A}(\beta; \delta)$ are ideals of $A$ for any subsets $\gamma$, $\delta$ of $U$.

Conversely, assume that the nonempty $\gamma$-inclusive and the $\delta$-exclusive set of $\langle (\alpha, \beta); A \rangle$ are ideals of $A$ for any subsets $\gamma$, $\delta$ of $U$. Let $x, a \in A$ be such that $\alpha(x) = \gamma$ and $\beta(a) = \delta$. Then $x \in i_{A}(\alpha; \gamma)$ and $a \in e_{A}(\beta; \delta)$. Since $i_{A}(\alpha; \gamma)$ and $e_{A}(\beta; \delta)$ are ideals of $A$ by assumption, we have $0 \in i_{A}(\alpha; \gamma) \cap e_{A}(\beta; \delta)$. Hence $\alpha(x) = \gamma \subseteq \alpha(0)$ and $\beta(a) = \delta \supseteq \beta(0)$. Let $x, y, a, b \in A$ be such that $x \in i_{A}(\alpha; \gamma)$, $y \in i_{A}(\alpha; \gamma)$, $a \in e_{A}(\beta; \delta)$, and $b \in e_{A}(\beta; \delta)$. Then $\gamma \subseteq \gamma(x \ast y) \cap e_{A}(\beta; \delta)$.

Therefore (from 6) that $x \in i_{A}(\alpha; \gamma)$ and $a \in e_{A}(\beta; \delta)$. Thus $\alpha(x) \subseteq \gamma$ and $\alpha(a) \subseteq \delta$. Therefore $\gamma \subseteq \gamma(x \ast y) \cap e_{A}(\beta; \delta)$ and $\beta(a) \subseteq \beta(a \ast b) \cup \beta(b)$. Therefore $\langle (\alpha, \beta); A \rangle$ is a DFS-ideal of $A$. 

\(\Box\)
Corollary 2. For a DFS-set \( ((\alpha, \beta); E) \) of \( E \), the following are equivalent:

1. \( ((\alpha, \beta); E) \) is a DFS-ideal of \( E \).
2. The nonempty \( \gamma \)-inclusive set and \( \delta \)-exclusive set of \( ((\alpha, \beta); E) \) are ideals of \( E \) for any subsets \( \gamma \) and \( \delta \) of \( U \).

Corollary 3. If \( ((\alpha, \beta); E) \) is a DFS-ideal of \( E \), then the nonempty double-framed including set of \( ((\alpha, \beta); E) \) is an ideal of \( E \).

For any DFS-set \( ((\alpha, \beta); E) \) of \( E \), let \( ((\alpha^*, \beta^*); E) \) be a DFS-set of \( E \) defined by

\[
\alpha^*: E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \alpha(x) & \text{if } x \in i_E(\alpha; \gamma), \\ \eta & \text{otherwise}, \end{cases}
\]

\[
\beta^*: E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \beta(x) & \text{if } x \in e_E(\beta; \delta), \\ \rho & \text{otherwise}, \end{cases}
\]

where \( \gamma, \delta, \eta \) and \( \rho \) are subsets of \( U \) with \( \eta \subseteq \alpha(x) \) and \( \rho \supseteq \beta(x) \).

Theorem 3. If \( ((\alpha, \beta); E) \) is a DFS-ideal of \( E \), then so is \( ((\alpha^*, \beta^*); E) \).

Proof. Assume that \( ((\alpha, \beta); E) \) is a DFS-ideal of \( E \). Then the nonempty \( \gamma \)-inclusive set \( i_E(\alpha; \gamma) \) and the \( \delta \)-exclusive set \( e_E(\beta; \delta) \) of \( ((\alpha, \beta); E) \) are ideals of \( E \) for every subsets \( \gamma \) and \( \delta \) of \( U \). If \( x \in i_E(\alpha; \gamma) \cap e_E(\beta; \delta) \), then

\[
\alpha^*(0) = \alpha(0) \supseteq \alpha(x) = \alpha^*(x)
\]

and

\[
\beta^*(0) = \beta(0) \subseteq \beta(x) = \beta^*(x).
\]

If \( x \notin i_E(\alpha; \gamma) \), then \( \alpha^*(x) = \eta \). Hence \( \alpha^*(0) \supseteq \eta = \alpha^*(x) \).

If \( x \notin e_E(\beta; \delta) \), then \( \beta^*(x) = \rho \). Hence \( \beta^*(0) \supseteq \rho = \beta^*(x) \).

