On Some Properties of Spectral Radius for Brualdi-Li Matrix

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Abstract: Let $B_{2m}$ denote the Brualdi-Li matrix, and let $\rho(B_{2m})$ denote the spectral radius of Brualdi-Li matrix. We obtain some properties of $\rho(B_{2m})$.

Keywords: Brualdi-Li Matrix, Spectral Radius, Reducible Matrix, Tournament Matrix

1 Introduction

A tournament matrix of order $n$ is a $(0,1)$ matrix $T$ satisfying the equation $T + T^t = J - I$, where $J$ is the all ones matrix, $I$ is the identity matrix, and $T^t$ is the transpose of $T$. The tournament matrices are inspired in the round robin competitions. Tournament matrices (and their generalizations) appear in a variety of combinatorial applications (e.g., in biology, sociology, statistics, and networks).

Brualdi and Li matrix of order $2m$ is defined by

$$B_{2m} = \begin{pmatrix} U_m & U_m^t \\ I + U_m^t U_m \end{pmatrix},$$

where $U_m$ is strictly lower triangular tournament matrix (all of whose entries below the main diagonal are equal to one). A matrix $A$ of order $n$ is said to be a reducible matrix if there exists a permutation matrix $P$ such that

$$PAP^t = \begin{pmatrix} A_1 & A_2 \\ 0 & A_2 \end{pmatrix},$$

where $A_1$ and $A_2$ are square (non-vacuous), or if $n = 1$ and $A = O$. A matrix is called irreducible if it is not reducible. The spectral radius of a matrix $A_{n \times n}$ defined as $\rho(A) = \max \{|\lambda_1|,|\lambda_2|,\ldots,|\lambda_n|\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A_{n \times n}$. If an nonnegative matrix $A$ is irreducible and it has exactly one eigenvalue of modulus $\rho(A)$, then the matrix is called a primitive matrix. Obviously, Brualdi and Li matrix $B_{2m}$ ($m \geq 2$) is primitive matrix.

In 1983 Brualdi and Li conjectured that the maximal spectral radius for tournaments of order $2m$ is attained by the Brualdi-Li matrix [1]. This conjecture has recently been confirmed in [2]. The several interesting properties of Brualdi-Li matrix are studied. In this paper we investigate some properties of spectral radius for Brualdi-Li Matrix.

2 Preliminaries

The notation and terminology used in this paper will basically follow those in [3].

Let $I_m = (1,1,\ldots,1)^t_{m \times 1}, 0_m = (0,0,\ldots,0)^t_{m \times 1},$ and

$$U_m = \begin{pmatrix} 0 & 0 \ldots 0 \\ 1 & 0 \ldots 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 \cdots 0 \\ 1 & 1 \cdots 1 \end{pmatrix}_{m \times m},$$

where $m \geq 2$ is an integer.

Lemma 2.1[3] Let $n$ be a nonnegative integer, and $A$ be a primitive matrix of order $n$. Then

$$\lim_{k \rightarrow \infty} \left( \frac{A}{\rho} \right)^k I_n = S,$$

where $\rho = \rho(A) > 0, S > 0$ is an eigenvector of $A$ corresponding to the eigenvalue of $\rho(A)$.

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Let $b(2m,k) = 1_{2m}^t \mathcal{B}_{2m}^k 1_{2m}$, 
\[b_j(2m,k) = 1_{2m}^t \mathcal{B}_{2m}^k a_{1j},\text{and}\]
\[b_r(2m,k) = 1_{2m}^t \mathcal{B}_{2m}^k a_{r1},\text{then}\]
\[b(2m,k) + 1 = 1_{2m}^t \mathcal{B}_{2m}^{k+1} 1_{2m} = 1_{2m}^t \mathcal{B}_{2m}^{k+1} (m-1)_{1m} = mb(2m,k) - b_1(2m,k)
= (m-1)b(2m,k) + b_r(2m,k).\]
It is easy to verify the following result.

**Lemma 2.2** Let $k, m \geq 2$ be an integer, and $\rho = \rho(\mathcal{B}_{2m})$. Then

1. \(\lim_{k \to \infty} \sqrt[n]{b(2m,k)} = \rho;\)
2. \(\lim_{k \to \infty} b(2m,k) = \rho;\)
3. \(\lim_{k \to \infty} b(2m,k) = m - \rho;\)
4. \(\lim_{k \to \infty} b(2m,k) = \rho - m + 1;\)
5. \(\lim_{k \to \infty} b(2m,k) = m - \rho - m + 1;\)

Let $\mathcal{B}_{2m}^k = \begin{pmatrix} B_{11}^k & \ldots & B_{kr}^k \\ B_{21}^k & \ldots & B_{2r}^k \end{pmatrix}$ and 
\[b_{ij}(2m,k) = 1_{2m}^t B_{ij}^k 1_{2m}, i, j = 1, 2, \]
where $B_{11}^k, B_{12}^k, B_{21}^k, B_{22}^k$ are matrices of order $m$.

