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On some Classical Properties of the Mixture of Burr XII and Lomax Distributions

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Abstract: The BurrXII and Lomax distributions are the most widely and important distributions used for life time purpose and for modeling the business failure time data. Burr XII distribution is mainly used to explain the allocation of wealth and wages among the people of the particular society. And Lomax is used to model the business failure time data. It is a transformed shape of Pareto distribution. In this research paper, the properties of the mixture of burr XII and Lomax distributions have been given. Classical properties such as Cumulative distribution function, failure rate, hazard rate, odd function, inverse hazard function and the cumulative hazard function, , r^{th} moment, , moments, mean and variance have been derived.

Keywords: Lomax distribution, burr XII, mixture distribution, mean and variance, CDF.

1 Introduction

Mixture distributions have been widely used by many statisticians and mathematicians for the purpose of the discussion, estimation of survey and their application in different fields of life. [2] Provided that the inverse weibull (IW) mixture models which have the negative weights can give the result of the system under special kind of circumstances. [9] Also proposed the argumentation about the characteristic of aging of the failure rate models which have one mode and it has also the inverse Weibull distribution. In the observed data of finance the utilization of the mixture normal distribution is well known. [3] Provided in his study that the mixture of the normal distribution to have space for the non-normality of the financial time series data. The research was carried out on the financial data of Bursa Malaysia stock market. The parameters were estimated by using the commonly used maximum likelihood method through EM algorithm. Different kind of studies has been carried out on the modeling of assets return by using the mixture of the normal. [6] Proposed in their research for first time the use of the mixture of the normal distribution for tackling the heavy tailed data. [8] Proved in his research that when all the segments have the same mean, the mixture of the normal distribution is leptokurtic. [4] for the first time proposed the significant property of the EM algorithm which is used nowadays for the mixture models for the purpose of estimation by using maximum likelihood method.

2 BURR Distributions

Burr distribution is the well-known probability distribution. It was first proposed by [1] in which the classical properties and the computational methods for the estimation of maximum likelihood estimates to the life time censored data was proposed.

2.1. BURR XII distribution

[7] Provided the three parameter Burr XII distributions with the following cumulative distribution function and the probability density function for $x>0$

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$$F(x;s,k,c)=1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}, \quad k,c,s>0 \tag{2.1.1}$$

And $$f(x;s,k,c)=cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k-1}, k,c,s>0 \tag{2.1.2}$$

Respectively, where k and c are the shape parameters and s is the scale parameter.

2.2. LOMAX Distribution

Lomax in 1954 proposed Pareto Type – II distribution also known as Lomax distribution and used it for the analysis of the business failure life time data. The 3 parameter Lomax distribution was introduced by [10] with the following CDF and pdf;

$$F(x,\alpha,\lambda)=1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}, \quad \mu\le x\le\infty \tag{2.2.1}$$

And, $$f(x,\alpha,\lambda)=\frac{\alpha}{\lambda}\left[1+\frac{x-\mu}{\lambda}\right]^{-(\alpha+1)} \quad ; \quad \mu\le x\le\infty \tag{2.2.2}$$

3 Probability Density Function

The probability density function of the mixture of burr XII and Lomax distribution has the following form

$$f(x;k,s,c,\alpha,\lambda)=p_1 f_1(x;k,s,c)+p_2 f_2(x;\alpha,\lambda) \tag{3.1}$$

Where p_1 and p_2 are the mixing proportions and $p_1+p_2=1$

$f_1(x;k,s,c)$ is the pdf of the Burr XII distribution and $f_2(x;\alpha,\lambda)$ is the pdf of the Lomax distribution.

Therefore the mixture of these densities is as follow:

$$f(x;k,s,c,\alpha,\lambda)=p_1 cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k-1}+p_2\frac{\alpha}{\lambda}\left[1+\frac{x-\mu}{\lambda}\right]^{-(\alpha+1)} \tag{3.2}$$

