Right Triangular Spherical Dihedral f–Tilings with Two Pairs of Congruent Sides

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Abstract: In this paper we present the study of dihedral f-tilings by spherical right triangles on two distinct cases of adjacency and with two pairs of congruent sides. Some aspects of the combinatorial structure are given.

Keywords: Dihedral f-tilings, combinatorial properties, spherical trigonometry.

1 Introduction

By a dihedral folding tiling (f-tiling, for short) of the sphere \( S^2 \) whose prototiles are spherical right triangles, \( T_1 \) and \( T_2 \), we mean a polygonal subdivision \( \tau \) of \( S^2 \) such that each cell (tile) of \( \tau \) is congruent to \( T_1 \) or \( T_2 \) and the vertices of \( \tau \) satisfy the angle-folding relation, i.e., each vertex of \( \tau \) is of even valency and the sums of alternating angles around each vertex are \( \pi \). In fact, the crease pattern associated to the subjacent graph of a spherical f-tiling satisfy the Kawasaki’s condition at any vertex \( v \). In this paper we shall discuss dihedral f-tilings by spherical right triangles, considering two distinct cases of adjacency. We assume that from all the sides of the prototiles involved there are two pairs of congruent sides. The 3-dimensional representations of the obtained f-tilings are presented as well as the combinatorial structure.

f-tilings are intrinsically related to the theory of isometric foldings of Riemannian manifolds, introduced by S. A. Robertson [6] in 1977. The classification of f-tilings was initiated by Ana Breda [1], with a complete classification of all spherical monohedral f-tilings. Later on, in 2002, Y. Ueno and Y. Agaoka [10] have established the complete classification of all triangular monohedral tilings (without any restrictions on angles).

The study of dihedral f-tilings by spherical right triangles is a very extensive and exhaustive work and some particular cases were recently obtained in papers [4, 5]. A list of all dihedral f-tilings of the sphere by triangles and parallelograms including the combinatorial structure of each tiling can be found in [2]. Robert Dawson has also been interested in special classes of spherical tilings, see [7–9] for instance.

We shall denote by \( \Omega(T_1, T_2) \) the set, up to an isomorphism, of all dihedral f-tilings of \( S^2 \) whose prototiles are \( T_1 \) and \( T_2 \).

From now on \( T_1 \) is a spherical right triangle of internal angles \( \frac{\pi}{2}, \alpha \) and \( \beta \), with edge lengths \( a \) (opposite to \( \beta \)), \( b \) (opposite to \( \alpha \)) and \( c \) (opposite to \( \frac{\pi}{2} \)), and \( T_2 \) is a spherical right triangle of internal angles \( \frac{\pi}{2}, \gamma \) and \( \delta \), with edge lengths \( d \) (opposite to \( \delta \)), \( e \) (opposite to \( \gamma \)) and \( f \) (opposite to \( \frac{\pi}{2} \)) (see Figure 1). We will assume throughout the text that \( T_1 \) and \( T_2 \) are distinct triangles, i.e., \((\alpha, \beta) \neq (\gamma, \delta)\) and \((\alpha, \beta) \neq (\delta, \gamma)\). The case \( \alpha = \beta \) or \( \gamma = \delta \) was analyzed in [4], and so we will assume further that \( \alpha \neq \beta \) and \( \gamma \neq \delta \).

Fig. 1 Prototiles: spherical right triangles \( T_1 \) and \( T_2 \)

Relations between faces, edges, vertices and angles of any dihedral f-tiling of \( S^2 \), with prototiles \( T_1 \) and \( T_2 \), are stated in proposition 1.

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Proposition 1. [3, Proposition 2.1] Let \( \tau \in \Omega (T_1, T_2) \). If \( N_1 > 0 \) and \( N_2 > 0 \) denote the number of spherical right triangles of \( \tau \) congruent to \( T_1 \) and \( T_2 \), respectively, and \( E \) and \( V \) denote the number of edges and vertices of \( \tau \), respectively, then:

\[
\begin{align*}
(i) & \quad N_1 + N_2 = 2V - 4 = 2E \geq 8; \\
(ii) & \quad 3V = 6 + E; \\
(iii) & \quad \text{there are, at least, six vertices of valency four;} \\
(iv) & \quad \text{the cases } (\alpha + \beta \geq \pi \text{ and } \gamma + \delta \geq \pi) \text{ and } (\alpha + \beta > \pi \text{ and } \gamma + \delta > \pi) \text{ cannot occur.}
\end{align*}
\]

It follows straight away that
\[
\alpha + \beta > \frac{\pi}{2} \quad \text{and} \quad \gamma + \delta > \frac{\pi}{2},
\]
and also \( a \neq b \) and \( d \neq e \).

In order to get any dihedral f-tiling \( \tau \in \Omega (T_1, T_2) \), we find it useful to start by considering one of its local configurations, beginning with a common vertex to two tiles of \( \tau \) in adjacent positions.

In the diagrams that follows it is convenient to label the tiles according to the following procedures:

(i) We begin the configuration of a tiling \( \tau \in \Omega (T_1, T_2) \) with a right triangle \( T_1 \), labelled by 1; then we label with 2 a right triangle \( T_2 \), adjacent to \( T_1 \);

(ii) For \( j \geq 3 \), the location and orientation of tile \( j \) can be deduced from the configuration of tiles \( (1, 2, 3, \ldots, j - 1) \) and from the hypothesis that the configuration is part of a complete f-tiling (except in the cases indicated).

2 F-tilings by right triangles

In the following subsections we will consider separately two distinct cases of adjacency, assuming that any element of \( \Omega (T_1, T_2) \) has at least two cells such that they are in adjacent positions and in one of the situations illustrated in Figure 2 (the remaining two cases of adjacency are not in the scope of this paper).

![Fig. 2 Distinct cases of adjacency](image)

We omit the analysis of the cases where all the edges of \( T_1 \) and \( T_2 \) have different lengths, except the adjacent sides. This study is very extensive and is outside of the scope of this work.

2.1 Case of Adjacency A

Suppose that any element of \( \Omega (T_1, T_2) \) has at least two cells congruent, respectively, to \( T_1 \) and \( T_2 \), such that they are in adjacent positions as illustrated in Figure 2–A. We have \( b \neq e \), as \( b = e \) implies \( T_1 \equiv T_2 \). As \( a = d \), using trigonometric formulas, we obtain

\[
\cos \beta = \cos \delta = \frac{\sin \alpha}{\sin \gamma}
\]

In the following subsections we will consider separately the cases \( a = c, b = e \), and \( b = f \). Note that each one of the cases \( a = f (= d) \) and \( c = f \) imply \( a = c \). The cases \( e = f \) and \( c = e \) are analogous to the cases \( b = c \) and \( b = f \) (the same f-tilings are obtained), respectively, where the roles of the angles \( (\alpha, \beta) \) and \( (\gamma, \delta) \) are interchanged.

