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Application of Semiorthogonal B-Spline Wavelets for the Solutions of Linear Second Kind Fredholm Integral Equations

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Abstract: In this paper, the linear semiorthogonal compactly supported B-spline wavelets together with their dual wavelets have been applied to approximate the solutions of Fredholm integral equations of the second kind. Properties of these wavelets are first presented; these properties are then utilized to reduce the computation of integral equations to some algebraic equations. The method is computationally attractive, and application of it has been demonstrated through illustrative examples.

Keywords: Scaling functions, B-spline wavelets, Semiorthogonal, Fredholm Integral equation of second kind.

1 Introduction

Wavelets theory is a relatively new and emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for waveform representations and segmentations, time frequency analysis, and fast algorithms for easy implementation [1]. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms. Wavelets can be separated into two distinct types, orthogonal and semiorthogonal [1, 2]. The research works available in open literature on integral equation methods have shown a marked preference for orthogonal wavelets [3]. This is probably because the original wavelets, which were widely used for signal processing, were primarily orthogonal. In signal processing applications, unlike integral equation methods, the wavelet itself is never constructed since only its scaling function and coefficients are needed. However, orthogonal wavelets either have infinite support or a nonsymmetric, and in some cases fractal, nature. These properties can make them a poor choice for characterization of a function. In contrast, the semiorthogonal wavelets have finite support, both even and odd symmetry, and simple analytical expressions, ideal attributes of a basis function [3].

Numerical methods for approximating the solution of Fredholm integral equation of second kind are limitedly known. In the present paper, we apply compactly supported linear semiorthogonal B-spline wavelets, specially constructed for the bounded interval to solve the second Kind linear Fredholm integral equations of the form:

$$y(x) = f(x) + \int_0^1 K(x,t)y(t)dt, \quad 0 \leqslant x \leqslant 1, \tag{1}$$

where $K(x,t)$ and $f(x)$ are known functions and $y(x)$ is unknown function to be determined.

In recent years, the applications of methods based on wavelets have influenced many areas of applied mathematics. In areas such as the numerical solutions of differential equations, partial differential equations and fractional differential equations, wavelets are recognized as a powerful tool. Another area in which the wavelet is gaining considerable attention is the study of integral equations. It is found that semiorthogonal wavelets are best suited for integral equation applications.

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The present method consists of reducing equation (1) to a set of algebraic equations by expanding the unknown function as linear B-spline wavelets with unknown coefficients. The properties of these wavelets are then utilized to evaluate the unknown coefficients.

2 B-spline scaling functions and wavelet functions

When semiorthogonal wavelets are constructed from B-spline of order m, the lowest octave level $j = j_0$ is determined in [4-6] by

$$2^{j_0} \geq 2m-1 \tag{2}$$

so as to give a minimum of one complete wavelet on the interval $[0,1]$. In this paper, we will use a wavelet generated by a linear B-spline, $m = 2$, the second order cardinal B-spline function. From (2), the second-order B-spline lowest level, which must be an integer, is determined to $j_0 = 2$. This constrains all octave levels to $j \geq 2$.

As in the case with all semiorthogonal wavelets, the second-order B-spline also serves as scaling functions. The second-order B-spline scaling functions are given by [7, 8]

$$\varphi_{j,k}(x) = \begin{cases} x_j - k, & k \leq x_j \leq k+1 \\ 2-(x_j-k), & k+1 \leq x_j \leq k+2, \\ 0, & otherwise \end{cases} \tag{3}$$

$$for \;\; k = 0, \ldots, 2^j - 2$$

with the respective left and right side boundary scaling functions are

$$\varphi_{j,k}(x) = \begin{cases} 2-(x_j-k), & 0 \leq x_j \leq 1 \\ 0, & otherwise \end{cases} \tag{4}$$

$$for \;\; k = -1$$

$$\varphi_{j,k}(x) = \begin{cases} x_j - k, & k \leq x_j \leq k+1 \\ 0, & otherwise \end{cases} \tag{5}$$

$$for \;\; k = 2^j - 1$$

The actual coordinate position x is related to $x_j = 2^j x$.

