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Generating BCOPs* to Compute Natural Frequencies of an Elliptical Plate With Half of the Boundary Simply Supported and the Rest Free

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Boundary characteristic orthogonal polynomials in two variables (BCOPs) have been built up over an elliptical domain occupied by a thin elastic plate. Half of the plate boundary, $y \leq 0$, is taken simply supported while the other half is kept free. The orthogonal polynomial sequence (OPS) is generated from a set of linearly independent functions satisfying the essential boundary conditions of the problem and tabulated in advance, once and for all, for various values of the aspect ratio $r = b/a$ of the plate with the desired precision. Free flexural vibrations of the plate have been examined by using these polynomials in the Rayleigh-Ritz method. Successive approximations have been worked out to ensure convergence. Comparisons have been made with known results in literature. Three-dimensional mode shapes and the associated contour lines have been plotted in some selected cases.

Keywords: Orthogonal polynomials, vibration, elliptical plates, non-uniform boundary conditions.

NOMENCLATURE

BCOPs	boundary characteristic orthogonal polynomials;
OPS	orthogonal polynomial sequence
CF	half of the plate boundary $y \leq 0$ is clamped and the rest free;
PDEs	partial differential equations
a, b	semi major and minor axes of the elliptical domain;

*Boundary Characteristic Orthogonal Polynomials

r	aspect ratio b/a;
x, y	cartesian coordinates;
X, Y	non-dimensional coordinates;
R	domain occupied by the plate in xy-coordinates;
$W(x,\, y)$	displacement;
ρ	density of the material of the plate;
E	Young's modulus of the material of the plate;
ω	angular natural frequency;
ν	Poisson ratio;
λ	non-dimensional frequency parameter;
∇^2	Laplacian operator;
N	number of terms used in the approximation (approximation order);
c_j	the unknown coefficients used in the expansion;
S	set of N-linearly independent functions $F_i(x, y)$;
m_i, n_i	non-negative integers;
$\phi_i(x,\, y)$	orthogonal functions over R;
$\hat{\phi}_i(x,\, y)$	orthonormal functions over R;
f, g	functions of x and y;
$\langle f, g\rangle$	inner product of f and g;
$\|f\|$	norm of f;
$[a_{ij}]\,, [b_{ij}]$	$N \times N$ matrices.

1 Introduction

The topic of orthogonal polynomials has a very rich history going back to 19th century when mathematicians and physicists tried to solve the most important differential equations of mathematical physics. Since then orthogonal polynomials have developed to a standard subject within mathematics which is driven by applications. The applications are numerous both within mathematics (e.g. statistics, combinatorics, harmonic analysis, number theory) and other sciences such as physics, biology, computer science, and chemistry. Following the publication of Szego's well known treatise [1] there has been tremendous growth of literature covering various aspects of the subject. Orthogonal polynomial sequence (OPS) have been widely used by Singh and Chakraverty [2-6] to solve the vibration problems of plates of different shapes under a variety of uniform conditions on the boundary. Some important books on orthogonal polynomials are Suetin and Pankrattiey [7], Beckmenn [8], Chihara [9], and Gautschi et. al. [10].

The geometric configurations of certain design problems may force an engineer to use plates of different geometrical shapes but with non-uniform conditions on the boundary occasionally. That is why plates with non-uniform boundary conditions are key component in civil and mechanical engineering and industrial design. Analytical solutions for bending of rectangular plates with mixed boundary conditions have been examined by Boborykin [11]. In general, there are no analytical solutions to the vibration problems of plates with discontinuous boundary conditions even for plates of simple geometrical shapes like rectangles, Wei et al. [12].

As far as the vibration problem of an elliptical plate with non-uniform boundary conditions is concerned (Fig. 1), very little is available in literature and it is mostly on circular plates with uniform thickness. Some important references are Hirano and Okazaki [13], Sundararajan [14], Hemmig [15], Bartlett [16], Leissa et al. [17], and Narita and Leissa [18,19]. Recently S.M. Hassan [20] has solved the same problem by using the usual, traditional, polynomials in x and y that satisfies the essential boundary conditions of the plate as basis functions in the Rayleigh-Ritz procedure and gave much of explicit numerical results for the first time.