2.1.1 \( a = c \)

The condition \( a = c \) leads to \( \beta = \delta = \frac{\pi}{2} = a = f, \alpha \neq b \) and \( \gamma = e \).

Proposition 2. If there are two cells in adjacent positions as illustrated in Figure 2–A, with \( a = c \), then for each \( k_1, k_2, k_1, k_2 \geq 1 \), \( \Omega (T_1, T_2) = \{ P_{\alpha/k_1, k_2}^\alpha, P_{\beta/k_1, k_2}^\beta \} \), where \( P_{\alpha/k_1, k_2}^\alpha \) and \( P_{\beta/k_1, k_2}^\beta \), \( \alpha \in (0, \frac{\pi}{2}) \), \( \beta \neq \gamma \), are non-isomorphic dihedral f-tilings, with \( 1 \leq j \leq \phi(k_1, k_2) \), and \( 1 \leq j \leq \phi(k_1, k_2) \) for some integers \( \phi(k_1, k_2) \) and \( \phi(k_1, k_2) \), respectively; these values correspond to the number of distinct f-tilings in each class and satisfy

\[
\phi(k_1, k_2) \leq \sum_{k=0}^{n} \binom{n}{k}^2, \quad \text{with } n = k_1 + k_2.
\]

Proof.

Suppose that any element of \( \Omega (T_1, T_2) \) has at least two cells congruent, respectively, to \( T_1 \) and \( T_2 \), such that they are in adjacent positions as illustrated in Figure 2–A.

We will consider separately the cases \( \alpha < \beta \) and \( \alpha > \beta \).

1. Suppose firstly that \( \alpha < \beta \). Consequently, \( \alpha < \frac{\pi}{2}, \gamma \neq \alpha \). Therefore, the configuration illustrated in Figure 2–A is extended to the one given in Figure 3.

At vertices \( v_1 \) and \( v_2 \), that are in antipodal positions, we have

\[
k_1 \alpha + k_2 \gamma = \pi \quad \text{or} \quad \pi = k_1 \alpha + k_2 \gamma = \pi,
\]

\( k_1, k_2, k_1, k_2 \geq 1 \).

For each \( \alpha \in (0, \frac{\pi}{2}) \), \( \alpha \neq \gamma \), and each pair \( (k_1, k_2) \), with \( k_1, k_2 \geq 1 \), the condition \( k_1 \alpha + k_2 \gamma = \pi \) leads to a class of f-tilings \( P_{\alpha/k_1, k_2}^\alpha \), where \( 1 \leq j \leq \phi(k_1, k_2) \); \( \phi(k_1, k_2) \) is the number of distinct f-tilings for the pair \( (k_1, k_2) \), and \( 2(k_1 + k_2) \) is the valency of the vertices \( v_1 \).
and \( v_2 \). The value of \( \phi(k_1, k_2) \) could be obtained counting circular permutations with two groups of indistinguishable objects. \( \phi(k_1, k_2) \) satisfy

\[
\phi(k_1, k_2) \leq \sum_{k=0}^{n} \binom{n}{k}^2, \text{ with } n = k_1 + k_2.
\]

For instance, fixing \( \alpha \in \left(0, \frac{\pi}{2}\right) \),

- if \( k_1 = k_2 = 1 \), there is a single f-tiling, \( \mathcal{P}^{0}_{1(1,1)} \), whose planar and 3D representations are given in Figure 4;
- if \( k_1 = 1 \) and \( k_2 = 2 \), there are two distinct f-tilings, \( \mathcal{P}^{0}_{1(1,2)} \) and \( \mathcal{P}^{0}_{2(1,2)} \); planar and 3D representations are given in Figure 5 and Figure 6, respectively.

Similarly, the case \( \frac{\pi}{2} + k_1 \alpha + k_2 \gamma = \pi \), with \( k_1, k_2 \geq 1 \), leads, for each \( \alpha \in \left(0, \frac{\pi}{2}\right) \), \( \alpha \neq \gamma \), and each pair \((k_1, k_2)\), to a class of f-tilings \( \mathcal{P}^{0}_{j(k_1, k_2)} \), where \( 1 \leq j \leq \phi(k_1, k_2) \); \( \phi(k_1, k_2) \) is the number of distinct f-tilings for the pair \((k_1, k_2)\), and \( 2(k_1 + k_2 + 1) \) is the valency of the vertices \( v_1 \) and \( v_2 \). The value of \( \phi(k_1, k_2) \) could be obtained counting circular permutations with three groups of indistinguishable objects. For instance, the case \( k_1 = k_2 = 1 \) give rise to six distinct f-tilings, whose planar and 3D representations are given in Figures 7–12, respectively.

2. Suppose now that \( \alpha > \beta \). As mentioned before, we have \( \beta = \delta = \frac{\pi}{2} = a = f \), and consequently \( \alpha > \beta = \frac{\pi}{2} \).

Analogously to the previous case, the configuration illustrated in Figure 2–A is extended to the one given in Figure 3.

At vertices \( v_1 \) and \( v_2 \) (in antipodal positions), we must have \( \alpha + k \gamma = \pi \), \( k \geq 1 \). For each \( \gamma \in \left(0, \frac{\pi}{2}\right) \) and \( k \geq 1 \), the condition \( \alpha + k \gamma = \pi \) leads to a class of \( \left\lceil \frac{k+1}{\alpha} \right\rceil \) f-tilings, where \( \lceil n \rceil \) denotes the smallest integer greater than or equal to \( n \). This class is a subset of \( \left\{ \mathcal{P}^{\alpha}_{j(k_1, k_2)} \right\} \), with \( k_1 \geq 1 \) and \( k_2 = 1 \), and where the roles of \( \alpha \) and \( \gamma \) are interchanged.
Proof.
As illustrated in Figure 2–A, Proposition 3.

The subcases $\alpha < \beta$ and $\alpha > \beta$.

**Proposition 3.** If there are two cells in adjacent positions as illustrated in Figure 2–A, with $b = c$ and $\alpha < \beta$, then $\Omega(T_1, T_2) = \emptyset$.