The second order B-spline wavelets are given by [7, 8]

$$\psi_{j,k}(x) = \frac{1}{6}\begin{cases} x_j - k & k \leq x_j \leq k+\frac{1}{2} \\ 4-7(x_j-k) & k+\frac{1}{2} \leq x_j \leq k+1 \\ -19+16(x_j-k) & k+1 \leq x_j \leq k+\frac{3}{2} \\ 29-16(x_j-k) & k+\frac{3}{2} \leq x_j \leq k+2 \\ -17+7(x_j-k) & k+2 \leq x_j \leq k+\frac{5}{2} \\ 3-(x_j-k) & k+\frac{5}{2} \leq x_j \leq k+3 \\ 0 & otherwise, \end{cases} \tag{6}$$

$$for \;\; k = 0, \ldots, 2^j - 3$$

with the respective left and right hand side boundary wavelets are

$$\psi_{j,k}(x) = \frac{1}{6}\begin{cases} -6+23x_j & 0 \leq x_j \leq \frac{1}{2} \\ 14-17x_j & \frac{1}{2} \leq x_j \leq 1 \\ -10+7x_j & 1 \leq x_j \leq \frac{3}{2} \\ 2-x_j & \frac{3}{2} \leq x_j \leq 2 \\ 0 & otherwise \end{cases} \tag{7}$$

$$for \;\; k = -1$$

$$\psi_{j,k}(x) = \frac{1}{6}\begin{cases} 2-(k+2-x_j) & k \leq x_j \leq k+\frac{1}{2} \\ -10+7(k+2-x_j) & k+\frac{1}{2} \leq x_j \leq k+1 \\ 14-17(k+2-x_j) & k+1 \leq x_j \leq k+\frac{3}{2} \\ -6+23(k+2-x_j) & k+\frac{3}{2} \leq x_j \leq k+2 \\ 0 & otherwise \end{cases} \tag{8}$$

$$for \;\; k = 2^j - 2$$

Some of the important properties relevant to the present work are given in [9, 10] as:

1. *Vanishing moment*: A wavelet $\psi(x)$ is said to be have a vanishing moment of order m if

$$\int_{-\infty}^{\infty} x^p \psi(x)dx = 0; \; p = 0, 1, \ldots, m-1.$$

All wavelets must satisfy the above condition for $p = 0$. Linear B-spline wavelet has 2 vanishing moments. That is

$$\int_{-\infty}^{\infty} x^p \psi_4(x)dx = 0, \quad p = 0, 1.$$

For a good approximation and data compression, vanishing moments property is necessary condition.

2. *Semiorthogonality*: The wavelets $\psi_{j,k}$ form a semiorthogonal basis if

$$\langle \psi_{j,k}, \psi_{s,i} \rangle = 0; \; j \neq s; \; \forall j,k,s,i \in Z.$$

Linear B-spline wavelets are semiorthogonal.

3 Function approximation

A function $f(x)$ defined over interval [0, 1] may be approximated by B-spline wavelets as [2]

$$f(x) = \sum_{k=-1}^{2^{j_0}-1} c_{j_0,k}\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_{k=-1}^{2^j-2} d_{j,k}\psi_{j,k}(x). \tag{9}$$

In particular, for $j_0 = 2$, if the infinite series in equation (9) is truncated at M , then eq. (9) can be written as [7,8]

$$f(x) \approx \sum_{k=-1}^{3} c_k \varphi_{2,k}(x) + \sum_{j=2}^{M} \sum_{k=-1}^{2^j-2} d_{j,k} \psi_{j,k}(x) = C^T \Psi(x). \quad (10)$$

where $\varphi_{2,k}$ and $\psi_{j,k}$ are scaling and wavelet functions, respectively, here C and Ψ are $(2^{M+1}+1)\times 1$ vectors given by

$$C = [c_{-1}, c_0, \ldots, c_3, d_{2,-1}, \ldots, d_{2,2}, \ldots, d_{M,-1}, \ldots, d_{M,2^M-2}]^T, \quad (11)$$

$$\Psi = [\varphi_{2,-1}, \ldots, \varphi_{2,3}, \psi_{2,-1}, \ldots, \psi_{2,2}, \ldots, \psi_{M,-1}, \ldots, \psi_{M,2^M-2}]^T, \quad (12)$$

with

$$c_k = \int_0^1 f(x) \tilde{\varphi}_{2,k}(x) dx, \;\; k = -1, 0, \ldots, 3,$$

$$d_{j,k} = \int_0^1 f(x) \tilde{\psi}_{j,k}(x) dx, \;\; j = 2, \ldots, M, \, k = -1, \ldots, 2^j - 2, \quad (13)$$