The present work deals with generating BCOPs in two variables so that at least the essential boundary conditions of the problem are satisfied. These will be used in the Rayleigh-Ritz method to solve the vibration problem of the specified plate. The author has noticed that this procedure is easier, more suitable for use on digital computers and greatly simplifies the resulting eigenvalue problem. The most significant advantage of BCOPs lies in computing the OPS and tabulating them in advance once and for all and no need to calculate them again and again. That is why this approach has become popular and caught the attention of many people working in that field.

2 Building up the boundary characteristic orthogonal polynomials

Following the same approach used by Bhat [21] and Liew et al. [22] for rectangular plates, one starts with a suitable set of linearly independent functions

$$S = \{F_i(x,\, y) = u f_i(x,\, y), \qquad i = 1,\ 2,\ \cdots,\ N\}, \tag{2.1}$$

which vanishes on the lower half $(y \leq 0)$ of the plate boundary. To obtain an orthogonal set we define the inner product of two functions f and g by

$$\langle f, g\rangle = \iint\limits_R f(x,y)\; g(x,y)\, dx\, dy, \tag{2.2}$$

and the norm of a function f by

$$\|f\| = \sqrt{\langle f,\ f\rangle} = \left[\iint\limits_R f^2(x,y)\, dx\, dy\right]^{\frac{1}{2}}, \tag{2.3}$$

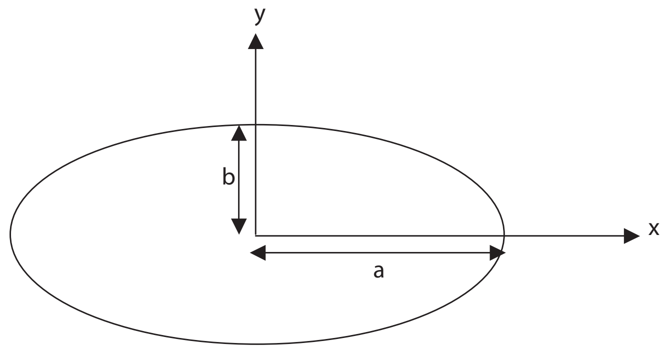

Figure 1.1: The elliptical domain R occupied by the plate.

where R is the domain occupied by the plate in the $xy-$plane. Then, the orthogonal functions $\phi_{\text{Im}}(x, y)$ are generated by using the famous Gram-Schmidt orthogonalization process, the algorithm for which may be summarized as follows:

$$\left.\begin{array}{lcl} \phi_1 &=& F_1, \\ \phi_i &=& F_i - \sum\limits_{j=1}^{i-1} \alpha_{ij}\,\phi_j, \\ \text{where} && \\ \alpha_{ij} &=& \langle F_i,\,\phi_j\rangle / \langle \phi_j,\,\phi_j\rangle, \quad j = 1,\ 2,\ \cdots,\ i-1 \end{array}\right\}, \qquad i = 2,\ 3,\ \cdots,\ N. \tag{2.4}$$

The functions ϕ_i can be normalized by using

$$\hat{\phi}_i = \phi_i / \|\phi_i\| = \phi_i / \langle \phi_i, \phi_i \rangle^{\frac{1}{2}}. \tag{2.5}$$

Computations of α_{ij} are greatly simplified if u and f_i are taken as simple polynomials in x and y. The functions ϕ_i and $\hat{\phi}_i$ can be expressed in terms of $f_1, f_2, \cdots$ if desired [6]. Thus coefficients β_{ij} and $\hat{\beta}_{ij}$ can be found such that:

$$\phi_i = u \sum_{j=1}^{i} \beta_{ij}\, f_j, \qquad \hat{\phi}_i = u \sum_{j=1}^{i} \hat{\beta}_{ij}\, f_j, \qquad i = 1,\ 2,\ \cdots,\ N \tag{2.6}$$