**Proof.**
Suppose that any element of $\Omega(T_1, T_2)$ has at least two cells congruent, respectively, to $T_1$ and $T_2$, such that they are in adjacent positions as illustrated in Figure 2–A, with $\alpha < \beta$. With the labelling used in Figure 13, we have $\theta_1, \theta_2 \in \{\beta, \pi/2\}$. Nevertheless, as $\beta > \alpha = \pi/2$, we get an impossibility at vertex $v$.

**Proposition 4.** If there are two cells in adjacent positions as illustrated in Figure 2–A, with $b = c$ and $\alpha > \beta$, then $\Omega(T_1, T_2) \neq \emptyset$ iff

(i) $\alpha = \pi/4$, $\beta = \pi/2 = \gamma$ and $\delta = \pi/7$

(ii) $\alpha = \pi/2$, $\beta + \gamma = \pi/2$ and $k\delta = \pi$, with $k \geq 3$.

The first case leads to two different dihedral $f$-tilings denoted by $S$ and $T$, respectively. Planar and 3D representations of $S$ and $T$ are given in Figure 17 and Figure 18, respectively.

In the last case, for each $k \geq 3$, we obtain a single tiling, denoted by $R^k$, with $\beta = \arccos \sqrt{\cos \pi/k}$ and $\gamma = \pi/2 - \beta$. A planar representation of $R^k$ is given in Figure 19(b) and 3D representations, for $k = 3$ and $k = 4$, are given in Figure 20.

**Proof.**
Suppose that any element of $\Omega(T_1, T_2)$ has at least two cells congruent, respectively, to $T_1$ and $T_2$, such that they are in adjacent positions as illustrated in Figure 2–A, with $\alpha > \beta$. With the labelling used in Figure 14(a), we have $\theta_1 = \delta$ or $\theta_2 = \gamma$.

If $\theta_1 = \delta$ (Figure 14(b)), then $\theta_2 = \beta$ or $\theta_2 = \beta$. But in either cases an incompatibility between sides cannot be avoided around vertex $v$.

Therefore $\theta_1 = \gamma$ (Figure 15(a)). Analyzing the edge lengths, at vertex $v$ we must have

\[
\frac{\pi}{2} + k\gamma = \pi \quad \text{or} \quad \frac{\pi}{2} + k_1\gamma + k_2\beta = \pi,
\]

with $k \geq 2$ and $k_1, k_2 \geq 1$.

1. Suppose firstly that $\frac{\pi}{2} + k\gamma = \pi$, with $k \geq 2$ (Figure 15(b)). Note that $\theta_2 = \frac{\pi}{2}$ (tiles 7 and 9) as $\theta_2 = \delta$...
At vertex $v$, we denote such an f-tiling by $T$. The corresponding 3D representation is given in Figure 18(b). At vertex $v$, we have necessarily $2\delta + \gamma = \pi$ or $3\delta = \pi$.

If $2\delta + \gamma = \pi$, we get the local configuration illustrated in Figure 16(a). At vertex $v'$ we have $\frac{\pi}{2} + \delta + k'\beta = \pi$, for some $k' \geq 1$. Nevertheless, we obtain an impossibility, as an incompatibility between sides cannot be avoided around vertex $v'$.

If $3\delta = \pi$, i.e., $\delta = \frac{\pi}{3}$, then $\gamma > \frac{\pi}{3}$, and so $\gamma = \frac{\pi}{3}$ $(k = 2)$. Moreover, equation (2) implies $\beta = \frac{\pi}{6}$. The last configuration is extended to the one illustrated in Figure 16(b). Now $\theta_3 = \beta$ or $\theta_3 = \frac{\pi}{2}$.

The first case gives rise to a single f-tiling whose planar representation is illustrated in Figure 17(a). We denote such an f-tiling by $S$. The corresponding 3D representation is given in Figure 17(b).

If $\theta_3 = \frac{\pi}{2}$, the last configuration is extended in a unique way to the global planar representation given in Figure 18(a). We denote such an f-tiling by $T$. The corresponding 3D representation is given in Figure 18(b).

2. Suppose now that $\frac{\pi}{2} + k_1\gamma + k_2\beta = \pi$, with $k_1, k_2 \geq 1$.

2.1 We consider firstly the case $k_1 = 1$, i.e., when $\frac{\pi}{2} + \gamma + k_2\beta = \pi$, $k_2 \geq 1$.

2.1.1 If $k_2 = 1$, we obtain the configuration illustrated in Figure 19(a). With the labelling of this figure, we have $\theta_2 = \beta$ or $\theta_2 = \frac{\pi}{2}$.

If $\theta_2 = \beta$, at vertex $v_1$ we must have $\frac{\pi}{2} + \gamma + \gamma = \pi$, as $\frac{\pi}{2} + \gamma + \beta = \pi$ give rise to an incompatibility between sides. Therefore $\gamma = \beta = \frac{\pi}{4} = \delta$, and consequently at vertex $v_2$ we get $3\delta = \pi$. The last configuration is then extended in a unique way to a planar representation that corresponds to the previous f-tiling $T$.

If $\theta_2 = \frac{\pi}{2}$, we get the local configuration illustrated in Figure 19(b). We have $k\delta = \pi$, with $k \geq 3$. Using (2), we obtain $\beta = \arccos \sqrt{\cos \frac{\pi}{6}}$ and $\gamma = \frac{\pi}{2} - \beta$. We denote such a family of f-tilings by $R^k$, $k \geq 3$. The corresponding 3D representations for $k = 3$ and $k = 4$ are given in Figure 20.

2.1.2 If $k_2 \geq 2$ and $\delta \leq \frac{\pi}{2}$, then $\gamma > \frac{\pi}{2}$ and we get the configuration illustrated in Figure 21(a). Note that $\theta_2 = \beta$, as $\theta_2 = \frac{\pi}{2}$ implies $\frac{\pi}{2} + \gamma + k_2\beta = \pi$ and an incompatibility between sides cannot be avoided around vertex $v_1$. Nevertheless we reach a contradiction at vertex $v_2$.