where $\tilde{\varphi}_{2,k}(x)$ and $\tilde{\psi}_{j,k}(x)$ are dual functions of $\varphi_{2,k}$ and $\psi_{j,k}$, respectively. These can be obtained by linear combinations of $\varphi_{2,k}$, $k = -1, \ldots, 3$ and $\psi_{j,k}$, $j = 2, \ldots, M$, $k = -1, \ldots, 2^j - 2$, as follows. Let

$$\Phi = [\varphi_{2,-1}(x), \varphi_{2,0}(x), \varphi_{2,1}(x), \varphi_{2,2}(x), \varphi_{2,3}(x)]^T, \quad (14)$$

$$\bar{\Psi} = [\psi_{2,-1}(x), \psi_{2,0}(x), \ldots, \psi_{M,2^M-2}(x)]^T. \quad (15)$$

Using eq. (3-5) and eq. (14), we get

$$\int_0^1 \Phi \Phi^T dx = P_1 = \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 & 0 \\ 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 \\ 0 & 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{24} & \frac{1}{12} \end{bmatrix}, \quad (16)$$

and from eq. (6-8) and eq. (15), we have

$$\int_0^1 \bar{\Psi} \bar{\Psi}^T dx = P_2 = \begin{bmatrix} N_{4\times 4} & & & & \\ & \frac{1}{2} N_{8\times 8} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \frac{1}{2^{M-2}} N_{2^M \times 2^M} \end{bmatrix}, \quad (17)$$

where P_1 and P_2 are 5×5 and $(2^{M+1}-4)\times(2^{M+1}-4)$ matrices, respectively, and N is a five diagonal matrix given by

$$N = \begin{bmatrix} \frac{2}{27} & \frac{1}{96} & -\frac{1}{864} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \frac{1}{96} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} & 0 & \cdot & \cdot & \cdot & 0 \\ -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{1}{96} & -\frac{1}{864} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} \\ 0 & \cdot & \cdot & \cdot & 0 & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{1}{96} \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & -\frac{1}{864} & \frac{1}{96} & \frac{2}{27} \end{bmatrix}. \quad (18)$$

Suppose $\tilde{\Phi}$ and $\tilde{\bar{\Psi}}$ are the dual functions of Φ and $\bar{\Psi}$, respectively, given by

$$\tilde{\Phi} = [\tilde{\phi}_{2,-1}(x), \tilde{\phi}_{2,0}(x), \tilde{\phi}_{2,1}(x), \tilde{\phi}_{2,2}(x), \tilde{\phi}_{2,3}(x)]^T, \quad (19)$$

$$\tilde{\bar{\Psi}} = [\tilde{\psi}_{2,-1}(x), \tilde{\psi}_{2,0}(x), \ldots, \tilde{\psi}_{M,2^M-2}(x)]^T. \quad (20)$$

Using eqs. (14)-(15) and (19)-(20), we have

$$\int_0^1 \tilde{\Phi} \Phi^T dx = I_1, \quad \int_0^1 \tilde{\bar{\Psi}} \bar{\Psi}^T dx = I_2, \quad (21)$$

where I_1 and I_2 are 5×5 and $(2^{M+1}-4)\times(2^{M+1}-4)$ identity matrices, respectively. Then eqs. (16), (17) and (21) yield

$$\tilde{\Phi} = P_1^{-1} \Phi, \quad \tilde{\bar{\Psi}} = P_2^{-1} \bar{\Psi}. \quad (22)$$

4 Fredholm integral equations of second kind

In this section, linear Fredholm integral equation of the second kind of the form (1) has been solved by using B-spline wavelets. For this, we use eq. (10) to approximate $y(x)$ as

$$y(x) = C^T \Psi(x), \quad (23)$$

where $\Psi(x)$ is defined in eq. (12), and C is $(2^{M+1}+1)\times 1$ unknown vector defined similarly as in eq. (11). We also expand $y(x)$ and $K(x,t)$ by B-spline dual wavelets $\tilde{\Psi}$ defined as in eqs. (19-20) as

$$f(x) = C_1^T \tilde{\Psi}(x), \quad K(x,t) = \tilde{\Psi}^T(t) \Theta \tilde{\Psi}(x), \quad (24)$$

where

$$\Theta_{i,j} = \int_0^1 \left[\int_0^1 K(x,t) \Psi_i(t) dt \right] \Psi_j(x) dx. \quad (25)$$