3 Rayleigh-Ritz method

The Rayleigh-Ritz method consists of minimizing the Rayleigh quotient

$$\omega^2 = \frac{D \iint\limits_R \left[\left(\nabla^2 W\right)^2 + 2(1-\nu)\{W_{xy}^2 - W_{xx}\, W_{yy}\} \right] dx\, dy}{\rho h \iint\limits_R W^2 \, dx\, dy}, \tag{3.1}$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, E is Young's modulus, ρ is mass density, ν is the Poisson ratio of the material of the plate, h is the thickness of the plate which has been taken to be unity all over the plate in this problem, ω is the natural radian frequency, and ∇^2 is the Laplacian operator. Substituting the $N-$term approximation of the deflection function

$$W(x,y) = \sum_{j=1}^{N} c_j \; \phi_j(x,\; y). \tag{3.2}$$

into equation (3.1) and minimizing ω^2 as a function of the coefficients $c_1,\; c_2,\; \cdots,\; c_N$ yields

$$\sum_{j=1}^{N} \left(a_{ij} - \lambda^2\, b_{ij}\right)\, c_j = 0, \qquad ,i = 1,\; 2, \cdots,\; N, \tag{3.3}$$

where

$$a_{ij} = \iint\limits_R \left[\phi_i^{xx}\phi_j^{xx} + \phi_i^{yy}\phi_j^{yy} + \nu\left(\phi_i^{xx}\phi_j^{yy} + \phi_i^{yy}\phi_j^{xx}\right) + 2(1-\nu)\phi_i^{xy}\phi_j^{xy}\right] dx\, dy, \tag{3.4}$$

$$b_{ij} = \iint\limits_R \phi_i\, \phi_j \; dx\, dy, \tag{3.5}$$

$$\lambda^2 = a^4 \omega^2 \rho h / D. \tag{3.6}$$

In order to express the $N \times N$ matrices $[a_{ij}]$ and $[b_{ij}]$ in closed form, the following result is found to be very useful.

$$\int_0^1\!\!\int_0^z x^i y^j z^k \;\, dx\; dy = \frac{2r^{j+1}\Gamma\left[(i+1)/2\right]\Gamma\left[(j+k+3)/2\right]}{(j+1)\Gamma\left[(i+j+k)/2+2\right]}, \tag{3.7}$$

where Γ is the Gamma function, i and j are non-negative even integers, $(j+k+3)$ should be positive and $z = \sqrt{1-x^2}$. If i or j is odd the integral vanishes. Solving the generalized eigenvalue problem equation (3.3) for λ and c_j one gets the frequencies and mode shapes.

4 Numerical work and discussion

To generate BCOPs one can start with the linearly independent set $\{u\, f_i\}_{i=1}^N$ where

$$u = (y + r\, z)\, z^2 \quad \text{and} \quad f_i = x^{m_i}\, y^{n_i}, \qquad i = 1,\; 2, \cdots,\; N \tag{4.1}$$

here m_i and n_i are non-negative integers. The factor $z^2 = 1 - x^2$ in equation (4.1) has been introduced to overcome difficulties arising in some integrals. Hence the functions $u\, f_i$ also satisfy the same boundary conditions of the problem. One of the most obvious choices of f_i will be

$$\{f_i, \quad i = 1,\ 2,\ \cdots\} = \{1, x, y, x^2, x\,y, y^2, x^3, x^2y, xy^2, y^3,\ \cdots\}. \tag{4.2}$$

Finally BCOPs can be expressed in terms of f_i by computing β_{ij} in the expressions

$$\phi_i = u \sum_{j=1}^{i} \beta_{ij}\, f_j, \qquad \hat{\phi}_i = \sum_{j=1}^{i} \hat{\beta}_{ij}\, f_j. \tag{4.3}$$

The coefficients $\hat{\beta}_{ij}$ of the first fifteen polynomials are reported in Tables 1 up to 4. These correspond to $r = 0.5,\ 1.0,\ 1.5$ and 2.0, respectively. The tabulated results have been computed once for all and the reader can use them directly without repeating the calculations again and again. These calculations were done in double precision arithmetic but the results have been reported to six significant figures only. As a check on the accuracy of the results it has been verified that

$$\langle \hat{\phi}_i, \hat{\phi}_j \rangle \quad = \quad \begin{cases} 10^{-15} & \text{for} \quad i \neq j\,, \\ 1.0 & \text{for} \quad i = j\,. \end{cases} \tag{4.4}$$

