Common Fixed Point Results for Generalized Contractions on Ordered Partial Metric Spaces

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Abstract: In this paper, we consider a new class of pairs of generalized contractive type mappings defined in ordered partial metric spaces. Some coincidence and common fixed point results for these mappings are presented. An example is given to illustrate our obtained results.

Keywords: partially ordered set, partial metric, common fixed point, coincidence point, generalized contraction.

1 Introduction and Preliminaries

In spite of its simplicity, the Banach fixed point theorem still seems to be the most important result in metric fixed point theory. Fixed point theorems are very useful in the existence theory of differential equations, integral equations, functional equations and other related areas. Existence of a fixed point for contraction type mappings in partially metric spaces and its applications has been considered recently by many authors[1, 2, 4, 5, 7, 11, 21, 25, 28, 30, 32, 37]. Consistent with [6, 24], the following definitions and results will be needed in the sequel.

Definition 1.[24] A partial metric on a nonempty set X is a function p : X × X → R+ such that for all x, y, z ∈ X :
(P1) x = y ⇔ p(x, x) = p(y, y),
(P2) p(x, x) ≤ p(x, y),
(P3) p(x, y) = p(y, x),
(P4) p(x, z) + p(y, z) ≥ p(x, y) + p(z, z).

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Remark. It is clear that, if p(x, y) = 0, then from (P1) and (P2) x = y. But if x = y, p(x, y) may not be 0.

Example 1.[24] Let a function p : R+ × R+ → R+ be defined by p(x, y) = max{x, y} for any x, y ∈ R+. Then, (R+, p) is a partial metric space.

Example 2.[24] If X = {[a, b] : a, b ∈ R, a ≤ b}, then p : X × X → R+ defined by p([a, b], [c, d]) = max{b, d} − min{a, c} defines a partial metric on X.

If p is a partial metric on X, then the function p* : X × X → R+ given by

\[ p^*(x, y) = 2p(x, y) − p(x, x) − p(y, y) \] \hspace{1cm} (1)

is a metric on X.

Definition 2.[24, 26, 27] Let (X, p) be a partial metric space. Then

(i) A sequence \( \{x_n\} \) in a partial metric space (X, p) converges to a point x ∈ X if and only if \( p(x, x) = \lim_{n \to \infty} p(x, x_n) \).

(ii) A sequence \( \{x_n\} \) in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) \( \lim_{m,n \to \infty} p(x_m, x_n) \).

(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence \( \{x_n\} \) in X converges to a point x ∈ X, that is \( p(x, x) = \lim_{m,n \to \infty} p(x_m, x_n) \).

Remark. It is easy to see that, every closed subset of a complete partial metric space is complete.

Example 3.[20] If X = [0, 1] ∪ [2, 3] and define p : X × X → [0, ∞) by

\[ p(x, y) = \begin{cases} \max\{x, y\} & \text{if } \{x, y\} \cap [2, 3] \neq \emptyset, \\
|x - y| & \text{if } \{x, y\} \subset [0, 1]. \end{cases} \]

Then (X, p) is a complete partial metric space.

Lemma 1.[24, 25, 26] Let (X, p) be a partial metric space. Then

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(a) \( \{x_n\} \) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, p^*)\).

(b) A partial metric space \((X, p)\) is complete if and only if the metric space \((X, p^*)\) is complete. Furthermore,

\[
\lim_{n \to \infty} p^*(x_n, x) = 0
\]

if and only if

\[
\lim_{n \to \infty} p(x, x_n) = \lim_{n,m \to \infty} p(x_m, x_n).
\]

**Lemma 2.** A mapping \( f : X \to X \) is said to be continuous at \( a \in X \) if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( f(B(a, \delta)) \subseteq B(f(a), \varepsilon) \).

The following result is easy to check.