Let \( x, y \in E \). If \( x \neq y \) then \( x \notin i_E(\alpha; \gamma) \) or \( y \notin i_E(\alpha; \gamma) \), then \( \alpha^*(x) = \eta \) or \( \alpha^*(y) = \eta \). Hence \( \alpha^*(x) \supseteq \eta = \alpha^*(x \cup y) \cap \alpha^*(y) \).

If \( x \neq y \notin i_E(\alpha; \gamma) \) or \( y \notin i_E(\alpha; \gamma) \), then \( \alpha^*(x) = \eta \) or \( \alpha^*(y) = \eta \).

Hence \( \alpha^*(x) \supseteq \eta = \alpha^*(x \cup y) \cap \alpha^*(y) \).

Now, if \( x \neq y \neq x \in e_E(\beta; \delta) \), then \( x \in e_E(\beta; \delta) \). Thus

\[
\beta^*(x) = \beta(x) \subseteq \beta(x \cup y) = \beta^*(x \cup y) \cup \beta^*(y).
\]

If \( x \neq y \notin e_E(\beta; \delta) \) or \( y \notin e_E(\beta; \delta) \), then \( \beta^*(x \cup y) = \rho \) or \( \beta^*(y) = \rho \). Hence

\[
\beta^*(y) \supseteq \rho = \beta^*(x \cup y) \cup \beta^*(y).
\]

Therefore \( ((\alpha^*, \beta^*); E) \) is a DFS-ideal of \( E \).

The following example shows that the converse of Theorem 3 is not true in general.

Example 4. Suppose that there are ten houses in the initial universe set \( U \) given by

\[
U = \{ h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10} \}.
\]

Let a set of parameters \( E = \{ e_0, e_1, e_2, e_3 \} \) be a set of status of houses in which

- \( e_0 \) stands for the parameter “beautiful”.
- \( e_1 \) stands for the parameter “cheap”.
- \( e_2 \) stands for the parameter “in good location”.
- \( e_3 \) stands for the parameter “in green surroundings”.

with the following binary operation:

\[
\alpha : E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{ h_2, h_4, h_6, h_8, h_{10} \} & \text{if } x = e_0, \\ \{ h_3, h_6, h_9 \} & \text{if } x = e_1, \\ \{ h_8 \} & \text{if } x = e_2, \\ \{ h_1, h_3, h_6, h_9 \} & \text{if } x = e_3. \end{cases}
\]

\[
\beta : E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{ h_6 \} & \text{if } x = e_0, \\ \{ h_3, h_6, h_9 \} & \text{if } x = e_1, \\ \{ h_2, h_4, h_6, h_8, h_{10} \} & \text{if } x = e_2, \\ U & \text{if } x = e_3. \end{cases}
\]

Then \( (E, \ast, e_0) \) is a BCI-algebra (see [9]). Consider a DFS-set \( ((\alpha, \beta); E) \) of \( E \) as follows:

\[
\alpha^*: E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \alpha(x) & \text{if } x \in i_E(\alpha; \gamma), \\ \emptyset & \text{otherwise}, \end{cases}
\]

\[
\beta^*: E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \beta(x) & \text{if } x \in e_E(\beta; \delta), \\ U & \text{otherwise}. \end{cases}
\]

Note that \( i_E(\alpha; \{ h_8 \}) = \{ e_0, e_1, e_3 \} \) is not an ideal of \( E \) since \( e_2 \ast e_3 = e_1 \notin i_E(\alpha; \{ h_8 \}) \) and \( e_2 \notin i_E(\alpha; \{ h_8 \}) \). Using Corollary 2, \( ((\alpha, \beta); E) \) is not a DFS-ideal of \( E \).