These results are of great importance in solving the vibration problems of plates, in general, through using the BCOPs as basis functions in the approximation solution used in Rayleigh-Ritz method. Following these procedures and using the non-dimensional variables

$$X = x/a, \quad \text{and} \quad Y = y/a, \tag{4.5}$$

the first six frequencies of the plate vibration have been computed and reported in Table 5 for all values of the aspect ratio r. The program can generate such polynomials and calculate frequencies for any given value of $r > 0$. The approximation order N has been increased from 1 up to 15 to ensure convergence of results. All the computations have been worked out for $\nu = 0.3$. Comparison of these results with those in [20] shows a complete agreement with them. No more results are available up to now for making other comparisons. The trend of convergence of the first six frequencies computed by using BCOPs for $r = 0.5$ can be depicted from Table 6. All the results reported have been converged to at least four significant figures.

5 Mode shapes

Figures $5.2(a)$ to (f) depict the first six mode shapes and the associated contour lines for a plate of uniform thickness. Half of the plate boundary ($y \leq 0$) is taken clamped while the upper half is kept free. The figures have been plotted for some selected values of the plate parameters namely $r = 0.5$ and $\nu = 0.3$. Figures corresponding to $\nu = 0.33$ are roughly the same. Other more figures corresponding to different aspect ratios are available in [20]. Tools of Computer Graphics under Turbo C++ have been used to produce my own software for that purpose.

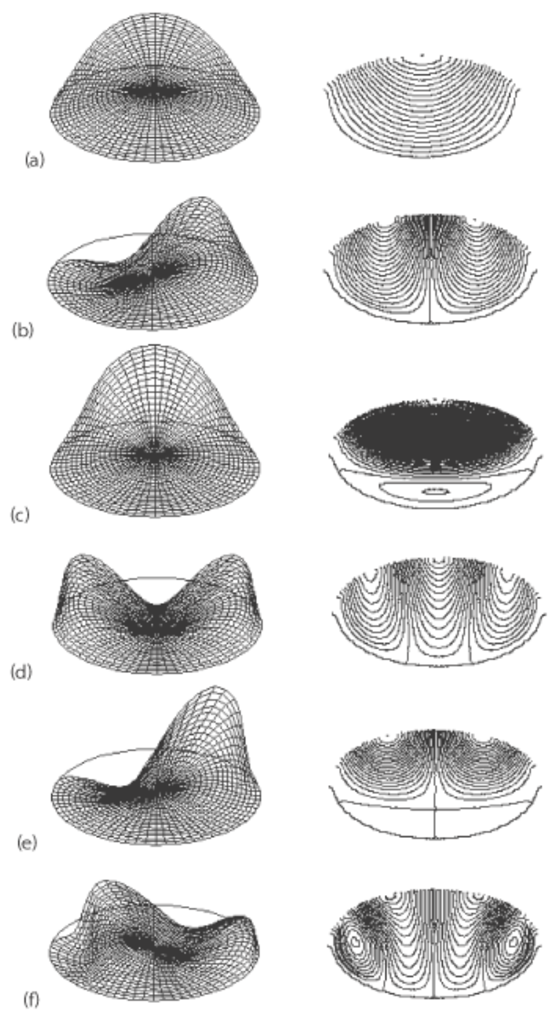

Figure 5.1: First six mode shapes and the associated contour lines for $r = 0.5$ and $\nu = 0.3$.

6 Conclusion

The numerical results presented here are of great importance in solving PDEs via using BCOPs as basis functions in the approximation solution. Interested readers can use this directly without repeating the calculations again and again for similar problems. Those polynomials are not only simplifying the problem but also minimizing the effects of ill-conditioning which frequently occurs in such problems.