On the other hand, if $k_2 \geq 2$ and $\delta > \frac{\pi}{2}$, we have necessarily $3\delta = \pi$ as illustrated in Figure 21(b). If $\theta_2 = \delta$, we reach an incompatibility between sides around vertex $v$. Now, $\theta_3 = \frac{\pi}{2}$ or $\theta_3 = \beta$. These two distinct cases give rise to the local configurations illustrated in Figure 22(a) and Figure 22(b), respectively. In both cases, at vertex $v$ we must have $\frac{\pi}{2} + k\gamma = \pi$, for some $k \geq 2$. As $\delta = \frac{\pi}{2}$, we have $\gamma > \frac{\pi}{6}$, and so $k = 2$ and $\gamma = \frac{\pi}{4}$. But then,
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\[ \alpha \delta \]

iff \[ \emptyset \]

for some \( k > v \) at vertex \( \theta \) the one given in Figure 23(b). Now, we have

The last configuration is then extended in a unique way to \( \theta \)

Therefore we have

3 between sides cannot be avoided around this vertex.

2.1.3 \( b = f \)

**Proposition 5.** If there are two cells in adjacent positions as illustrated in Figure 2–A and \( b = f \), then \( \Omega(T_1, T_2) \neq \emptyset \) iff

\[ (i) \delta + \alpha + \gamma = \pi \text{ and } k/\beta \geq 3, k \geq 3, \text{ or} \]

\[ (ii) 2\alpha + \gamma = \pi, \alpha + 3/\beta = \pi \text{ and } \delta = \pi/3 \text{ or} \]

\[ (iii) \alpha + 2/\gamma = \pi, \alpha + 2/\beta = \pi \text{ and } \delta = \pi/3 \text{ or} \]

\[ (iv) \alpha + \gamma + \pi = \pi, \beta = \pi/3 \text{ and } \delta = \pi/3 \text{ or} \]

\[ (v) 2\alpha + 2/\gamma = \pi, \alpha + \gamma + 2/\beta = \pi \text{ and } \delta = \pi/3. \]

For each \( k \geq 3 \), the first case leads to a single f-tiling, denoted by \( \mathcal{V}^k \). A planar representation is given in Figure 30. 3D representations for \( k = 3 \) and \( k = 4 \) are given in Figure 31.

The case (ii) leads to a single f-tiling, denoted by \( \mathcal{Z} \). In Figure 38(b) and Figure 39 are given the corresponding planar and 3D representations.

The case (iii) leads to a single f-tiling, denoted by \( \mathcal{L} \), whose planar and 3D representations are presented in Figure 42. The case (iv) leads to a single f-tiling, denoted by \( \mathcal{U} \), whose planar and 3D representations are presented in Proposition Figure 50(a) and Figure 50(b), respectively.

In the last situation, there is a single f-tiling, denoted by \( \mathcal{U} \). In Figure 44 are given the corresponding planar and 3D representations.

**Proof.**

Suppose that any element of \( \Omega(T_1, T_2) \) has at least two cells congruent, respectively, to \( T_1 \) and \( T_2 \), such that they
are in adjacent positions as illustrated in Figure 2–A. If 

\[ b = f, \text{ then} \]

\[ \cos \alpha = \cos \beta \cos \gamma \quad \text{and} \quad \sin \beta = \sin \alpha \sin \delta. \]

We also have \( v \neq a, b, c \) and \( c \neq a, b \).

1. If \( \beta > \alpha \), then we must have \( \beta > \frac{\pi}{2} \), and consequently \( \delta > \frac{\pi}{2} \), otherwise \( \sin \alpha < \sin \beta = \sin \alpha \sin \delta \) and \( \sin \delta > 1 \), which is an incongruence. Therefore we get the

configuration illustrated in Figure 24(a). Nevertheless, there is no way to satisfy the angle-folding relation around vertex \( v \).

2. Suppose now that \( \beta < \alpha \).

2.1 If \( \alpha > \frac{\pi}{2} \), as \( \cos \beta \cos \gamma = \cos \alpha < \cos \beta \), we conclude that \( \beta < \frac{\pi}{2} < \alpha \), and consequently \( \gamma > \frac{\pi}{2} > \delta \). The configuration illustrated in Figure 2–A is then extended in a unique way to the one presented in Figure 24(b). We must have \( \alpha + \beta < \pi \) (\( \alpha + \beta = \pi \)
implies $\beta = \frac{\pi}{2}$, but at vertex $v$ there is no way to satisfy the angle-folding relation.

2.2 If $\alpha < \frac{\pi}{2}$, then $\beta < \alpha < \frac{\pi}{2}$ and also $\gamma, \delta < \frac{\pi}{2}$. As $\cos \alpha = \cos \beta \cos \gamma$, we conclude that $\gamma < \alpha$. Moreover, $\sin \beta = \sin \alpha \sin \delta$ implies $\beta < \delta$.

With the labelling of Figure 25(a), we have $\theta_1 = \delta$ or $\theta_1 = \frac{\pi}{2}$.

2.2.1 If $\theta_1 = \delta$ (Figure 25(b)), then at vertex $v$ we must have $\frac{\pi}{2} + k_1 \delta + k_2 \beta = \pi$, with $k_1 \geq 1$, $k_2 \geq 0$ and $k_1 + k_2 \geq 2$. It is easy to observe that in any choice for $k_1$ and $k_2$ an incompatibility between sides cannot be avoided around vertex $v$.

2.2.2 Suppose now that $\theta_1 = \frac{\pi}{2}$ (Figure 26(a)). Now, we have $\theta_2 = \beta$, $\theta_2 = \delta$ or $\theta_2 = \gamma$.

2.2.2.1 If $\theta_2 = \beta$ (Figure 26(b)), then at vertex $v_1$ we must have $\alpha + k_1 \beta + k_2 \gamma = \pi$, with $k_1, k_2 \geq 1$. Note that if at vertex $v_1$ we have $\alpha + \beta + \alpha = \pi$, $\alpha + k_1 \beta + k_2 \delta = \pi$, $k \geq 2, k_1, k_2 \geq 1$, an incompatibility between sides cannot be avoided around this vertex. Moreover, if $\alpha + \beta + k \gamma = \pi, k \geq 1$, we get $\delta > \alpha > \beta > \gamma$ and consequently a contradiction is achieved at vertex $v_2$.

Considering the possible angle combinations at vertex $v_1$, it is easy to observe that the case $k_1 = k_2 = 1$ leads to a contradiction. Therefore $k_1 + k_2 > 2$.

If $\delta > \gamma$, then at vertex $v_2$ we must have $\frac{\pi}{2} + \delta + k_1 \beta = \pi$, with $k \geq 1$, which implies $\alpha > \delta > \gamma > \beta$. Nevertheless, at vertex $v_3$ we get $\frac{\pi}{2} + \alpha + \rho > \pi, \forall \rho \in \{\frac{\pi}{2}, \alpha, \beta, \gamma, \delta\}$.