Note that \( i_E(\alpha; \gamma) = \{ e_0, e_1 \} = e_E(\beta; \delta) \) for \( \gamma = \{ h_2, h_4, h_6, h_8, h_{10} \} \) and \( \delta = \{ h_3, h_6, h_9 \} \). Let \( ((\alpha^*, \beta^*); E) \) be a DFS-set of \( E \) defined by

\[
\alpha^*: E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \alpha(x) & \text{if } x \in i_E(\alpha; \gamma), \\ \emptyset & \text{otherwise}, \end{cases}
\]

\[
\beta^*: E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \beta(x) & \text{if } x \in e_E(\beta; \delta), \\ U & \text{otherwise}. \end{cases}
\]

that is,

\[
\alpha^*: E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x = e_0, \\ \{ h_2, h_4, h_6, h_8, h_{10} \} & \text{if } x = e_1, \\ \emptyset & \text{if } x = e_2, e_3. \end{cases}
\]

\[
\beta^*: E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{ h_6 \} & \text{if } x = e_0, \\ \{ h_3, h_6, h_9 \} & \text{if } x = e_1, \\ U & \text{if } x = e_2, e_3. \end{cases}
\]

It is routine to verify that \( ((\alpha^*, \beta^*); E) \) is a DFS-ideal of \( E \).
Let \(\langle \alpha, \beta; A \rangle\) and \(\langle \alpha, \beta; B \rangle\) be DFS-sets of \(A\) and \(B\), respectively. The \((\alpha^\gamma, \beta^\gamma)\)-product (see [9]) of \(\langle \alpha, \beta; A \rangle\) and \(\langle \alpha, \beta; B \rangle\) is defined to be a DFS-set \(\langle \alpha^\gamma_{A\times B}, \beta^\gamma_{A\times B}; A \times B \rangle\) of \(A \times B\) in which
\[
\alpha^\gamma_{A\times B} : A \times B \to \mathcal{P}(U), (x, y) \mapsto \alpha(x) \cap \alpha(y),
\]
\[
\beta^\gamma_{A\times B} : A \times B \to \mathcal{P}(U), (x, y) \mapsto \beta(x) \cup \beta(y).
\]

**Theorem 4.** For any BCK/BCI-algebras \(E\) and \(F\) as sets of parameters, let \(\langle \alpha; E \rangle\) and \(\langle \alpha; F \rangle\) be DFS-ideals of \(E\) and \(F\), respectively. Then the \((\alpha^\gamma, \beta^\gamma)\)-product of \(\langle \alpha; E \rangle\) and \(\langle \alpha; F \rangle\) is a DFS-ideal of \(E \times F\).

**Proof.** Note that \((E \times F, \circledast, (0, 0))\) is a BCK/BCI-algebra. For any \((x, y) \in E \times F\), we have
\[
\alpha_{E \times F}(0, 0) = \alpha(0) \cap \alpha(0) \geq \alpha(x) \cap \alpha(y) = \alpha_{E \times F}(x, y),
\]
\[
\beta_{E \times F}(0, 0) = \beta(0) \cup \beta(0) \subseteq \beta(x) \cup \beta(y) = \beta_{E \times F}(x, y).
\]
Let \((x, y), (a, b) \in E \times F\). Then
\[
\alpha_{E \times F}((x, y) \circledast (a, b)) \cap \alpha_{E \times F}(a, b) = \alpha_{E \times F}(x \times a, y \times b) \cap \alpha_{E \times F}(a, b)
\]
\[
= (\alpha(x \times a) \cap (\alpha(y) \times b)) \cap (\alpha(a) \cap \alpha(b))
\]
\[
= (\alpha(x \times a) \cap (\alpha(y) \times b)) \cap (\alpha(a) \cap \alpha(b))
\]
\[
\subseteq \alpha(x) \cap \alpha(y) = \alpha_{E \times F}(x, y).
\]
and
\[
\beta_{E \times F}((x, y) \circledast (a, b)) \cup \beta_{E \times F}(a, b) = \beta_{E \times F}(x \times a, y \times b) \cup \beta_{E \times F}(a, b)
\]
\[
= (\beta(x \times a) \cup (\beta(y) \times b)) \cup (\beta(a) \cup \beta(b))
\]
\[
= (\beta(x \times a) \cup (\beta(y) \times b)) \cup (\beta(a) \cup \beta(b))
\]
\[
\supseteq \beta(x) \cup \beta(y) = \beta_{E \times F}(x, y).
\]
Hence \(\langle \alpha_{E \times F}, \beta_{E \times F}; E \times F \rangle\) is a DFS-ideal of \(E \times F\).