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j	$i \rightarrow 1$	2	3	4	5	6	7	8	j
1	$0.186877E+01$	0.0	$-0.239495E+01$	$-0.176166E+01$	0.0	$-0.943525E+00$	0.0	$0.165881E+01$	1
		$0.590958E+01$	0.0	0.0	$-0.749146E+01$	0.0	$-0.102574E+02$	0.0	2
15	$0.884178E+03$		$0.101259E+02$	$0.967631E+00$	0.0	$-0.149536E+02$	0.0	$-0.936833E+01$	3
14	0.0	$0.820852E+03$		$0.153280E+02$	0.0	$0.255115E+01$	0.0	$-0.188997E+02$	4
13	$0.150448E+03$	0.0	$0.510042E+03$		$0.348416E+02$	0.0	$0.524837E+01$	0.0	5
12	0.0	$0.917652E+02$	0.0	$0.238933E+03$		$0.469465E+02$	0.0	$0.426781E+01$	6
11	$0.270037E+01$	0.0	$0.279475E+02$	0.0	$0.830202E+02$		$0.365155E+02$	0.0	7
10	$-0.342073E+03$	0.0	$0.171570E+02$	0.0	$-0.716275E+00$	$0.206375E+03$		$0.959387E+02$	8
9	0.0	$-0.276417E+03$	0.0	$0.254160E+02$	0.0	0.0	$0.174520E+03$		9
8	$-0.327320E+02$	0.0	$-0.138474E+03$	0.0	$0.186765E+02$	$0.229814E+02$	0.0		8
7	0.0	$-0.120603E+02$	0.0	$-0.434167E+02$	0.0	0.0	$0.953194E+01$		7
6	$-0.925591E+02$	0.0	$-0.426743E+02$	0.0	$0.599932E+00$	$-0.744698E+02$	0.0		6
5	0.0	$-0.535548E+02$	0.0	$-0.587826E+02$	0.0	0.0	$-0.513060E+02$		5
4	$-0.338835E+01$	0.0	$-0.171735E+02$	0.0	$-0.392802E+02$	$-0.323124E+01$	0.0		4
3	$0.288088E+02$	0.0	$0.982735E+01$	0.0	$-0.156032E+01$	$-0.119749E+02$	0.0		3
2	0.0	$0.103433E+02$	0.0	$0.973151E+01$	0.0	0.0	$-0.444045E+01$		2
1	$0.672687E+00$	0.0	$0.113374E+01$	0.0	$0.178909E+01$	$0.270746E+01$	0.0		1
j	15	14	13	12	11	10	9	$\leftarrow i$	j

Table 5.1: Coefficients $\hat{\beta}_{ij}$ of 15-orthogonal polynomials $\hat{\phi}_i$ for $r = 0.5$, $N = 15$, $\nu = 0.3$

j	$i \to 1$	2	3	4	5	6	7	8	j
1	$0.660711E+00$	0.0	$-0.846742E+00$	$-0.622840E+00$	0.0	$-0.333587E+00$	0.0	$0.586477E+00$	1
		$0.208935E+01$	0.0	0.0	$-0.264863E+01$	0.0	$-0.362652E+01$	0.0	2
15	$0.195378E+02$		$0.179003E+01$	$0.171055E+00$	0.0	$-0.264345E+01$	0.0	$-0.165610E+01$	3
14	0.0	$0.362769E+02$		$0.541926E+01$	0.0	$0.901968E+00$	0.0	$-0.668206E+01$	4
13	$0.132978E+02$	0.0	$0.450818E+02$		$0.615918E+01$	0.0	$0.927790E+00$	0.0	5
12	0.0	$0.162220E+02$	0.0	$0.422379E+02$		$0.414953E+01$	0.0	$0.377224E+00$	6
11	$0.954725E+00$	0.0	$0.988093E+01$	0.0	$0.293521E+02$		$0.129102E+02$	0.0	7
10	$-0.151176E+02$	0.0	$0.758238E+00$	0.0	$-0.316552E-01$	$0.912059E+01$		$0.169597E+02$	8
9	0.0	$-0.244320E+02$	0.0	$0.224648E+01$	0.0	0.0	$0.154256E+02$		9
8	$-0.578626E+01$	0.0	$-0.244789E+02$	0.0	$0.330156E+01$	$0.406258E+01$	0.0		8
7	0.0	$-0.426395E+01$	0.0	$-0.153501E+02$	0.0	0.0	$0.337005E+01$		7
6	$-0.818115E+01$	0.0	$-0.377191E+01$	0.0	$0.530270E-01$	$-0.658226E+01$	0.0		6
5	0.0	$-0.946724E+01$	0.0	$-0.103914E+02$	0.0	0.0	$-0.906970E+01$		5
4	$-0.119796E+01$	0.0	$-0.607174E+01$	0.0	$-0.138876E+02$	$-0.114241E+01$	0.0		4
3	$0.509273E+01$	0.0	$0.173725E+01$	0.0	$-0.275828E+00$	$-0.211688E+01$	0.0		3
2	0.0	$0.365689E+01$	0.0	$0.344061E+01$	0.0	0.0	$-0.156994E+01$		2
1	$0.237831E+00$	0.0	$0.400838E+00$	0.0	$0.632539E+00$	$0.957232E+00$	0.0		1
j	15	14	13	12	11	10	9	$\leftarrow i$	j