Now that $b(2m,k+1) = 1_{2m}^t \mathcal{B}_{2m}^{k+1} 1_{2m}$
\[= 1_{2m}^t \mathcal{B}_{2m}^{k+1} (m-1)_{1m} = (m - 1)(b(2m,k) + b_{12}(2m,k)) + m(b_{11}(2m,k) + b_{22}(2m,k));\]
\[b(2m,k) + 1 = b_{11}(2m,k) + b_{22}(2m,k) + 1 \]
\[= (m - 1)b_{11}(2m,k) + b_{12}(2m,k) + (m - 1)b_{21}(2m,k) + b_{22}(2m,k),\]
then we have
\[b_{11}(2m,k) = b_{22}(2m,k),\] 
\[b(2m,k-1) = b_{11}(2m,k) + b_{22}(2m,k) + 1\]
\[= (m - 1)b_{11}(2m,k) + b_{12}(2m,k) + (m - 1)b_{21}(2m,k) + b_{22}(2m,k),\]
Leading to the following result.

**Lemma 2.3** Let $k, m \geq 2$ be an integer. Then

1. \[\begin{pmatrix} b(2m,k) \\ b(2m,k) + 1 \\ b(2m,k+2) \end{pmatrix} = \begin{pmatrix} b_1(2m,k) \\ b_2(2m,k) \\ b_3(2m,k) \end{pmatrix};\]

2. \[\begin{pmatrix} b(2m,k) \\ b(2m,k) + 1 \\ b(2m,k+2) \end{pmatrix} = \begin{pmatrix} b_1^2(2m,k) \\ b_1(2m,k) \\ b_3(2m,k) \end{pmatrix},\]
\[= \begin{pmatrix} 1 & 1 & -1 \\ m & 0 & -1 \\ m^2 & -2m & 1 \end{pmatrix} \begin{pmatrix} b(2m,k) \\ b(2m,k) + 1 \\ b(2m,k+2) \end{pmatrix}.\]

**Lemma 2.4** Let $m \geq 2$ be an integer, 
\[\rho = \rho(\mathcal{B}_{2m}),\] and
\[v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m \] be an eigenvector of $\mathcal{B}_{2m}$ corresponding to the eigenvalue of $\rho(\mathcal{B}_{2m}), \Sigma_{j=1}^m v_j + \Sigma_{j=1}^m w_j = 1$. Then
\[1. \rho = m - \Sigma_{j=1}^m v_j + w_1 = m - 1 + \Sigma_{j=1}^m v_j;\]
\[2. \nu_m = \rho + 1 - m;\]
\[3. v_k = \frac{1}{2^{k+1}} - \frac{2m+1}{2^{k+1}} \frac{1}{\rho + 1}. \]
\[k = 1, 2, \ldots, m.\]

3 Some properties for spectral radius of Brualdi-Li matrix

**Theorem 3.1** Let $m \geq 2$ be an integer, and $\rho = \rho(\mathcal{B}_{2m})$. Then

1. \[\lim_{k \to \infty} b_{11}(2m,k) = -m(m-1)b(2m,k) + 2b(2m,k) + 1 - b(2m,k+2),\]
\[= (m - 1)b(2m,k) - b(2m,k+2);\]

2. \[\lim_{k \to \infty} b_{12}(2m,k) = m(m-1)b(2m,k) + 2b(2m,k) + 1 - b(2m,k+2),\]
\[= -m(m-1)b(2m,k) + 2b(2m,k) + 1 - b(2m,k+2);\]

3. \[\lim_{k \to \infty} b_{21}(2m,k) = m(m-1)b(2m,k) + 2b(2m,k) + 1 - b(2m,k+2),\]
\[= -m(m-1)b(2m,k) + 2b(2m,k) + 1 - b(2m,k+2);\]

Proof By Lemma 2.2, we have

1. \[\lim_{k \to \infty} b_{11}(2m,k) = -m(m-1)b(2m,k) + 2b(2m,k) + 1 - b(2m,k+2),\]
\[= (m - 1)b(2m,k) - b(2m,k+2);\]

2. \[\lim_{k \to \infty} b_{12}(2m,k) = m(m-1)b(2m,k) + 2b(2m,k) + 1 - b(2m,k+2),\]
\[= -m(m-1)b(2m,k) + 2b(2m,k) + 1 - b(2m,k+2);\]

Using a similar approach, we have obtained (2) and (3).