Thus, we must have $\alpha > \gamma > \delta > \beta$ and consequently $k_2 = 1$ and $k_1 \geq 2$.

At vertex $v_2$ (Figure 26(b)) we must have $\frac{\pi}{2} + k \delta = \pi$, with $k \geq 2$, or $\frac{\pi}{2} + k_1 \delta + k_2 \beta = \pi$, with $k_1 \geq 2$ and $k_2 \geq 1$.

If $\frac{\pi}{2} + k \delta = \pi, k \geq 2$ (Figure 27(a)), then at vertex $v_4$ we have $\alpha + \alpha + \rho = \pi$, with $\rho \in \{\alpha, \gamma, \delta\}$. If $\alpha = \beta$ (Figure 27(b)), we reach a contradiction at vertex $v_5$. For the remaining cases, the system (3) is impossible. Note that $\rho = \alpha$ implies $k_1 = k \geq 2, \rho = \gamma$ implies $k_2 = 1$ and $\rho = \delta$ implies $k_1 = 2$.

\[
\begin{align*}
\cos \beta \sin \gamma &= \cos \delta \sin \alpha \\
\cos \alpha &= \cos \beta \cos \gamma \\
\sin \beta &= \sin \alpha \sin \delta \\
\end{align*}
\]

If $\frac{\pi}{2} + k_1 \delta + k_2 \beta = \pi$, with $k_1 \geq 2$ and $k_2 \geq 1$ (Figure 28(a)), at vertex $v_5$ we get a contradiction, as for
2.2.2.2 If $\theta_3 = \delta$ (Figure 28(b)), then at vertex $v_1$ we must have $\alpha + \delta + k\gamma = \pi$, with $k \geq 1$. Note that if at vertex $v_1$ we have $\alpha + \delta + \frac{\pi}{2} = \pi$, then $\frac{\pi}{2} + \gamma + \beta = \pi$, which implies that system (3) is impossible. On the other hand, if $\alpha + \delta + \alpha = \pi$ or $\alpha + k_1\delta + k_2\beta = \pi$, $k_1 \geq 1$, $k_2 \geq 0$, $k_1 + k_2 \geq 2$, an incompatibility between sides cannot be avoided around this vertex.

With the labelling of Figure 29(a), we have $(\theta_3, \theta_4) = (\delta, \delta), (\theta_3, \theta_4) = (\delta, \frac{\pi}{2}), (\theta_3, \theta_4) = (\frac{\pi}{2}, \delta)$ or $(\theta_3, \theta_4) = (\frac{\pi}{2}, \frac{\pi}{2})$.

(i) If $(\theta_3, \theta_4) = (\delta, \delta)$ (Figure 29(b)), then

- if $\delta > \gamma$, at vertex $v_4$ we must have $\frac{\pi}{2} + \delta + k\beta = \pi$, $k \geq 1$. But an incompatibility between sides cannot be avoided around this vertex;
- if $\delta < \gamma$, at vertex $v_3$, we must have $\frac{\pi}{2} + \gamma + \beta = \pi = \delta + \alpha + \gamma$, which implies that system (3) has no solution.

(ii) If $(\theta_3, \theta_4) = (\delta, \frac{\pi}{2})$ (Figure 29(a)), we consider the subcases $k = 1$ and $k > 1$.

If $k = 1$, we get the local configuration illustrated in Figure 30, where $k\beta = \pi$, with $k \geq 3$. We denote such family of $f$-tilings by $\mathcal{R}^k$, $k \geq 3$. The corresponding 3D representations for $k = 3$ and $k = 4$ are given in Figure 31.
If \( k > 1 \) (Figure 32(a)), then \( \delta > \alpha > \beta > \gamma \) and \( \delta > \frac{\pi}{2} \). At vertex \( v_2 \) we have necessarily \( \delta + \delta + \rho = \pi \), with \( \rho \in \{\alpha, \beta, \gamma, \delta\} \). In all these cases, and for the possible values of \( k \), we obtain a solution of system (3). Nevertheless, we get a contradiction at vertex \( v_0 \), as there is no way to satisfy the angle-folding relation around this vertex.

(iii) Consider now \( (\theta_3, \theta_4) = (\frac{\pi}{2}, \delta) \) (Figure 32(b)). If \( k = 1 \) the system (3) is impossible (at vertex \( v_1 \) we have \( \delta + \alpha + \gamma = \pi = \frac{\pi}{2} + \gamma + \beta \)). The case \( k > 1 \) leads to a contradiction at vertex \( v_4 \), as \( \frac{\pi}{2} + \delta + \rho > \pi \), \( \forall \rho \in \{\alpha, \beta, \gamma, \delta\} \).

(iv) Observing Figure 33(a) we conclude that in the case \( (\theta_3, \theta_4) = (\frac{\pi}{2}, \delta) \) we get a contradiction at vertex \( v_4 \).

2.2.2.3 Finally we analyze the case \( \theta_2 = \gamma \) (Figure 33(b)), considering separately the cases \( \gamma > \delta \) and \( \gamma < \delta \).

2.2.2.3.1 If \( \gamma > \delta \) (\( \alpha > \gamma > \delta > \beta \)), then at vertex \( v_1 \) we must have \( \alpha + \gamma + \gamma = \pi, \alpha + \gamma + \alpha = \pi, \alpha + \gamma + \delta = \pi \) or \( \alpha + \gamma + k\beta = \pi, k \geq 1 \).

(i) Suppose firstly that \( \alpha + \gamma + \gamma = \pi \), as illustrated in Figure 34(a). At vertex \( v_2 \) we have one of the following conditions:

(i1) \( \frac{\pi}{2} + \delta + \delta = \pi \);

(i2) \( \alpha + \delta + \delta + \beta = \pi \);

(i3) \( \gamma + \delta + \delta + \delta = \pi \);

Each one of the conditions (i1)–(i3), jointly with \( \alpha + \gamma + \gamma = \pi \), give rise to a solution of the system (3). But with these solutions we obtain a contradiction at vertex \( v_3 \), as there is no way to satisfy the angle-folding relation around this vertex.
Fig. 25 Local configurations

Fig. 26 Local configurations

The condition (i4) implies that at vertex $v_2$ we must have $\alpha + \gamma + \delta + \beta = \pi$. This leads to an impossible system (3).

If condition (i5) holds then we get the configuration illustrated in Figure 34(b). Note that $\theta_5$ must be $\gamma$, otherwise $\theta_5 = \frac{\pi}{2}$ and $\frac{\pi}{2} + \alpha + \rho > \pi \forall \rho \in \{\alpha, \beta, \gamma\}$. At vertex $v_4$ we have necessarily $\frac{\pi}{2} + \gamma + k\beta = \pi$, $k \geq 1$, and so the system (3) becomes impossible.