Let \(\langle \alpha, \beta; A \rangle\) and \(\langle f, g; B \rangle\) be DFS-sets of \(A\) and \(B\), respectively. Then \(\langle \alpha, \beta; A \rangle\) is called a double-framed soft subset (briefly, DFS-subset) of \(\langle f, g; B \rangle\), denoted by \(\langle \alpha, \beta; A \rangle \subseteq \langle f, g; B \rangle\), (see [9]) if
\[
(i) \quad A \subseteq B,
\]
\[
(ii) \quad \left(\alpha(e) \text{ and } f(e) \text{ are identical approximations}\right) \text{ for all } e \in A.
\]

**Theorem 5.** Let \(\langle \alpha, \beta; A \rangle\) be a DFS-subset of \(\langle f, g; B \rangle\). If \(\langle f, g; B \rangle\) is a DFS-ideal of \(B\), then \(\langle \alpha, \beta; A \rangle\) is a DFS-ideal of \(A\).

**Proof.** Let \(x \in A\). Then \(x \in B\), and so
\[
\alpha(0) = f(0) \geq f(x) = \alpha(x), \quad \beta(0) = g(0) \leq g(x) = \beta(x).
\]
Let \(x, y \in A\). Then \(x \times y \in B\). Hence
\[
\alpha(x) = f(x) \geq f(x \times y) \cap f(y) = \alpha(x \times y) \cap \alpha(y)
\]
and
\[
\beta(x) = g(x) \leq g(x \times y) \cup g(y) = \beta(x \times y) \cup \beta(y).
\]
Therefore \(\langle \alpha, \beta; A \rangle\) is a DFS-ideal of \(A\).
\((\langle \alpha, \beta \rangle; A)\) and \((\langle f, g \rangle; A)\) is defined to be the DFS-set \(\langle \langle f \cap g \rangle; A \rangle\) of \(A\) where
\[
\alpha \cap f : A \rightarrow \mathcal{P}(U), \quad x \mapsto \alpha(x) \cap f(x), \\
\beta \cup g : A \rightarrow \mathcal{P}(U), \quad x \mapsto \beta(x) \cup g(x).
\]

It is denoted by
\[
\langle \langle \alpha, \beta \rangle; A \rangle \cap \langle \langle f, g \rangle; A \rangle = \langle \langle \alpha \cap f, \beta \cup g \rangle; A \rangle
\]
(see [9]).

**Theorem 6.** The int-uni DFS-set \(\langle \langle \alpha, \beta \rangle; A \rangle \cap \langle \langle f, g \rangle; A \rangle\) of two DFS-ideals \(\langle \langle \alpha, \beta \rangle; A \rangle\) and \(\langle \langle f, g \rangle; A \rangle\) of \(A\) is a DFS-ideal of \(A\).

**Proof.** For any \(x \in A\), we have
\[
(\alpha \cap f)(0) = \alpha(0) \cap f(0) \supseteq \alpha(x) \cap f(x) = (\alpha \cap f)(x), \\
(\beta \cup g)(0) = \beta(0) \cup g(0) \subseteq \beta(x) \cup g(x) = (\beta \cup g)(x).
\]

For any \(x, y \in A\), we have
\[
(\alpha \cap f)(x) = \alpha(x) \cap f(x) \\
\supseteq (\alpha(x \ast y) \cap \alpha(y)) \cap (f(x \ast y) \cap f(y)) \\
= (\alpha \ast f)(x \ast y) \cap (\alpha \ast f)(y) \\
= (\beta \cup g)(x \ast y) \cap (\beta \cup g)(y).
\]

Therefore \(\langle \langle \alpha, \beta \rangle; A \rangle \cap \langle \langle f, g \rangle; A \rangle\) is a DFS-ideal of \(A\).

**Corollary 4.** The int-uni DFS-set \(\langle \langle \alpha, \beta \rangle; E \rangle \cap \langle \langle f, g \rangle; E \rangle\) of two DFS-ideals \(\langle \langle \alpha, \beta \rangle; E \rangle\) and \(\langle \langle f, g \rangle; E \rangle\) of \(E\) is a DFS-ideal of \(E\).

**Definition 5.** Let \((U, E) = (U, X)\) where \(X\) is a BCI-algebra. A DFS-ideal \(\langle \langle \alpha, \beta \rangle; E \rangle\) of \(E\) is said to be closed if it satisfies:
\[
(\forall x \in E)(\alpha(0 \ast x) \supseteq \alpha(x), \beta(0 \ast x) \subseteq \beta(x)). \tag{10}
\]