Where $x>0$ and $k,s,c,\alpha,\lambda>0$, $p_1+p_2=1$

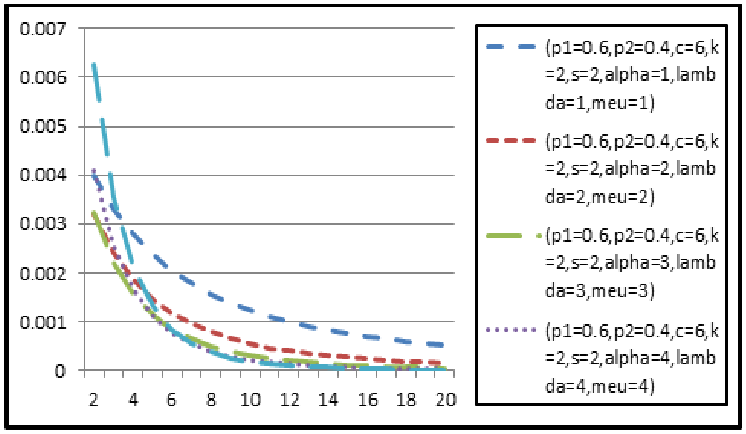

Figure. 3.1. Graph for the Probability density of the mixture of Burr XII and Lomax Distribution for $p_1=0.6$, $p_2=0.4$, $c=6$, $k=2$, $s=2$ and 2, $\alpha=1, 2, 3, 4$, $\lambda=1, 2, 3, 4$ and $\mu=1, 2, 3, 4$.

From the figure 3.1 it can be observed as the values of the parameters of the Lomax distribution increases simultaneously, there is a decreasing trend in the probability distribution. Moreover the figure shows that the mixture distribution is positively skewed.

4 Area under the Curve

Since Burr XII and Lomax distribution are the complete probability density functions as already mentioned in the literature so their mixture will also be the complete pdf.

5 Cumulative Distribution Function

The cumulative distribution function for the mixture of *burr XII* and *Lomax distributions* can be expressed in the following form

$$F(x;k,s,c,\alpha,\beta)=p_1 F_1(x;k,s,c)+p_2F_2(x;\alpha,\lambda) \tag{5.1}$$

Where $F_1(x;k,s,c)$ is the cdf of the *Bur XII* distribution and has the following form

$$F_1(x;k,s,c)=1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-k} \quad , \ x>0\ ,\ k,c,s>0 \tag{5.2}$$

And $F_2(x;\alpha,\lambda)$ is the cdf of the Lomax distribution and can be expressed in the following form

$$F_2(x,\alpha,\lambda)=1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}, \qquad \mu\le x\le\infty \tag{5.3}$$

So the expression (5.1) becomes

$$F(x;k,s,c,\alpha,\lambda)=p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right] \tag{5.4}$$

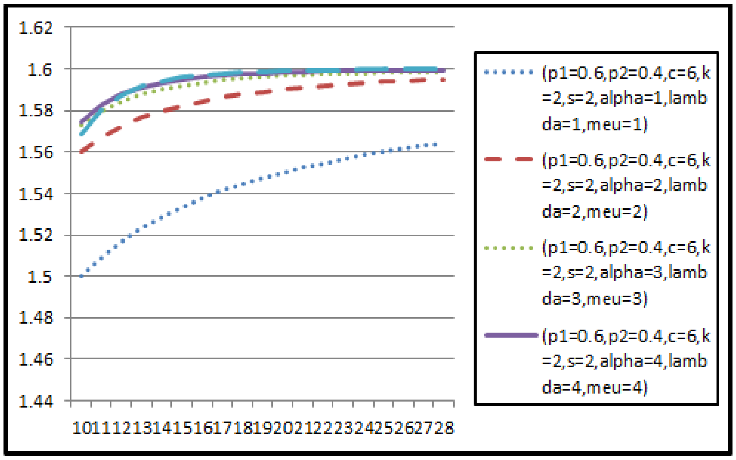

Figure. 5.1. Graph for the CDF of the mixture of Burr XII and Lomax Distribution for p1=0.6, p_2=0.4, c=6, k=2, s=2 and 2, α =1, 2, 3, 4, λ =1, 2, 3, 4 and μ =1, 2, 3, 4

The figure of the CDF is showing the increasing pattern as the time increases. It is also showing that as the parameters of Lomax distribution is increasing simultaneously, there is an increasing trend in the CDF.