Due to the angles relations and the system (3), if $k_2 = 0$ in condition (i6), then $k_1$ must be 5 and we obtain the local configuration illustrated in Figure 35(a). But we get a contradiction at vertex $v_3$ as there is no way to satisfy the angle-folding relation around this vertex. If $k_2 > 0$ (Figure 35(b)), then at vertex $v_3$ we must have $\alpha + \alpha + \rho = \pi$, with $\rho \in \{\beta, \delta\}$. But in either cases an incompatibility between sides cannot be avoided around this vertex.

(ii) If $\alpha + \gamma + \alpha = \pi$ (Figure 36(a)), then at vertex $v_2$ we have one of the conditions (i1)–(i6) considered in the previous case.

If condition $\frac{\pi}{2} + \gamma + \delta = \pi$ (i1) holds, then we obtain one of the configurations illustrated in Figure 36(b) and Figure 37(a). In both cases we get a contradiction at vertex $v_3$, as an incompatibility between sides cannot be avoided around this vertex (note that, due to the solution of system (3), in the first case at vertex $v_3$ we must have $\frac{\pi}{2} + \delta + \gamma = \pi$ and in the second $\alpha + \beta + \beta + \beta = \pi$).

If $\alpha + \gamma + \delta + \beta = \pi$ (Figure 37(b)), then we obtain an impossibility at vertex $v_3$, since $\frac{\pi}{2} + \alpha + \rho > \pi \forall \rho \in \{\alpha, \beta, \gamma\}$.

Fig. 27 Local configurations

Fig. 28 Local configurations

If $\gamma + \delta + \delta = \pi$, $\alpha + \delta + \delta + \beta = \pi$ or $\gamma + \delta + \delta + \beta = \pi$ (i3)–(i5), then, due to the solution of system (3) for each case, we get a contradiction at vertex $v_3$ (Figure 36(a)).

If $k_1 \delta + k_2 \beta = \pi$, $k_1 \geq 2$, $k_2 \geq 0$, $k_1 + k_2 > 3$ (i6), then, taking into account the vertices $v_2$–$v_4$ (Figure 36(a))
Case give rise to a single $f$-tiling whose planar representation is illustrated in Figure 38(a) (at vertex $v_4$) or $(k_1 = 4$ and $k_2 = 0)$. The first case leads to a contradiction, as illustrated in Figure 38(a) (at vertex $v_4$ we have $\frac{\pi}{2} + \alpha + \rho > \pi$, $\forall \rho \in \{\alpha, \beta, \delta, \gamma\}$). The second case give rise to a single $f$-tiling whose planar representation is illustrated in Figure 38(b). We denote such $f$-tiling by $Z$. The corresponding 3D representation is given in Figure 39.
(iii) If at vertex $v_1$ (Figure 33(b)) we have $\alpha + \gamma + \delta = \pi$, then we get $\frac{\pi}{2} + \beta + \gamma = \pi$, which implies that system (3) has no solution.

(iv) Finally, if $\alpha + \gamma + k\beta = \pi$, $k \geq 1$ (Figure 40(a)), and considering the possibilities for $\theta_3$, we have $\theta_4 = \delta$ or $\theta_1 = \frac{\pi}{2}$ (recall that $\frac{\pi}{2} + \alpha + \rho > \pi$, $\forall \rho \in \{\alpha, \beta, \delta, \gamma\}$).

If $\theta_4 = \delta$, then $\theta_3 = \delta$ and there is no way to satisfy the angle-folding relation around vertex $v_2$.

The case $\theta_4 = \frac{\pi}{2}$, as illustrated in Figure 40(b), is analogous to the one studied in 2.2.2.2 (ii), where the roles of the angles $(\alpha, \beta)$ and $(\gamma, \delta)$ are interchanged.

2.2.2.3.2 If $\gamma < \delta$, with the labelling of Figure 33(b), at vertex $v_2$ we must have one of the following conditions:

\[\begin{align*}
\text{(i)} & \quad \delta + \delta + \alpha = \pi; \\
\text{(ii)} & \quad \delta + \delta + \delta = \pi; \\
\text{(iii)} & \quad \delta + \delta + k\beta = \pi, \quad k \geq 1; \\
\text{(iv)} & \quad \delta + \delta + \gamma = \pi; \\
\text{(v)} & \quad \delta + \delta + \gamma + k\beta = \pi, \quad k \geq 1; \\
\text{(vi)} & \quad \delta + \delta + \gamma \geq \pi.
\end{align*}\]

If condition (i1) holds, then at vertex $v_2$ (Figure 33(b)) we have $\alpha + \gamma + \delta = \pi$, which implies that system (3) has no solution.

In the case (i2) (Figure 41(a)), we have to consider at vertex $v_1$ the following possibilities:

\[\begin{align*}
\text{(j1)} & \quad \alpha + \gamma + \frac{\pi}{2} = \pi; \\
\text{(j2)} & \quad \alpha + \gamma + \alpha = \pi; \\
\text{(j3)} & \quad \alpha + \gamma + \alpha + \alpha = \pi; \\
\text{(j4)} & \quad \alpha + \gamma + \delta + \gamma = \pi; \\
\text{(j5)} & \quad \alpha + \gamma + \gamma = \pi; \\
\text{(j6)} & \quad \alpha + \gamma + \alpha + \beta = \pi; \\
\text{(j7)} & \quad \alpha + \gamma + k\gamma = \pi, \quad k \geq 1; \\
\text{(j8)} & \quad \alpha + \gamma + \gamma + \beta = \pi; \\
\text{(j9)} & \quad \alpha + \gamma + \delta = \pi; \\
\text{(j10)} & \quad \alpha + \gamma + k\beta = \pi, \quad k \geq 1.
\end{align*}\]

Both conditions (i2)–(j1) and (i2)–(j5) imply $\beta = \frac{\pi}{4}$, $\alpha = \arctan(\sqrt{2})$, $\gamma = \frac{\pi}{2} - \alpha$ and $\delta = \frac{\pi}{2}$, and lead to one dihedral $f$-tiling denoted by $U$, whose planar and 3D representations are presented in Proposition 6 (Figure 50(a) and Figure 50(b), respectively), with the roles of the angles $(\alpha, \gamma)$ and $(\beta, \delta)$ interchanged.
The condition (i2) together with one of the conditions (j2)–(j4) leads to a contradiction at vertex \( v_4 \), as we must have a alternating sum containing two angles \( \beta \) and it is not possible due to the respective solutions of system (3).