**Example 6.** Let \((U, E)\) be a soft universe which is given in Example 5. Consider a DFS-set \(\langle \langle \alpha, \beta \rangle; E \rangle\) of \(E\) as follows:
\[
\alpha : E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 
U & \text{if } x = e_0, \\
\{h_1, h_2\} & \text{if } x = e_1, \\
\{h_1, h_3, h_4\} & \text{if } x = e_2, \\
\{h_1, h_3\} & \text{if } x = e_3, \\
\{h_1\} & \text{if } x = e_4,
\end{cases} \\
\beta : E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 
\{h_5\} & \text{if } x = e_0, \\
\{h_6, h_5\} & \text{if } x = e_1, \\
\{h_5, h_3\} & \text{if } x = e_2, \\
\{h_3, h_4, h_5\} & \text{if } x \in \{e_3, e_4\}.
\end{cases}
\]

Then \(\langle \langle \alpha, \beta \rangle; E \rangle\) is a closed DFS-ideal of \(E\).

**Example 7.** Let \((U, E) = (U, X)\) where \(X = \{2^n \mid n \in \mathbb{Z}\}\) is a BCI-algebra with a binary operation “\(\ast\)” (usual division). Let \(\langle \langle \alpha, \beta \rangle; E \rangle\) be a DFS-set of \(E\) defined as follows:
\[
\alpha : E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 
\gamma_1 & \text{if } n \geq 0, \\
\gamma_2 & \text{if } n < 0,
\end{cases} \\
\beta : E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 
\delta_1 & \text{if } n \geq 0, \\
\delta_2 & \text{if } n < 0,
\end{cases}
\]
where \(\gamma_1, \gamma_2, \delta_1\) and \(\delta_2\) are subsets of \(U\) with \(\gamma_1 \supseteq \gamma_2\) and \(\delta_1 \subseteq \delta_2\). Then \(\langle \langle \alpha, \beta \rangle; E \rangle\) is a DFS-ideal of \(E\) which is not closed since
\[
\alpha(1 \ast 2^3) = \alpha(2^{-3}) = \gamma_2 \not\supseteq \gamma_1 = \alpha(2^3)
\]
and/or
\[
\beta(1 \ast 2^3) = \beta(2^{-3}) = \delta_2 \not\subseteq \delta_1 = \beta(2^3).
\]

**Theorem 7.** Let \((U, E) = (U, X)\) where \(X\) is a BCI-algebra. Then a DFS-ideal of \(E\) is closed if and only if it is a DFS-algebra of \(E\).

**Proof.** If \(\langle \langle \alpha, \beta \rangle; E \rangle\) is a closed DFS-ideal of \(E\), then \(\alpha(0 \ast x) \supseteq \alpha(x)\) and \(\beta(0 \ast x) \subseteq \beta(x)\) for all \(x \in E\). It follows from the (3), (III) and (9) that
\[
\alpha(x \ast y) \supseteq \alpha((x \ast y) \ast x) \cap \alpha(x) \\
= \alpha(0 \ast x) \cap \alpha(x) \\
\supseteq \alpha(x) \cap \alpha(x)
\]
and
\[
\beta(x \ast y) \subseteq \beta((x \ast y) \ast x) \cup \beta(x) \\
= \beta(0 \ast x) \cup \beta(x) \\
\subseteq \beta(x) \cup \beta(y)
\]
for all \(x, y \in E\). Hence \(\langle \langle \alpha, \beta \rangle; E \rangle\) is a DFS-algebra of \(E\).

Conversely, let \(\langle \langle \alpha, \beta \rangle; E \rangle\) be a DFS-ideal of \(E\) which is also a DFS-algebra of \(E\). Then
\[
\alpha(0 \ast x) \supseteq \alpha(0) \cap \alpha(x) = \alpha(x), \\
\beta(0 \ast x) \subseteq \beta(0) \cup \beta(x) = \beta(x)
\]
for all \(x \in E\). Therefore \(\langle \langle \alpha, \beta \rangle; E \rangle\) is closed.

Let \(X\) be a BCI-algebra and \(B(X) := \{x \in X \mid 0 \leq x\}\). For any \(x \in X\) and \(n \in \mathbb{N}\), we define \(x^n\) by
\[
x^0 = x, \quad x^{n+1} = x \ast (0 \ast x^n).
\]
If there is an \(n \in \mathbb{N}\) such that \(x^n \in B(X)\), then we say that \(x\) is of finite periodic (see [18]), and we denote its period \(|x|\) by
\[
|x| = \min \{n \in \mathbb{N} \mid x^n \in B(X)\}.
\]
Otherwise, \(x\) is of infinite period and denoted by \(|x| = \infty\).

