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Inverse of the Vandermonde and Vandermonde confluent matrices

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The inverse of the Vandermonde and confluent Vandermonde matrices are presented. In the case of the Vandermonde matrix, we present a decomposition in three factors, one of them a diagonal matrix. The evaluation of such inverse matrices is a key point to find functions of a matrix, namely exponential functions (evolution operators) and logarithmic functions (entropies) in quantum mechanical topics.

Keywords: Wigner Function, Divergent Series

1 Introduction

Although Vandermonde systems arise in many approximation and interpolation problems [**?**, 2], they also appear when we need to solve systems of differential equations [3], such as when we interact a multi-level atom with a classical or quantum field, a trapped ion with a laser field (see for instance [4]), etc. When an atom interacts with a quantized field they get entangled [5]. This produces that, by analyzing the density matrix of the atom or the density matrix of the quantized field, we can determine when they disentangle. To do so we need either to calculate the entropy of the sub-systems and this requires evaluation of functions of (density) matrices.

In the particular case of the entropy, we need to calculate logarithmic functions of the sub-system's density matrices [5]. Vandermonde matrices, and in particular, their inverse, are helpful to determine such functions. A more common function is the exponential function of a matrix, as a Hamiltonian may be written usually in matrix form, and therefore the solution of Scrödinger equations involve the use of evolution operators, i.e. exponentials of Hamiltonians (see for instance [6], also for the case when superoperators are considered, and [7] for time dependent Hamiltonians). The key point for the evalution of such functions is to find the inverse of a Vandermonde matrix or of the confluent Vandermonde matrix (in case there are repeated eigenvalues). The purpose of the present paper is precisely this.

2 Vandermonde matrices

A matrix $N \times N$ of the form

$$V = \begin{pmatrix} 1 & 1 & 1 & . & . & . & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & . & . & . & \lambda_{N-1} & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & . & . & . & \lambda_{N-1}^2 & \lambda_N^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & . & . & . & \lambda_{N-1}^3 & \lambda_N^3 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \lambda_3^{N-1} & . & . & . & \lambda_{N-1}^{N-1} & \lambda_N^{N-1} \end{pmatrix} \tag{2.1}$$

or

$$V_{i,j} = \lambda_j^{i-1} \qquad i = 1, 2, 3, ..., N; \qquad j = 1, 2, 3, ..., N \tag{2.2}$$

is said to be a Vandermonde matrix [**?**, 8].

The determinant of the Vandermonde matrix can be expressed as $\det(V) = \prod\limits_{1 \leq i \leq j \leq N} (\lambda_j - \lambda_i)$. Therefore, if the numbers $\lambda_1, \lambda_2, ..., \lambda_N$ are distinct, V is a nonsingular matrix [8].

When two or more λ_i are equal, the corresponding matrix is singular. In that case, one may use a generalization called confluent Vandermonde matrix [**?**, 9], which makes the matrix non-singular, while retaining most properties. If $\lambda_i = \lambda_{i+1} = ... = \lambda_{i+k}$ and $\lambda_i \neq \lambda_{i-1}$, then the $(i+k)$th column is given by

$$C_{i+k,j} = \left\{ \begin{array}{cc} 0 & j \leq k \\ \dfrac{(j-1)!}{(j-k-1)!} x_i^{j-k-1} & j > k \end{array} \right. \tag{2.3}$$

The confluent Vandermonde matrix looks as

$$C = \begin{pmatrix} 1 & 1 & ... & 1 & 0 & 0 & ... & 1 & 1 \\ \lambda_1 & \lambda_2 & ... & \lambda_i & 1 & 0 & ... & \lambda_{m-1} & \lambda_m \\ \lambda_1^2 & \lambda_2^2 & ... & \lambda_i^2 & 2\lambda_i & 0 & ... & \lambda_{m-1}^2 & \lambda_m^2 \\ \lambda_1^3 & \lambda_2^3 & ... & \lambda_i^3 & 3\lambda_i^2 & . & ... & \lambda_{m-1}^3 & \lambda_m^3 \\ ... & ... & ... & ... & ... & \dfrac{(i-1)!}{(i-k-1)!}\lambda_i^{i-k-1} & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ \lambda_1^{n-1} & \lambda_2^{n-1} & ... & \lambda_i^{n-1} & (n-1)\,\lambda_i^{n-2} & \dfrac{(n-1)!}{(n-k-1)!}\lambda_i^{n-k-1} & ... & \lambda_{m-1}^{n-1} & \lambda_m^{n-1} \end{pmatrix}. \tag{2.4}$$

Another way to write the $(i+k)$ column is using the derivative, as follows

$$C_{i,j+k} = \frac{dC_{i,j+k-1}}{d\lambda_j}. \tag{2.5}$$

3 The inverse of the Vandermonde matrix

In applications, a key role is played by the inverse of the Vandermonde and confluent Vandermonde matrices [**?**, **?**, **?**, **?**, 2, 3, 13, 14]. Both matrices, Vandermonde and confluent Vandermonde, can be factored into a lower triangular matrix L' and an upper triangular matrix U' where V or C is equal to $L'U'$. The factorization is unique if no row or column interchanges are made and if it is specified that the diagonal elements of U' are unity.