Also taking into account the solution of system (3) when conditions (i2)–(j6) hold, we obtain a contradiction at vertex \( v_5 \) (Figure 41(b)).

If \( \alpha + \gamma + k\gamma = \pi, k \geq 1 \) (j7), then we must have \( k = 1 \) (the other possible values for \( k \) lead to a contradiction at vertex \( v_4 \), where we have at least two angles \( \beta \) in the alternated angle sums). In this case we obtain a dihedral \( f \)-tiling denoted by \( \mathcal{L} \), whose planar and 3D representations are presented in Figure 42.

If condition (j8) holds, then the last configuration extends in a unique way to the one illustrated in Figure 43(a). At vertex \( v_5 \) we reach a contradiction since there is no way to satisfy the angle-folding relation.

If condition (j9) holds, then at vertex \( v_1 \) (Figure 33(b)) we have \( \gamma + \alpha + \delta = \pi = \frac{\pi}{2} + \beta + \gamma \), which implies that system (3) has no solution.

Finally, we consider \( \alpha + \gamma + k\beta = \pi, k \geq 1 \) (j10). If \( k = 1 \), then we obtain the previous \( f \)-tiling \( \mathcal{L} \). If \( k > 1 \) and \( \theta_3 = \alpha \) (Figure 43(b)), then \( \frac{\pi}{2} + \delta + k\beta = \pi, k \geq 1 \), and \( \alpha > \delta > \gamma > \beta \). If \( \theta_4 = \delta \) we obtain a contradiction.
at vertex \( v_4 \), as \( \frac{\pi}{2} + \alpha + \rho > \pi, \forall \rho \in \{ \frac{\pi}{2}, \alpha, \beta, \gamma, \delta \} \).

If \( \theta_5 = \delta \), then at vertex \( v_5 \) we have \( \alpha + \alpha + \rho = \pi \), with \( \rho \in \{ \gamma, \beta \} \). But in either cases the system (3) has no solution. The case \( \theta_3 = \beta \) leads to a dihedral f-tiling denoted by \( \mathcal{U} \), whose planar and 3D representations are presented in Figure 44.

In the case (i3) (Figure 45(a)), we have \( \alpha > \delta > \gamma > \beta \) and at vertex \( v_2 \) we must have \( \frac{\pi}{2} + \gamma + \gamma = \pi, \frac{\pi}{2} + \gamma + \gamma + k\beta = \pi \) or \( \frac{\pi}{2} + \gamma + k\beta = \pi, k \geq 1 \). In the two first cases, we have \( \theta_3 = \alpha \) and \( \alpha + \alpha + \rho = \pi \), with \( \rho \in \{ \alpha, \beta, \gamma, \delta \} \).

But considering each one of these possibilities for \( \rho \), we conclude that system (3) has no solution. In the last case we obtain \( \frac{\pi}{2} + \gamma + k\beta = \pi, \frac{\pi}{2} + \gamma + k\beta = \pi, k \geq 1 \), at vertex \( v_3 \) and consequently at vertex \( v_1 \) we get \( \frac{\pi}{2} + \alpha + \gamma > \pi \).

If \( \delta + \alpha + \gamma = \pi \) (44), the last configuration extends to the one illustrated in Figure 45(b). At vertex \( v_3 \) we must have \( \frac{\pi}{2} + \delta + k\beta = \pi \) or \( \frac{\pi}{2} + \delta + k\gamma = \pi, k \geq 1 \). For each \( k \geq 1 \), the condition \( \delta + \alpha + k\beta = \pi \) implies that the system (3) is impossible. In the second case, if \( k = 1 \) we obtain the previous f-tiling \( \mathcal{L} \); if \( k > 1 \) (Figure 46(a)), as \( \beta \) increases as \( k \) increases and \( \delta > \alpha \) increases, at vertex \( v_4 \) we have necessarily \( \beta + \beta + k\gamma, k \geq 1 \). But an incompatibility between sides cannot be avoided around this vertex.

If condition (i5) holds (Figure 46(b)), then \( \alpha > \delta > \gamma > \beta \) and at vertex \( v_2 \) we must have \( \alpha + \alpha + \rho = \pi \), with \( \rho \in \{ \alpha, \beta, \gamma, \delta \} \). Any of the choices leads to a system (3) impossible.

Finally, if \( \delta + \delta + k\beta = \pi, k \geq 1 \) (Figure 47), then \( \frac{\pi}{2} + \alpha + \gamma = \pi \) and \( k = 1 \). But due to the solution of system (3) a contradiction is achieved at vertex \( v_3 \).  

2.2 Case of Adjacency B

Suppose that any element of \( \Omega(T_1, T_2) \) has at least two cells congruent, respectively, to \( T_1 \) and \( T_2 \), such that they are in adjacent positions as illustrated in Figure 2–B. As \( b = c \), using trigonometric formulas, we obtain

\[
\cos \alpha = \frac{\cos \gamma}{\sin \beta} \quad \sin \beta = \frac{\sin \delta}{\sin \gamma}.
\]

In the following subsection we will consider the case \( a = f \). The case \( c = d \) is analogous interchanging the roles of the angles \( (\alpha, \beta) \) and \( (\gamma, \delta) \). The case \( a = c \) implies \( \sin \beta = 1 \), and so \( \beta = \frac{\pi}{2} = a = c \). It is easy to see that
the corresponding study is analogous to the one presented in Proposition 3 and Proposition 4 (the same f-tilings are obtained), where the roles of the angles \((\alpha, \beta)\) and \((\gamma, \delta)\) are interchanged. Both cases \(a = d\) and \(c = f\) imply \(T_1 \equiv T_2\), which is not possible. The cases \(b = c\), \(c = e\) and \(d = f\) are analogous to the case \(a = c\). The case \(e = f\) implies \(b = c\).

2.2.1 \(a = f\)

If \(a = f\), then

\[
\cos \beta = \cos \alpha \cos \delta \quad \text{and} \quad \sin \alpha = \sin \beta \sin \gamma.
\]

In this case we must have \(\alpha < \beta\). In fact, if \(\alpha > \beta\) and

(i) \(\alpha < \frac{\pi}{2}\), then \(\sin \alpha = \sin \beta \sin \gamma > \sin \beta\) and \(\sin \gamma > 1\), which is an incongruence.