6 Reliability Function

The reliability function which can also be called the survival function is the property of the random variable and is linked with the failure of some system within a specified time. It is defined as

$$R(x)=1-F(x) \tag{6.1}$$

The reliability or the survival function for the mixture of Burr XII and Lomax distributions is expressed in the following form by substituting (5.4) in (6.1)

$$R(x)=1-\left(p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right]\right) \tag{6.2}$$

7 Hazard Function

Hazard function can be expressed as the ratio of the probability density function and reliability function and can be given in the following form

$$h(x)=\frac{f(x)}{R(x)} \tag{7.1}$$

The hazard rate for the mixture distribution can be obtained by putting (3.2) and (6.2) in (7.1) and can be expressed in the following form.

$$h(x)=\frac{p_1cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}+p_2\frac{\alpha}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\alpha+1)}}{1-\left(p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right]\right)} \tag{7.2}$$

8 Cumulative Hazard Function

The cumulative hazard function can be expressed as:

$$\wedge(x) = -\log R(x) \tag{8.1}$$

The cumulative hazard function for the given mixture distribution can be established by putting (6.2) in (8.1) and has the following form:

$$\wedge(x) = -\log\left[1-\left(p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right]\right)\right] \tag{8.2}$$

9 Reversed Hazard Function

The Reversed hazard rate can be expressed as the ratio of the probability density function and the cumulative distribution function *i.e.*

$$r(x) = \frac{f(x)}{F(x)} \tag{9.1}$$

The reversed hazard rate for the mixture distribution can be expressed as by putting (2.2) and (3.4) in (7.1) and has the following form.

$$r(x) = \frac{p_1 cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1} + p_2\frac{\alpha}{\lambda}\left[1+\frac{x-\mu}{\lambda}\right]^{-(\alpha+1)}}{p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right]} \tag{9.2}$$

10 Odds Function

The odds function denoted by $O(x)$ is the ratio of cumulative distribution function and the reliability function and has the following form:

$$O(x) = \frac{p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right]}{1-\left(p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right]\right)} \tag{10.1}$$

The odds function for the mixture distribution can be obtained by putting (5.4) and (6.2) in (11.1) and following expression is obtained.

$$O(x)=\frac{p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right]}{1-\left(p_1\left[1-\left\{1+\left(\frac{x}{s}\right)^c\right\}^{-k}\right]+p_2\left[1-\left\{1+\left(\frac{x-\mu}{\lambda}\right)\right\}^{-\alpha}\right]\right)} \tag{10.2}$$

11 r^th Moment about Origin

r^th moment for the real valued function can be defined as

$$\mu_r'=E\left(x^r\right)$$

$$\mu_r'=\int x^r f(x)dx$$

$$\mu_r'=ks\int_0^\infty x^r\left(p_1cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}+p_2\frac{\alpha}{\lambda}\left[1+\frac{x-\mu}{\lambda}\right]^{-(\alpha+1)}\right)dx \tag{11.1}$$

$$\mu_r'=p_1s^r\frac{\Gamma\left(k-\frac{r}{c}\right)\Gamma\left(\frac{r}{c}+1\right)}{\Gamma k}+p_2\alpha\sum_{i=0}^{r}\binom{r}{i}\mu^{r-i}\lambda^i\beta(i+1,\alpha-i) \tag{11.2}$$

12 Raw Moments about Origin

Putting $r = 1, 2, 3$ and 4 in (11.2) first four raw moments are:

$$\mu_1'=p_1s\frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}+1\right)}{\Gamma k}+p_2\alpha\sum_{i=0}^{1}\binom{1}{i}\mu^{1-i}\lambda^i\beta(i+1,\alpha-i)\ (mean) \tag{12.1}$$

$$\mu_2'=p_1s^2\frac{\Gamma\left(k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}+1\right)}{\Gamma k}+p_2\alpha\sum_{i=0}^{2}\binom{2}{i}\mu^{2-i}\lambda^i\beta(i+1,\alpha-i) \tag{12.2}$$

$$\mu_3'=p_1s^3\frac{\Gamma\left(k-\frac{3}{c}\right)\Gamma\left(\frac{3}{c}+1\right)}{\Gamma k}+p_2\alpha\sum_{i=0}^{3}\binom{3}{i}\mu^{3-i}\lambda^i\beta(i+1,\alpha-i) \tag{12.3}$$

$$\mu_4'=p_1s^4\frac{\Gamma\left(k-\frac{4}{c}\right)\Gamma\left(\frac{4}{c}+1\right)}{\Gamma k}+p_2\alpha\sum_{i=0}^{4}\binom{4}{i}\mu^{4-i}\lambda^i\beta(i+1,\alpha-i) \tag{12.4}$$