Table 5.2: Coefficients $\hat{\beta}_{ij}$ of 15-orthogonal polynomials $\hat{\phi}_i$ for $r = 1.0$, $N = 15$, $\nu = 0.3$

j	$i \to 1$	2	3	4	5	6	7	8	j
1	$0.359646E+00$	0.0	$-0.460908E+00$	$-0.339031E+00$	0.0	$-0.181582E+00$	0.0	$0.319238E+00$	1
		$0.113730E+01$	0.0	0.0	$-0.144173E+01$	0.0	$-0.197403E+01$	0.0	2
15	$0.210074E+01$		$0.649578E+00$	$0.620736E-01$	0.0	$-0.959275E+00$	0.0	$-0.600979E+00$	3
14	0.0	$0.585085E+01$		$0.294987E+01$	0.0	$0.490969E+00$	0.0	$-0.363725E+01$	4
13	$0.321707E+01$	0.0	$0.109064E+02$		$0.223509E+01$	0.0	$0.336683E+00$	0.0	5
12	0.0	$0.588674E+01$	0.0	$0.153276E+02$		$0.100387E+01$	0.0	$0.912600E-01$	6
11	$0.519687E+00$	0.0	$0.537850E+01$	0.0	$0.159772E+02$		$0.720741E+01$	0.0	7
10	$-0.243822E+01$	0.0	$0.122291E+00$	0.0	$-0.510545E-02$	$0.147100E+01$		$0.615447E+01$	8
9	0.0	$-0.591071E+01$	0.0	$0.543480E+00$	0.0	0.0	$0.373183E+01$		9
8	$-0.209976E+01$	0.0	$-0.888308E+01$	0.0	$0.119810E+01$	$0.147426E+01$	0.0		8
7	0.0	$-0.232100E+01$	0.0	$-0.835556E+01$	0.0	0.0	$0.183442E+01$		7
6	$-0.197922E+01$	0.0	$-0.912520E+00$	0.0	$0.128286E-01$	$-0.159241E+01$	0.0		6
5	0.0	$-0.343554E+01$	0.0	$-0.377090E+01$	0.0	0.0	$-0.329128E+01$		5
4	$-0.652088E+00$	0.0	$-0.33503E+01$	0.0	$-0.755948E+01$	$-0.621852E+00$	0.0		4
3	$0.184809E+01$	0.0	$0.630425E+00$	0.0	$-0.100095E+00$	$-0.768190E+00$	0.0		3
2	0.0	$0.199056E+01$	0.0	$0.187283E+01$	0.0	0.0	$-0.854566E+00$		2
1	$0.129459E+00$	0.0	$0.218189E+00$	0.0	$0.344311E+00$	$0.521051E+00$	0.0		1
j	15	14	13	12	11	10	9	$\leftarrow i$	j

Table 5.3: Coefficients $\hat{\beta}_{ij}$ of 15-orthogonal polynomials $\hat{\phi}_i$ for $r = 1.5$, $N = 15$, $\nu = 0.3$