**Definition 3.** Let \((X, p)\) be a partial metric space. \( T : X \to X \) is continuous if and only if given a sequence \( \{x_n\} \subseteq X \) and \( x \in X \) such that \( p(x, x_n) \to 0 \) as \( n \to \infty \), then \( p(Tx, Tx_n) \to 0 \) as \( n \to \infty \).

**Lemma 4.** Consider \( X = [0, \infty) \) endowed with the partial metric \( p : X \times X \to [0, \infty) \) defined by \( p(x, y) = \max\{x, y\} \) for all \( x, y \geq 0 \). Let \( F : X \to X \) be a non-decreasing function. If \( F \) is continuous with respect to the standard metric \( d(x, y) = |x - y| \) for all \( x, y \geq 0 \), then \( F \) is continuous with respect to the partial metric \( p \).

**Definition 3.** Let \( X \) be a set, \( T \) and \( g \) are selfmaps of \( X \). A point \( x \) in \( X \) is called a coincidence point of \( T \) and \( g \) if \( Tg(x) = gx \).

**Definition 5.** A mapping \( F : X \to X \) is said to be weakly contractive if \( \forall x \in X \) there exists \( k(x) \in [0, 1) \) such that \( d(Fx, Fy) \leq k(x) d(x, y) \).

Aydi [8] obtained the following result.

**Theorem 1.** Let \((X, \leq) \) be a partially ordered set and let \( p \) be a partial metric on \( X \) such that \((X, p)\) is complete. Let \( f : X \to X \) be a non-decreasing with respect to \( \leq \). Suppose that the following conditions hold: for \( y \leq x \), we have

\[
(i) \quad p(f(x), f(y)) \leq p(x, y) - \varphi(p(x, y)) \tag{3}
\]

where \( \varphi : [0, \infty) \to [0, \infty) \) is a continuous and non-decreasing function such that it is positive in \([0, \infty)\), \( \varphi(0) = 0 \) and \( \lim_{t \to \infty} \varphi(t) = \infty \);

(ii) there exist \( x_0 \in X \) such that \( x_0 \leq f(x_0) \);

(iii) \( f \) is continuous in \((X, p)\), or;

(iv) if a non-decreasing sequence \( \{x_n\} \) converges to \( x \in X \), then \( x_n \leq x \) for all \( n \). Then \( f \) has a fixed point \( u \in X \). Moreover, \( p(u, u) = 0 \).

Choudhury [13] introduced the following definition.

**Definition 7.** A mapping \( T : X \to X \), where \((X, d)\) is a metric space is said to be weakly \( C \)-contractive if \( d(Tx, Ty) \leq \frac{1}{2} [d(x, Ty) + d(y, Tx)] - \varphi(d(x, y), d(y, Tz)) \),

where \( \varphi : [0, \infty) \times [0, \infty) \to [0, \infty) \) is a continuous function such that \( \varphi(x, y) = 0 \) if and only if \( x = y = 0 \).

2 Main results

Set \( \Psi([0, \infty)) = \{ \psi : [0, \infty) \to [0, \infty) : \psi \) is continuous and nondecreasing mapping with \( \psi(t) = 0 \) if and only if \( t = 0 \} \). Our first main result is the following.

**Theorem 2.** Let \((X, \leq) \) be a partially ordered set and suppose there is a partial metric \( p \) on \( X \) such that \((X, p)\) is a complete partial metric space. Assume there is a continuous function \( \varphi : [0, \infty) \to [0, \infty) \) with \( \varphi(t) < t \) for each \( t > 0 \) and also suppose \( T, g : X \to X \) are such that \( TX \subseteq gx \), \( T \) is a \( g \)-non-decreasing and for every two elements \( x, y \in X \) which \( gx \) and \( gy \) are comparable, we have

\[
\psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \psi(p(gx, gy), p(gx, Tx)), \tag{4}
\]

where

\[
M(x, y) = \max\{\psi(p(gx, gy)), \psi(p(gx, Tx)), \psi(p(gy, Ty))\},
\]

\[
\varphi\left(\frac{p(gx, Ty) + p(gy, Tx)}{2}\right)
\]

\( \psi \in \Psi([0, \infty)) \) and \( \varphi : [0, \infty) \times [0, \infty) \to [0, \infty) \) is continuous mapping such that \( \varphi(x, y) = 0 \) if and only if \( x = y = 0 \). Also suppose either

(i) \( T, g \) are two continuous self-mappings of \( X \) and \( \{T, g\} \) is partial compatible or
If there exists an \( x_0 \) in \( X \) with \( g(x_0) \leq T(x_0) \) then \( T \) and \( g \) have a coincidence point.