**Theorem 3.2** Let $m \geq 2$ be an integer, and $\rho = \rho(\mathcal{B}_{2m})$. Then

1. \[\lim_{k \to \infty} b_{11}(2m,k) = \frac{b_{11}(2m,k)}{b_{12}(2m,k)}, \text{and}\]
\[\lim_{k \to \infty} b_{12}(2m,k) = \frac{b_{12}(2m,k)}{b_{11}(2m,k)} = \rho;\]

2. \[\lim_{k \to \infty} b_{12}(2m,k) = \rho, \text{for } i, j = 1, 2.\]

Proof By Lemma 2.3(2,3),
\[\lim_{k \to \infty} b_{12}(2m,k) = \frac{b_{12}(2m,k)}{b_{12}(2m,k-1)} = \rho;\]
\[\lim_{k \to \infty} b_{11}(2m,k) - \frac{b_{11}(2m,k-1)}{b_{11}(2m,k-1)} = \rho.\]

Using a similar approach, we have obtained
\[\lim_{k \to \infty} b_{11}(2m,k) = \rho \text{ and } 2.\]

**Theorem 3.3** Let $m \geq 2$ be an integer, and $\rho = \rho(\mathcal{B}_{2m})$, and
\[v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m \] be an eigenvector of $\mathcal{B}_{2m}$ corresponding to the eigenvalue of $\rho(\mathcal{B}_{2m})$, where $\Sigma_{j=1}^m v_j + \Sigma_{j=1}^m w_j = 1$. Then
\[\Sigma_{j=1}^m (w_i - v_i) = (m - \rho)^2.\]
Proof. By Lemma 2.4,
\[
\sum_{i=1}^{m} i (w_i - v_i) = \sum_{i=1}^{m} \frac{1}{\rho} (w_i - v_i (2p+1)) = \sum_{i=1}^{m} i \left( \frac{1}{\rho} (w_i - v_i) \right) (2p+1) = \sum_{i=1}^{m} \frac{1}{\rho (p+1)} i \left( \frac{2p+1}{\rho} \right) = \sum_{i=1}^{m} \frac{1}{\rho (p+1)} \frac{(2p+1)}{\rho (p+1)} i \left( \frac{2p+1}{\rho} \right) (2p+1) = \sum_{i=1}^{m} \frac{1}{\rho (p+1)} i \left( \frac{2p+1}{\rho} \right) (2p+1).
\]

Now that \( \sum_{i=1}^{m} i (w_i - v_i) = \frac{m^2 + 1}{(m-1)(m+1)} \)

\[
\sum_{i=1}^{m} i (2p+1) = \frac{m(1 + \frac{1}{\rho})^{2m+3} - (m+1)(1 + \frac{1}{\rho})^{2m+1} + 1 + \frac{1}{\rho}}{(1 + \frac{1}{\rho})^{2m+1}}.
\]

We have
\[
\sum_{i=1}^{m} i (w_i - v_i) = \frac{(2p+1)(2m+3 - (m+1)(1 + \frac{1}{\rho})^{2m+1} + 1 + \frac{1}{\rho})}{\rho (p+1)} = \frac{\rho^3}{2p+1} \left( \frac{2p+1}{\rho} \right) (2p+1) = \frac{\rho^3}{2p+1} \left( \frac{2p+1}{\rho} \right) (1 + \frac{1}{\rho})^{2m+1}.
\]

\[
(1 + \frac{1}{\rho})^{2m+1} = \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}}.
\]

\[
\sum_{i=1}^{m} i (w_i - v_i) = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}} = \frac{\rho^3}{2p+1} (m + \frac{1}{\rho})^{2m+1} - (m + 1)(1 + \frac{1}{\rho})^{2m+1} + 1 + \frac{1}{\rho}.
\]

\[
\sum_{i=1}^{m} i (2p+1) = \frac{\rho^3}{2p+1} \left( \frac{2p+1}{\rho} \right) = \frac{\rho^3}{2p+1} \left( \frac{2p+1}{\rho} \right)^{2m+1} = \frac{\rho^3}{2p+1} \left( \frac{2p+1}{\rho} \right)^{2m+1}.
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\[
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\]

\[
(1 + \frac{1}{\rho})^{2m+1} = \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}}.
\]

\[
\sum_{i=1}^{m} i (w_i - v_i) = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}} = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}}.
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\[
\sum_{i=1}^{m} i (2p+1) = \frac{\rho^3}{2p+1} \left( \frac{2p+1}{\rho} \right) = \frac{\rho^3}{2p+1} \left( \frac{2p+1}{\rho} \right) = \frac{\rho^3}{2p+1} \left( \frac{2p+1}{\rho} \right).
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\[
\sum_{i=1}^{m} i (w_i - v_i) = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}} = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}}.
\]

\[
(1 + \frac{1}{\rho})^{2m+1} = \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}}.
\]

\[
\sum_{i=1}^{m} i (w_i - v_i) = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}} = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}}.
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\sum_{i=1}^{m} i (w_i - v_i) = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}} = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}}.
\]

\[
\sum_{i=1}^{m} i (w_i - v_i) = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}} = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}}.
\]

\[
\sum_{i=1}^{m} i (w_i - v_i) = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}} = \frac{\rho^3}{2p+1} \frac{\rho^{n+1}}{(1 + \frac{1}{\rho})^{2m+1}}.
\]

References