We provide conditions for a DFS-ideal to be closed.
Theorem 8. Let \((U, E) = (U, X)\) where \(X\) is a BCI-algebra in which every element is of finite period. Then every DFS-ideal of \(E\) is closed.

Proof. Let \(((\alpha, \beta); E)\) be a DFS-ideal of \(E\). For any \(x \in E\), assume that \(|x| = n\). Then \(x^n \in B(X)\). Note that

\[
(0 \ast x^{n-1}) \ast x = (0 \ast (0 \ast (0 \ast x^{n-2}))) \ast x \\
= (0 \ast x) \ast (0 \ast (0 \ast x^{n-2})) \\
= 0 \ast (x \ast (0 \ast x^{n-2}) \\
= 0 \ast x^n = 0,
\]

and so

\[
\alpha ((0 \ast x^{n-1}) \ast x) = \alpha(0) \supseteq \alpha(x)
\]

and

\[
\beta ((0 \ast x^{n-1}) \ast x) = \beta(0) \subseteq \beta(x)
\]

by (8). It follows from (9) that

\[
\alpha(0 \ast x^{n-1}) \supseteq \alpha((0 \ast x^{n-1}) \ast x) \cap \alpha(x) \supseteq \alpha(x),
\]

\[
\beta(0 \ast x^{n-1}) \subseteq \beta((0 \ast x^{n-1}) \ast x) \cup \beta(x) \subseteq \beta(x). \tag{11}
\]

Also, note that

\[
(0 \ast x^{n-2}) \ast x = (0 \ast (0 \ast (0 \ast x^{n-3}))) \ast x \\
= (0 \ast x) \ast (0 \ast (0 \ast x^{n-3})) \\
= 0 \ast (x \ast (0 \ast x^{n-3}) \\
= 0 \ast x^{n-1},
\]

which implies from (11) that

\[
\alpha((0 \ast x^{n-2}) \ast x) = \alpha(0 \ast x^{n-1}) \supseteq \alpha(x),
\]

\[
\beta((0 \ast x^{n-2}) \ast x) = \beta(0 \ast x^{n-1}) \subseteq \beta(x).
\]

Using (9), we have

\[
\alpha(0 \ast x^{n-2}) \supseteq \alpha((0 \ast x^{n-2}) \ast x) \cap \alpha(x) \supseteq \alpha(x),
\]

\[
\beta(0 \ast x^{n-2}) \subseteq \beta((0 \ast x^{n-2}) \ast x) \cup \beta(x) \subseteq \beta(x).
\]

Continuing this process, we have \(\alpha(0 \ast x) \supseteq \alpha(x)\) and \(\beta(0 \ast x) \subseteq \beta(x)\) for all \(x \in E\). Therefore \(((\alpha, \beta); E)\) is closed.

Acknowledgement

This work (RPP-2012-021) was supported by the fund of Research Promotion Program, Gyeongsang National University, 2012. Also, the second author was partially supported by the research grants from the Deanship of Scientific Research Unit, University of Tabuk, Tabuk, Kingdom of Saudi Arabia. The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

Y. B. Jun received the Ph. D. Degree at Kyung Hee University, and is working at Dept. of Mathematics Education, Gyeongsang National University as a professor. He has published a book, “BCK-algebras”, with Professor J. Meng, and published many research papers. His mathematical research areas are BCK/BCI-algebra, fuzzy algebraic structure and soft (rough) algebraic structure, and he is reviewing many papers in this areas. His Awards and Honors are: Academic Achievement Award (7 July 2006), Busan-Kyeongnam Branch of the Korean Mathematical Society, and listed in 8th edition of "Marquis Who’s Who in Science and Engineering”.

G. Muhiuddin is working at Department of Mathematics, Tabuk University, Tabuk 71491, Saudi Arabia as an Assistant Professor. He has received his Ph.D. degree in Pure Mathematics. His mathematical research areas are Algebras related to logic (BCK, BCI, BCC-algebras, Hilbert algebras, implication algebras), Fuzzy logical algebras and Category theory. He has reviewed many research papers in this area as well as other related areas.

Abdullah M. Al-roqi is working at Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia as an Associate Professor. He has received his Ph.D. degree from School Of Mathematics and Statistics, University of Birmingham, United Kingdom. His mathematical research areas are Algebras related to logic, Finite Group Theory, Soluble groups, Classification of finite simple groups and Representation Theory.