**Proof.** Note that if \( T, g \) have a coincidence point \( z \), then
\[
p(Tz, Tz) = p(gz, gz) = 0.
\]
Indeed, assume that \( p(gz, gz) > 0 \). Then from (4) with \( x = y = z \), we have
\[
\psi(p(gz, gz)) = \psi(p(Tz, Tz)) \leq \psi(M(z, z)) - \phi(p(gz, gz), p(Tz, Tz)),
\]
where
\[
M(z, z) = \max\{\phi(p(gz, gz)), \phi(p(Tz, Tz)), \phi(\frac{p(gz, Tz) + p(Tz, gz)}{2})\}
\]
\[
= \max\{\phi(p(gz, gz)), \phi(p(gz, gz)), \phi(\frac{p(gz, gz) + p(gz, gz)}{2})\}
\]
\[
= \phi(p(gz, gz)).
\]
Then we have
\[
\psi(p(gz, gz)) = \psi(p(Tz, Tz)) \leq \psi(\phi(p(gz, gz))) - \phi(p(gz, gz), p(gz, gz)) \leq \psi(p(gz, gz)) - \phi(p(gz, gz), p(gz, gz)) \phi(p(gz, gz), p(gz, gz)), \]
\[
= 0.
\]
Therefore, we assume that\( \phi(t) < t \) for \( t > 0 \), we have
\[
\psi(p(Tx_0, Tx_0)) \leq \psi(\phi(p(Tx_0, Tx_0))) - \phi(p(Tx_0, Tx_0), p(Tx_0, Tx_0)) \leq \psi(\phi(p(Tx_0, Tx_0)) - \phi(p(Tx_0, Tx_0), p(Tx_0, Tx_0)) \leq \psi(p(Tx_0, Tx_0)).
\]
Then (8) holds.

- If \( M(x_n, x_{n+1}) = \phi(p(Tx_n, Tx_{n+1})) \), then we have
\[
\psi(p(Tx_n, Tx_{n+1})) \leq \psi(p(Tx_n, Tx_{n+1})) - \phi(p(Tx_n, Tx_{n+1}), p(Tx_n, Tx_{n+1})) \leq \psi(p(Tx_n, Tx_{n+1})) - \phi(p(Tx_n, Tx_{n+1}), p(Tx_n, Tx_{n+1})) \leq \psi(p(Tx_n, Tx_{n+1})).
\]

Thus, we have
\[
p(Tx_n, Tx_{n+1}) \leq \psi(p(Tx_n, Tx_{n+1}) \leq \psi(p(Tx_n, Tx_{n+1})).
\]
On the other hand, by the triangular inequality in partial metric space, we have
\[
p(Tx_n, Tx_{n+1}) \leq p(Tx_n, Tx_{n+1}) \leq p(Tx_n, Tx_{n+1}) + p(Tx_n, Tx_{n+1}).
\]
Thus, we have
\[
p(Tx_n, Tx_{n+1}) \leq p(Tx_n, Tx_{n+1}) + p(Tx_n, Tx_{n+1}) \leq p(Tx_n, Tx_{n+1}) + p(Tx_n, Tx_{n+1}),
\]
which implies that
\[
p(Tx_n, Tx_{n+1}) \leq p(Tx_n, Tx_{n+1}).
\]
Therefore, we proved that (8) holds. Then, the sequence \( \{p(Tx_n, Tx_{n+1})\} \) of real numbers is monotone decreasing. Hence there exists a real number \( \delta \geq 0 \) such that
\[
\lim_{n \to \infty} p(Tx_n, Tx_{n+1}) = \delta. \tag{10}
\]