j	$i \to 1$	2	3	4	5	6	7	8	j
1	$0.233597E+00$	0.0	$-0.299369E+00$	$-0.220207E+00$	0.0	$-0.117941E+00$	0.0	$0.207351E+00$	1
		$0.738698E+00$	0.0	0.0	$-0.936433E+00$	0.0	$-0.128217E+01$	0.0	2
15	$0.431728E+00$		$0.316435E+00$	$0.302385E-01$	0.0	$-0.467301E+00$	0.0	$-0.292760E+00$	3
14	0.0	$0.160323E+01$		$0.191600E+01$	0.0	$0.318894E+00$	0.0	$-0.236246E+01$	4
13	$0.117537E+01$	0.0	$0.398470E+01$		$0.108880E+01$	0.0	$0.164012E+00$	0.0	5
12	0.0	$0.286766E+01$	0.0	$0.746667E+01$		$0.366770E+00$	0.0	$0.333422E-01$	6
11	$0.337546E+00$	0.0	$0.349344E+01$	0.0	$0.103775E+02$		$0.456444E+01$	0.0	7
10	$-0.668111E+00$	0.0	$0.335097E-01$	0.0	$-0.139897E-02$	$0.403077E+00$		$0.299808E+01$	8
9	0.0	$-0.215951E+01$	0.0	$0.198563E+00$	0.0	0.0	$0.136344E+01$		9
8	$-0.102288E+01$	0.0	$-0.432730E+01$	0.0	$0.583639E+00$	$0.718169E+00$	0.0		8
7	0.0	$-0.150573E+01$	0.0	$-0.542709E+01$	0.0	0.0	$0.119149E+01$		7
6	$-0.723118E+00$	0.0	$-0.333393E+00$	0.0	$0.468697E-02$	$-0.581795E+00$	0.0		6
5	0.0	$-0.167359E+01$	0.0	$-0.183696E+01$	0.0	0.0	$-0.160331E+01$		5
4	$-0.423544E+00$	0.0	$-0.214668E+01$	0.0	$-0.491002E+01$	$-0.403905E+00$	0.0		4
3	$0.900276E+00$	0.0	$0.307105E+00$	0.0	$-0.487600E-01$	$-0.374215E+00$	0.0		3
2	0.0	$0.129291E+01$	0.0	$0.121644E+01$	0.0	0.0	$-0.555057E+00$		2
1	$0.840859E-01$	0.0	$0.141718E+00$	0.0	$0.223636E+00$	$0.338433E+00$	0.0		1
j	15	14	13	12	11	10	9	$\leftarrow i$	j

Table 5.4: Coefficients $\hat{\beta}_{ij}$ of 15-orthogonal polynomials $\hat{\phi}_i$ for $r = 2.0$, $N = 15$, $\nu = 0.3$

R	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0.5	4.4564	13.3483	20.2486	26.9533	34.1236	47.5870
[19]	4.4564	13.3483	20.2486	26.9533	34.1236	47.5870
1.0	3.3232	7.9217	10.9992	16.7589	18.9810	24.2116
[19]	3.3232	7.9217	10.9992	16.7589	18.9810	24.2116
1.5	2.7306	6.1901	9.1492	10.0641	16.4892	20.4614
[19]	2.7306	6.1901	9.1492	10.0641	16.4892	20.4614
2.0	2.4295	5.5915	6.7208	9.5892	13.9436	15.4105
[19]	2.4295	5.5915	6.7208	9.5892	13.9436	15.4105

Table 5.5: First six frequencies of an elliptical plate with half of the boundary $(y \leq 0)$ simply supported and the rest free ($N = 15$, $\nu = 0.3$)

N	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
6	4.5628	13.6563	22.0900	28.5357	36.9029	78.8077
7	4.5628	13.5519	22.0900	28.5357	36.7069	50.6933
8	4.5217	13.5519	22.0800	27.7957	36.7069	50.6933
9	4.5217	13.5239	22.0800	27.7957	36.3558	49.9267
10	4.5215	13.5239	20.3715	27.7718	36.3558	49.9267
11	4.5121	13.5239	20.3697	26.9805	36.3558	49.9267
12	4.5121	13.3483	20.3697	26.9805	36.0561	47.7325
13	4.4565	13.3483	20.2591	26.9537	36.0561	47.7325
14	4.4565	13.3483	20.2591	26.9537	34.1236	47.5870
15	4.4564	13.3483	20.2486	26.9533	34.1236	47.5870

Table 5.6: Convergence of first six frequencies ($r = 0.5$, $\nu = 0.3$)

References

[1] G. Szego, *Orthogonal Polynomials*, 4th ed., Providence, RI: Amer. Math. Soc., 1975.