Then, we can write $V^{-1} = (U')^{-1}(L')^{-1}$. Denoting $(U')^{-1}$ as U, we have found that U is an upper triangular matrix whose elements are

$U_{i,j} = 0$ if $i > j$

$U_{i,j} = \prod_{k=1,\, k\neq i}^{j} \frac{1}{\lambda_i - \lambda_k}$ *otherwise.*

The matrix U can be decomposed as the product of a diagonal matrix D and other upper triangular matrix W. It is very easy to find that

$$D_{i,j} = \begin{cases} \prod_{k=1,\, k\neq i}^{N} \frac{1}{\lambda_i - \lambda_k} & i = j \\ 0 & i \neq j \end{cases} \tag{3.1}$$

and

$$W_{i,j} = \begin{cases} 0 & i > j \\ \prod_{k=j+1,\, k\neq i}^{N} (\lambda_i - \lambda_k) & \text{otherwise} \end{cases}. \tag{3.2}$$

The matrix $L = (L')^{-1}$ is a lower triangular matrix, whose elements are

$$L_{i,j} = \begin{cases} 0 & i < j \\ 1 & i = j \\ L_{i-1,j-1} - L_{i-1,j}\lambda_{i-1} & i = 2, 3, ..., N;\ j = 2, 3, ..., i-1 \end{cases}. \tag{3.3}$$

Summarizing, the inverse of the Vandermonde matrix can be written as $V^{-1} = DWL$.

4 The inverse of the confluent Vandermonde matrix

We will treat now the case of the confluent Vandermonde matrix. We suppose that just one of the values λ_i is repeated, and it is repeated m times. We make the usual LU decomposition, getting $C = L'_c U'_c$, where L'_c is a lower triangular matrix and U'_c an upper triangular matrix U'. Then, we can write $C^{-1} = (U'_c)^{-1} (L'_c)^{-1}$. Denoting $(U'_c)^{-1}$ as U_c, we have found that U_c is an upper triangular matrix whose elements are

$$(U_c)_{i,j} = 0 \qquad i \qquad i > j \tag{4.1}$$

$$(U_c)_{i,j} = \frac{\delta_{ij}}{(i-1)!} \qquad i = 1, 2, 3, ..., m; \qquad j = 1, 2, 3, ..., m \tag{4.2}$$

$$(U_c)_{i,j} = -\frac{1}{(i-1)!} \sum_{\alpha=m+1}^{j} \prod_{\beta=i,\beta\neq\alpha}^{j} \frac{1}{(\lambda_\alpha - \lambda_\beta)} \tag{4.3}$$
$$i = 1, 2, 3, ..., m;$$
$$j = m+1, m+2, ..., N$$

$$(U_c)_{i,j} = \prod_{\beta=1,\beta\neq\alpha}^{j} \frac{1}{(\lambda_i - \lambda_\beta)} \qquad i = m+1, m+2, ..., N; \qquad j = i, ..., N \tag{4.4}$$

where it is understood that $\lambda_m = \lambda_{m-1} = ... = \lambda_2 = \lambda_1$, and where the numbers $\lambda_m,\ \lambda_{m-1},\ ...,\ \lambda_2$ appear, they must be substituted by λ_1.

The matrix $L_c = (L'_c)^{-1}$ is a lower triangular matrix, whose elements are given by the following recurrence relation,

$$(L_c)_{i,j} = \left\{ \begin{array}{cc} 0 & if i < j \\ 1 & if i = j \\ (L_c)_{i,ji-1,j-1} - (L_c)_{i,ji-1,j}\lambda_{i-1} & if i = 2, 3, ..., N; j = 2, 3, ..., i-1 \end{array} \right. , \tag{4.5}$$

also here it is understood that $\lambda_m = \lambda_{m-1} = ... = \lambda_2 = \lambda_1$, and where the numbers $\lambda_m,\ \lambda_{m-1},\ ...,\ \lambda_2$ appear, they must be substituted by λ_1.

When more than one value is repeated, the inverse has blocks with the same structure that we have already found.

5 Conclusions

We have shown a form to determine the inverse of Vandermonde and confluent Vandermonde matrices. Although several studies exist for Vandermonde matrices, it is not so for systems with repeated eigenvalues, which lead to confluent matrices. Such inverse matrices are of importance in several quantum mechanical topics where it is needed to find functions of matrices, such as in quantum information processes, where entropies play a key role.

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