We show that \( \delta = 0 \). Suppose, to the contrary, that \( \delta > 0 \). Then from continuity \( \phi \) and (9) gives that
\[
\lim_{n \to \infty} \psi(p(Tx_n, Tx_{n+1})) \\
\leq \lim_{n \to \infty} \psi(\phi(p(Tx_{n-1}, Tx_n))) \\
- \lim_{n \to \infty} \phi(p(Tx_{n-1}, Tx_n), p(Tx_{n-1}, Tx_n)),
\]
which implies that
\[
\psi(\delta) \leq \psi(\phi(\delta)) - \phi(\delta, \delta),
\]
which is possible only when \( \delta = 0 \). Therefore, we proved that
\[
\lim_{n \to \infty} p(Tx_n, Tx_{n+1}) = 0. \tag{11}
\]

From \( p(Tx_n, Tx_n), p(Tx_{n+1}, Tx_{n+1}) \leq p(Tx_n, Tx_{n+1}) \) and (11), we have
\[
\lim_{n \to \infty} p(Tx_n, Tx_n) = \lim_{n \to \infty} p(Tx_{n+1}, Tx_{n+1}) = 0. \tag{12}
\]

From (11), (12) and (1), we have
\[
\lim_{n \to \infty} p^s(Tx_n, Tx_{n+1}) = 0. \tag{13}
\]

Now, we prove that
\[
\lim_{m,n \to \infty} p(Tx_m, Tx_n) = 0.
\]

If not, then there exists an \( \epsilon > 0 \) and subsequences \( \{x_{n(k)}\} \) and \( \{x_{m(k)}\} \) of \( \{x_n\} \) with \( n(k) > m(k) \geq k \) such that
\[
p(Tx_{n(k)}, Tx_{m(k)}) \geq \epsilon \quad \text{and} \quad p(Tx_{n(k)-1}, Tx_{m(k)}) < \epsilon.
\]

Then we have
\[
\begin{align*}
\epsilon & \leq p(Tx_{n(k)}, Tx_{m(k)}) \\
& \leq p(Tx_{n(k)}, Tx_{n(k)-1}) + p(Tx_{n(k)-1}, Tx_{m(k)}) \\
& \quad - p(Tx_{n(k)-1}, Tx_{m(k)}) \\
& \quad < p(Tx_{n(k)}, Tx_{n(k)-1}) + \epsilon - p(Tx_{n(k)-1}, Tx_{n(k)-1}).
\end{align*} \tag{14}
\]

Taking \( k \to \infty \) in (14) and using (11) and (12) we get
\[
\lim_{k \to \infty} p(Tx_{n(k)}, Tx_{m(k)}) = \epsilon. \tag{15}
\]

Thus from the definition \( p \) we have
\[
\begin{align*}
p(Tx_{m(k)}, Tx_{n(k)}) & \leq p(Tx_{n(k)}, Tx_{m(k)}) + p(Tx_{m(k)-1}, Tx_{n(k)-1}) \\
& \quad + p(Tx_{n(k)-1}, Tx_{m(k)}) \\
& \quad - p(Tx_{m(k)-1}, Tx_{m(k)}) - p(Tx_{n(k)-1}, Tx_{n(k)-1}),
\end{align*} \tag{16}
\]
\[
\begin{align*}
p(Tx_{m(k)-1}, Tx_{n(k)-1}) & \leq p(Tx_{n(k)-1}, Tx_{m(k)}) + p(Tx_{m(k)}, Tx_{n(k)}) \\
& \quad + p(Tx_{n(k)}, Tx_{n(k)-1}) \\
& \quad - p(Tx_{m(k)}, Tx_{m(k)}) - p(Tx_{n(k)}, Tx_{n(k)}), \tag{17}
\end{align*}
\]