[2] B. Singh and S. Chakraverty, Transverse vibration of completely-free elliptic and circular plates using orthogonal polynomials in the Rayleigh-Ritz method, *Int. J. Mech. Sci.*, **33**(1991), 741-751.

[3] B. Singh and S. Chakraverty, On the use of orthogonal polynomials in the Rayleigh-Ritz method for the study of transverse vibration of elliptic plates, *Computers and Structures*, **43**(1992), 439-443.

[4] B. Singh and S. Chakraverty, Transverse vibration of simply supported elliptic and circular plates using boundary characteristic orthogonal polynomials in two variables, *Journal of Sound and Vibration*, **152**(1992), 149-155.

[5] B. Singh and S. Chakraverty, Use of chracteristic orthogonal polynomials in two dimensions for transverse vibration of elliptic and circular plates with variable thickness, *Journal of Sound and Vibration*, **173**(1994), 289-299.

[6] B. Singh and S. Chakraverty, Boundary characteristic orthogonal polynomials in numerical approximation, *Communications in numerical methods in engineering*, **10**(1994), 1027-1043.

[7] P. K. Suetin and E. V. Pankrattiey, (Translator), *Orthogonal Polynomials in TwoVariables (Analytical Methods and Special Functions)*, CRC Press, 1999.

[8] P. Beckmann, *Orthogonal Polynomials for Engineers and Physists*, Golem Press 1973.

[9] T. S. Chihara, *An introduction to Orthogonal Polynomials*, Gordon and Breach, London, 1978.

[10] W. Gautschi, G.H. Golub and G. Opfer, (Eds.), *Applications and Computation of Orthogonal Polynomials*, Conference at the Mathematical Research Institute Oberwolfach, Germany, March 22-28 (1998). Basel, Switzerland: Birkhauser, 1999.

[11] V. G. Boborykin, Solving the elastic bending problem for a plate with mixed boundary conditions, *International Applied Mechanics*, **42**(2006), 582-588.

[12] G. W. Wei, Y.B. Zhao, and Y. Xiang, The determination of natural frequencies of rectangular plates with mixed boundary conditions by discrete singular convolution, *Int. J. Mech. Sci.*, **43**(2001), 1731-1746.

[13] Y. Hirano and K. Okazaki, Vibrations of a circular plate having partly clamped or partly simply supported boundary, *Bulletin of the JSME*, **19**(1976), 610-618.

[14] C. Sundararajan, An approximate solution for the fundamental frequency of plates. *Transactions of the ASME.*, **45**(1978), 936-938.

[15] F. G. Hemmig, *Investigation of natural frequencies of circular plates with mixed boundary conditions*, Air Force Institute of Technology, School of Engineering, Report No. GAE MC/75-11, 1975.

[16] C. C. Bartlet, The vibration and buckling of a circular plate clamped on one part of the boundary and simply-supported on the remainder, *Qurt. J. Mech. Appl. Math.*, **16**(1963), 431-440.

[17] A. W. Leissa, P. A. A. Laura and R. H. Gutierrez, Transverse vibrations of circular plates having non-uniform edge constraints, *J. Acoust. Soc. Am.*, **66**(1979), 180-184.

[18] Y. Narita and A. W. Leissa, Transverse vibration of simply supported circular plates having partial elastic constraints, *Journal of Sound and Vibration*, **70**(1980), 103-116.

[19] Y. Narita and A. W. Leissa, Flexural vibrations of free circular plates elastically constrained along parts of the edge, *Int. J. Solids and Structures*, **17**(1981), 83-92.

[20] S. M. Hassan, Numerical solution for frequencies and mode shapes of an elliptical plate with half of the boundary simply supported and the rest free, *Int. J. Mech. Sci.*, **46**(2004), 1747-1761.

[21] R. B. Bhat, Natural frequencies of rectangular plates using characteristing orthogonal polynomials in Rayleigh-Ritz method, *Journal of Sound and Vibration*, **102**(1985), 493-499.

[22] K. M Liew, K. Y. Lamb and S. T. Chow, Free vibration analysis of rectangular plates using orthogonal plate functions, *Computers and Structures*, **34**(1990), 79-85.