13 Moments about Mean

$$\mu_1=0 \tag{13.1}$$

$$\mu_2=\mu_2'-(\mu_1')^2=Variance$$

$$\mu_2 = p_1 s^2 \frac{\Gamma\left(k-\frac{2}{c}\right)\Gamma\left(\frac{2}{c}+1\right)}{\Gamma k} + p_2 \alpha \sum_{i=0}^{2} \binom{2}{i} \mu^{2-i} \lambda^i \beta(i+1, \alpha - i) - \left(p_1 s \frac{\Gamma\left(k-\frac{1}{c}\right)\Gamma\left(\frac{1}{c}+1\right)}{\Gamma k} + p_2 \alpha \sum_{i=0}^{1} \binom{1}{i} \mu^{1-i} \lambda^i \beta(i+1, \alpha - i) \right)^2$$

The third and fourth moments about mean are obtained by putting values in the following expressions

$$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + (\mu_1')^3$$

$$\textit{And}\ \ \mu_4 = \mu_4' - 4\mu_1' \mu_3' + 6(\mu_1')^2 (\mu_2') - 3(\mu_1')^4$$

14 Measure of Skewness and Kurtosis

Skewness is that measure that how much a distribution leans to the one side of the mean and kurtosis can be utilized as the measure of the flatness of the probability distribution.

Skewness and kurtosis can be expressed by the symbols β_1 and β_2 respectively and is denoted by the following relation

$$\beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3} \tag{14.1}$$

And

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} \tag{14.2}$$

Putting (13.3) and (13.2) in the (14.1) the expression of β_1 can be obtained and by putting (11.5) and (13.2) in (14.2) the expression for the β_2 can be obtained

15 Maximum Likelihood Estimation (MLE)

The likelihood function is expressed as below

$$L(\underline{X}) = \prod_{i=1}^{n} \left(p_1 c k s^{-c} x^{c-1} \left[1 + \left(\frac{x}{s}\right)^c \right]^{-k-1} + p_2 \frac{\alpha}{\lambda} \left[1 + \frac{x-\mu}{\lambda} \right]^{-(\alpha+1)} \right)$$

The *log* likelihood function is given as

$$\ln L(\underline{X}) = \sum_{i=1}^{n} \ln \left(p_1 c k s^{-c} x^{c-1} \left[1 + \left(\frac{x}{s}\right)^c \right]^{-k-1} + p_2 \frac{\alpha}{\lambda} \left[1 + \frac{x-\mu}{\lambda} \right]^{-(\alpha+1)} \right) \tag{15.1}$$

Partially differentiating the equation (16.1) with respect to $k, s, c, \alpha\, and\ \lambda$ the estimates can be obtained but the most suitable technique for estimating the parameters is EM algorithm.

The MLE for $\underline{\theta}$ can be obtained by solving the above mentioned system of nonlinear equations $I\ (\underline{\theta}) = 0$. The solution of these nonlinear equations is not in closed form. For testing of hypothesis and the estimation of the confidence interval on the parameters of the model, the information matrix is required. The Fisher (1921) information matrix for mixture distribution can be formed

Where

$$I(\underline{\theta})=\begin{vmatrix} I_{11} & I_{12} & I_{13} & I_{14} & I_{15} \\ I_{21} & I_{22} & I_{23} & I_{24} & I_{25} \\ I_{31} & I_{32} & I_{33} & I_{34} & I_{35} \\ I_{41} & I_{42} & I_{43} & I_{44} & I_{45} \\ I_{51} & I_{52} & I_{53} & I_{54} & I_{55} \end{vmatrix} \quad (15.2)$$

$$I_{11}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k^2}\right] \quad I_{12}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k\,\partial s}\right] \quad I_{13}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k\,\partial c}\right] \quad I_{14}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k\,\partial \alpha}\right]$$

$$I_{15}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial k\,\partial \beta}\right] \quad I_{22}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial s^2}\right] \quad I_{23}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial s\,\partial c}\right] \quad I_{24}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial s\,\partial \alpha}\right]$$

$$I_{25}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial s\,\partial \beta}\right] \quad I_{33}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial c^2}\right] \quad I_{34}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial c\,\partial \alpha}\right] \quad I_{35}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial c\,\partial \beta}\right]$$

$$I_{44}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial \alpha^2}\right] \quad I_{45}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial \alpha\partial \beta}\right] \quad I_{55}=-E\left[\frac{\partial^2 \ln L(\underline{X})}{\partial \beta^2}\right]$$

And *Var-Cov* matrix can be obtained as follow:

$$V=I^{-1}(\underline{\theta})$$

$$N_5(\underline{\theta}, J(\underline{\theta})^{-1}) \quad (15.3)$$

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