Taking \( k \to \infty \) in (16) and (17) and using (12), (13) and (15) we get
\[
\begin{align*}
\lim_{k \to \infty} p(Tx_{n(k)}, Tx_{m(k)}) & = \lim_{k \to \infty} p(Tx_{n(k)-1}, Tx_{m(k)-1}) \\
& = \epsilon.
\end{align*} \tag{18}
\]

\[
\begin{align*}
p(Tx_{m(k)-1}, Tx_{m(k)}) & \leq p(Tx_{n(k)-1}, Tx_{m(k)}) + p(Tx_{m(k)-1}, Tx_{m(k)}) \\
& \quad - p(Tx_{m(k)-1}, Tx_{m(k)}) - p(Tx_{n(k)-1}, Tx_{n(k)-1}), \tag{19}
\end{align*}
\]

Now using inequality (4), we have
\[
\begin{align*}
\psi(\epsilon) & \leq \psi(p(Tx_{n(k)}, Tx_{m(k)})) \\
& \leq \psi(M(x_{n(k)}, x_{m(k)})) \\
& \quad - \phi(p(gx_{n(k)}, gx_{m(k)}), p(gx_{n(k)}, Tx_{m(k)})) \\
& \quad - \phi(p(Tx_{n(k)-1}, Tx_{m(k)-1}), p(Tx_{n(k)-1}, Tx_{n(k)})), \tag{20}
\end{align*}
\]
where
\[
M(x_{n(k)}, x_{m(k)}) = \max\left\{ \phi(p(gx_{n(k)}, gx_{m(k)})), \phi(p(gx_{n(k)}, Tx_{m(k)})), \phi(\frac{p(gx_{n(k)}, Tx_{m(k)}) + p(gx_{m(k)}, Tx_{n(k)})}{2}) \right\}.
\]

Letting \( k \to \infty \) in the above inequality and using (18) and (19), we obtain
\[
\lim_{k \to \infty} M(x_{n(k)}, x_{m(k)}) = \max\{\phi(\epsilon), \phi(0), \phi(0), \phi(\frac{\epsilon + \epsilon}{2})\} = \phi(\epsilon).
\]

As \( k \to \infty \), inequality (20) becomes,
\[
\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon, 0) \]
\[
< \psi(\epsilon) - \phi(\epsilon, 0)
\]
which is a contradiction by virtue of a property of \( \phi \) and \( \psi \).

Thus, we obtain that \( \lim_{m,n \to \infty} p(Tx_m, Tx_n) = 0 \), i.e., \( \{Tx_n\} \) is a Cauchy sequence in \( (X, p) \) and hence in the metric space \( (X, p^s) \) by Lemma 1. Since \( (X, p) \) be a complete partial metric space, then, from Lemma 1, \( (X, p^s) \) is also complete, so the sequence \( \{Tx_n\} \) converges in the metric space \( (X, p^s) \), so there exist \( z \in X \) such that
\[
\lim_{n \to \infty} p(Tx_n, z) = \lim_{n \to \infty} p^s(gx_{n+1}, z) = 0.
\]

Again, from Lemma 1, we get
\[
\begin{align*}
p(z, z) & = \lim_{n \to \infty} p(Tx_n, z) = \lim_{n \to \infty} p(gx_{n+1}, z) \\
& = \lim_{n,m \to \infty} p(Tx_n, Tx_m) = 0. \tag{21}
\end{align*}
\]
because \( \lim_{n \to \infty} p(Tx_n, Tx_n) = 0 \).
Suppose that the assumption (i) holds. Now we show that
\( z \) is a coincidence point of \( T \) and \( g \).
Since \( T \) and \( g \) are continuous, from (21) and using Lemma 3, we get
\[
p(Tz, Tz) = \lim_{n \to \infty} p(T(Tx_n), Tz)
\]
and
\[
p(gz, gz) = \lim_{n \to \infty} p(g(Tx_{n+1}), gz).
\]
Since \( T \) and \( g \) are partial compatible mappings, this implies that
\[
p(gz, gz) = 0 \text{ and } \lim_{n \to \infty} p(T(g(x_{n+1})), g(T(x_{n+1}))) = 0.
\]