(ii) \(\alpha > \frac{\pi}{2}\), then \(\gamma > \frac{\pi}{2}\) (by (4)), and consequently we have \(\beta < \frac{\pi}{2} < \delta\) or \(\delta < \frac{\pi}{2} < \beta\). In both cases, all the
Local configurations

1. If $\theta_1 = \frac{\pi}{2}$, then $\frac{\pi}{2} + \beta + k\gamma = \pi$ or $\frac{\pi}{2} + \beta + k\delta = \pi$, for $k \geq 1$. The first case leads to a contradiction at vertex $v$ (there is no way to satisfy the angle-folding relation around this vertex). The condition $\frac{\pi}{2} + \beta + k\delta = \pi$, $k \geq 1$, leads to $\gamma > \beta > \alpha > \delta$ and to the configuration illustrated in Figure 48(b) (note that $\gamma \neq \frac{\pi}{2}$ and $\frac{\pi}{2} + \gamma + \rho > \pi$, $\forall \rho \in \{\frac{\pi}{2}, \alpha, \beta, \gamma, \delta\}$; for instance, tile 7 is a consequence of these conditions). We have necessarily $\gamma = \frac{\pi}{2}$, and so $k = 1$, $\alpha = \frac{\pi}{4}$, $\delta = \arccot \sqrt{2}$ and $\beta = \frac{\pi}{2} - \delta = \arctan \sqrt{2}$.

Now, $\theta_2 = \alpha$ or $\theta_2 = \beta$. In the first case, the last configuration extends in a unique way to the one illustrated in Figure 49. At vertex $v$ we reach a contradiction, as, due to the edge length, there is no way to satisfy the angle-folding relation around this vertex.

The second case gives rise to a single f-tiling whose planar representation is illustrated in Figure 50(b). We denote this f-tiling by $\mathcal{U}$. The corresponding 3D representation is given in Figure 50(b).

2. Suppose now that $\theta_1 = \gamma$ (Figure 51(a)). We have $(\theta_2, \theta_3) = (\gamma, \delta)$ or $(\theta_2, \theta_3) = (\delta, \gamma)$.

2.1 Consider that $(\theta_2, \theta_3) = (\gamma, \delta)$. We separate the subcases $\gamma > \delta$ and $\gamma < \delta$.

2.1.1 If $(\theta_2, \theta_3) = (\gamma, \delta)$ and $\gamma > \delta$, then at vertex $v_1$ we must have $\frac{\pi}{2} + \gamma + k\alpha = \pi$, $k \geq 1$, and consequently $\beta > \gamma > \delta > \alpha$, $\gamma + \beta + \pi$ and $k = 1$ (taking into account the edge lengths and the fact that $\gamma + \beta + \pi > \pi$, $\forall \rho \in \{\alpha, \beta, \gamma, \delta, \frac{\pi}{2}\}$). As $\sin \alpha = \sin \beta \sin \gamma$, we obtain $\gamma = \frac{\pi}{2}$, $\gamma = \frac{\pi}{2}$ or $\gamma = \frac{3\pi}{4}$, which is not possible (note that $\gamma = \frac{\pi}{2}$ implies $\beta = \frac{\pi}{2}$).

2.1.2 If $(\theta_2, \theta_3) = (\gamma, \delta)$ and $\gamma < \delta$, then at vertex $v_2$ we have necessarily $\frac{\pi}{2} + \delta + k\alpha = \pi$, $k \geq 1$, and consequently $\beta > \delta > \gamma > \alpha$. Observing Figure 51(b), we have $\theta_4 = \alpha$. Therefore, the subcase $\gamma < \delta$ is not possible.
2.2 Consider now that \((\theta_2, \theta_3) = (\delta, \gamma)\). As in the previous case, we separate the subcases \(\gamma > \delta\) and \(\gamma < \delta\).

2.2.1 If \((\theta_2, \theta_3) = (\delta, \gamma)\) and \(\gamma > \delta\), then at vertex \(v_2\) we must have \(\frac{\pi}{2} + \gamma + k\alpha = \pi = \delta + \gamma + k\alpha, k \geq 1\), which implies \(\delta = \frac{\pi}{2}\), which is not possible.

2.2.2 If \((\theta_2, \theta_3) = (\delta, \gamma)\) and \(\gamma < \delta\), then at vertex \(v_1\) we have necessarily \(\frac{\pi}{2} + \delta + k\alpha = \pi, k \geq 1\), which implies \(\beta > \delta > \gamma > \alpha\). Therefore \(k = 1\) and \(\frac{\pi}{2} + \delta + \alpha = \pi = \beta + \delta + \gamma\). Solving the system

\[
\begin{align*}
\cos \beta &= \cos \alpha \cos \delta \\
\sin \alpha &= \sin \beta \sin \gamma \\
\cos \alpha \sin \delta &= \cos \gamma \sin \beta
\end{align*}
\]

we obtain \(\delta = \frac{\pi}{2}\) or \(\gamma = \frac{\pi}{2}\), that is not possible. \(\square\)

3 Summary

In Table 1 is shown a list of the spherical dihedral \(f\)-tilings whose prototiles are spherical right triangles (with two pairs of congruent sides in two cases of adjacency), \(T_1\) and \(T_2\), of internal angles \(\frac{\pi}{2}, \alpha, \beta,\) and \(\frac{\pi}{2}, \gamma, \delta\), respectively. Our notation is as follows:

- \(\beta_0^k = \arccos \sqrt{\cos \frac{\pi}{2}}, k \geq 3; \alpha_1 = \beta_1 = \arctan \sqrt{2}\).
- \(\alpha_0^k\) and \(\delta_0^k\) are the solutions of system (3), with \(k \geq 3\).
- \(\beta = \frac{\pi}{k}\) and \(\gamma = \pi - \alpha - \delta\); \(\alpha_0\) is the solution of system (3), with \(\beta = \frac{\pi}{4}\) and \(\gamma = \pi - 2\alpha\).
- \(|V|\) is the number of distinct classes of congruent vertices;
- \(N_1\) and \(N_2\) are, respectively, the number of triangles congruent to \(T_1\) and \(T_2\), respectively, used in the dihedral \(f\)-tilings.

![Fig. 50](image)

![Fig. 51](image)

\(\beta\) or \(\theta_5 = \beta\). In both cases we reach a contradiction, as \(\frac{\pi}{2} + \beta + \rho > \pi, \forall \rho \in \{\alpha, \beta, \gamma, \delta, \frac{\pi}{2}\}\).
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