From eqs. (24) and (23), we get

$$\int_0^1 K(x,t) y(t) dt = \int_0^1 C^T \Psi(t) \tilde{\Psi}^T(t) \Theta \tilde{\Psi}(x) dt$$

$$= C^T \Theta \tilde{\Psi}(x) \quad (26)$$

since

$$\int_0^1 \Psi(t)\tilde{\Psi}^T(t)dt = I.$$

By applying eqs. (23)-(26) in eq. (1). we have

$$C^T\Psi(x) - C^T\Theta\tilde{\Psi}(x) = C_1^T\tilde{\Psi}(x). \tag{27}$$

By multiplying both sides of the eq. (27) with $\Psi^T(x)$ from the right and integrating with respect to x from 0 to 1, we get

$$C^TP - C^T\Theta = C_1^T, \tag{28}$$

since

$$\int_0^1 \tilde{\Psi}(x)\Psi^T(x)dx = I,$$

and P is a $(2^{M+1}+1)\times(2^{M+1}+1)$ square matrix given by

$$P = \int_0^1 \Psi(x)\Psi^T(x)dx = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{29}$$

Consequently, from equation (28), we get $C^T = C_1^T(P-\Theta)^{-1}$. Hence we can calculate the solution for $y(x) = C^T\Psi(x)$.

5 Illustrative examples

Example 1

Consider the equation

$$y(x) = \cos x + \frac{3}{2}x\sin x + \int_0^1 K(x,t)y(t)dt, \quad 0 \le x \le 1,$$

where

$$K(x,t) = \begin{cases} -3\sin(x-t), & 0 \le t \le x \\ 0, & x \le t \le 1 \end{cases}$$

The solution for $y(x)$ is obtained by the method explained in section 4. The numerical approximate results for $M = 2, M = 4$ together with their exact solutions $y(x) = \cos x$ and absolute errors are cited in Tables 1 and 2 respectively.

The error function is given by

Error function$= \left\| y_{exact}(x_i) - y_{approximate}(x_i) \right\|$

$$= \sqrt{\sum_{i=1}^{n}(y_{exact}(x_i) - y_{approximate}(x_i))^2}$$

Global error estmate=R.M.S.error

$$= \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}(y_{exact}(x_i) - y_{approximate}(x_i))^2}$$

Table 1: Approximate solutions for $M = 2$

x	$y_{approximate}$	y_{exact}	*Absoluteerror*
0	1.001300	1.000000	1.30173E-3
0.1	0.995052	0.995004	4.75992E-5
0.2	0.979500	0.980067	5.66575E-4
0.3	0.954792	0.955336	5.44546E-4
0.4	0.921120	0.921061	5.94170E-5
0.5	0.878726	0.877583	1.14300E-3
0.6	0.825360	0.825336	2.45777E-5
0.7	0.764394	0.764842	4.47947E-4
0.8	0.696316	0.696707	3.90444E-4
0.9	0.621667	0.621610	5.68924E-5
1	0.541039	0.540302	7.36347E-4

Table 2: Approximate solutions for $M = 4$

x	$y_{approximate}$	y_{exact}	*Absoluteerror*
0	1.000080	1.000000	8.13789E-5
0.1	0.995007	0.995004	3.28342E-6
0.2	0.980032	0.980067	3.50527E-5
0.3	0.955302	0.955336	3.42873E-5
0.4	0.921064	0.921061	2.80525E-6
0.5	0.877654	0.877583	7.14185E-5
0.6	0.825339	0.825336	2.96120E-6
0.7	0.764815	0.764842	2.72328E-5
0.8	0.696682	0.696707	2.51241E-5
0.9	0.621612	0.621610	1.63566E-6
1	0.540347	0.540302	4.44686E-5

In example 1, *Error estimates (or R.M.S. errors)* are 0.00064165 and 0.0000398951 for $M = 2$ and $M = 4$ respectively.

Example 2

Consider the equation

$$y(x) = x + \int_0^1 (xt^2 + x^2t)y(t)dt, \quad 0 \le x \le 1$$

The solution for $y(x)$ is obtained by the method explained in section 4. The numerical approximate results for $M = 2, M = 4$ together with their exact solutions $y(x) = \frac{180x+80x^2}{119}$ and absolute errors are cited in Tables 3 and 4 respectively.

In example 2, *Error estimates (or R.M.S. errors)* are 0.0010266 and 0.0000641496 for $M = 2$ and $M = 4$ respectively.

Table 3: Approximate solutions for $M = 2$

x	$y_{approximate}$	y_{exact}	$Absoluteerror$
0	-0.001751	0.000000	1.75070E-3
0.1	0.157913	0.157983	7.01720E-5
0.2	0.330182	0.329412	7.70007E-4
0.3	0.515056	0.514286	7.69838E-4
0.4	0.712534	0.712605	7.06777E-5
0.5	0.922618	0.924370	1.75154E-3
0.6	1.149510	1.149580	7.10762E-5
0.7	1.389000	1.388240	7.69042E-4
0.8	1.641100	1.640340	7.68812E-4
0.9	1.905810	1.905880	7.17658E-5
1	2.183120	2.184870	1.75269E-3

Table 4: Approximate solutions for $M = 4$

x	$y_{approximate}$	y_{exact}	$Absoluteerror$
0	-0.000109	0.000000	1.09419E-4
0.1	0.157979	0.157983	4.37731E-6
0.2	0.329460	0.329412	4.81431E-5
0.3	0.514334	0.514286	4.81424E-5
0.4	0.712601	0.712605	4.37929E-6
0.5	0.924260	0.924370	1.09422E-4
0.6	1.149580	1.149580	4.38085E-6
0.7	1.388280	1.388240	4.81393E-5
0.8	1.640380	1.640340	4.81384E-5
0.9	1.905880	1.905880	4.38354E-6
1	2.184760	2.184870	1.09427E-4

Table 5: Approximate solutions for $M = 2$

x	$y_{approximate}$	y_{exact}	$Absoluteerror$
0	0.001187	0.000000	0.001187
0.1	0.310083	0.309017	0.001065
0.2	0.584716	0.587785	0.003069
0.3	0.804107	0.809017	0.004910
0.4	0.951212	0.951057	0.000155
0.5	1.012920	1.000000	0.012924
0.6	0.951212	0.951057	0.000155
0.7	0.804107	0.809017	0.004910
0.8	0.584716	0.587785	0.003069
0.9	0.310083	0.309017	0.001065
1	0.001187	0.000000	0.001187

Table 6: Approximate solutions for $M = 4$

x	$y_{approximate}$	y_{exact}	$Absoluteerror$
0	1.823150E-5	0.000000	1.82315E-5
0.1	0.309012	0.309017	4.75611E-6
0.2	0.587571	0.587785	2.14100E-4
0.3	0.808735	0.809017	2.81802E-4
0.4	0.951092	0.951057	3.51844E-5
0.5	1.000800	1.000000	8.03434E-4
0.6	0.951092	0.951057	3.51844E-5
0.7	0.808735	0.809017	2.81802E-4
0.8	0.587571	0.587785	2.14100E-4
0.9	0.309012	0.309017	4.75611E-6
1	1.823150E-5	0.000000	1.82315E-5

Example 3

Consider the equation

$$y(x) = \left(1 - \frac{1}{\pi^2}\right)\sin(\pi x) + \int_0^1 K(x,t)y(t)dt, \quad 0 \leq x \leq 1,$$

where

$$K(x,t) = \begin{cases} x(1-t), & x \leq t \\ t(1-x), & t \leq x \end{cases}$$

The solution for $y(x)$ is obtained by the method explained in section 4. The numerical approximate results for $M = 2, M = 4$ together with their exact solutions $y(x) = \sin(\pi x)$ and absolute errors are cited in Tables 5 and 6 respectively.

In example 3, *Error estimates (or R.M.S. errors)* are 0.00466338 and 0.000285911 for $M = 2$ and $M = 4$ respectively.

6 Conclusion

In the present paper, linear Fredholm integral equations of second kind have been solved by using second order B-spline wavelets. The method is based upon compactly supported linear semiorthogonal B-spline wavelets. The dual wavelets for these B-spline wavelets have been also presented. Because of semiorthogonality, compact support and vanishing moments properties of B-spline wavelets, the matrices are very sparse. The illustrative examples have been included to demonstrate the validity and applicability of the technique. These examples show the accuracy and efficiency of the described method.

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