The condition (p4), we obtain
\[
p(Tz, gz) \leq p(Tz, T(Tx_n)) + p(T(Tx_n), gz)
\]
\[
\leq p(g(Tx_{n+1}), gz) - p(T(Tx_n), Tz)
\]
\[
= p(gz, Tz) + p(g(Tx_{n+1}), gz)
\]
\[
\leq p(Tz, Tz) - p(gz, gz).
\]

Letting \( n \to \infty \) in the above inequality and using (22) and (23), we have
\[
p(Tz, gz) \leq p(Tz, Tz) + p(gz, gz) = p(Tz, Tz).
\]

Now, we will prove that \( p(Tz, gz) = 0 \). Suppose that this is not the case. Then, from (4) with \( y = z \), we get
\[
\psi(p(Tz, Tz)) \leq \psi(M(z, z)) - \phi(p(gz, gz), p(Tz, Tz)),
\]
where
\[
M(z, z) = \max\{\psi(p(gz, gz)), \psi(p(Tz, Tz)), \phi(p(gz, Tz)), \phi(p(gz, Tz))\}
\]
\[
\phi(p(gz, Tz) + p(gz, Tz))
\]
\[
\leq \phi(p(gz, Tz)) < p(Tz, Tz).
\]
Therefore, from (24) and the above inequality, we have
\[
p(gz, Tz) = p(Tz, gz),
\]
a contradiction. Hence \( p(gz, Tz) = 0 \) which implies that
\( Tz = gz \), that is, \( z \) is a coincidence point of \( T \) and \( g \).
Suppose now that (ii) holds. Since \( \{Tx_n\} \subseteq gX \) and \( gX \) is closed, there exists \( x \in X \) such that \( z = gx \). From (7) and hypothesis (ii), we have
\[
gx \leq gx \text{ for all } n, gx \leq g(x).
\]

Now, we claim that \( x \) is a coincidence point of \( T \) and \( g \). We have
\[
p(gx, Tx) \leq p(gx, gx_{n+1}) + p(gx_{n+1}, Tx) - p(gx_{n+1}, gx_{n+1})
\]
\[
= p(gx_{n+1}, Tx) - p(gx_{n+1}, gx_{n+1}),
\]
\[
p(gx, Tx) \leq p(gx, gx) + p(gx, Tx) - p(gx, gx)
\]
\[
= p(gx, z) + p(gx, Tx) - p(gx, gx).
\]

Taking \( n \to \infty \) in the above inequality, we have
\[
p(gx, Tx) \leq \lim_{n \to \infty} p(Tx_n, Tx),
\]
\[
\lim_{n \to \infty} p(gx_n, Tx) \leq p(gx, Tx).
\]

By property of \( \psi \) and using (26), we have
\[
\psi(p(gx, Tx)) \leq \lim_{n \to \infty} \psi(p(Tx_n, Tx))
\]
\[
\leq \lim_{n \to \infty} \left[ \psi(M(x, x_n)) - \phi(p(gx, gx_n), p(gx, Tx)) \right]
\]
\[
\leq \lim_{n \to \infty} \psi(M(x, x_n)) - \phi(0, p(gx, Tx))
\]
\[
= \psi(M(x, x_n)) - \phi(0, p(gx, Tx))
\]
where
\[
\lim_{n \to \infty} M(x, x_n)
\]
\[
= \lim_{n \to \infty} \left[ \max\{\phi(p(gx, gx_n)), \phi(p(gx, Tx))\}, \phi\left(\frac{p(gx, Tx_n) + p(gx_n, Tx)}{2}\right)\right]
\]
\[
= \phi(p(gx, Tx)),
\]
hence
\[
\psi(p(gx, Tx)) \leq \psi(p(gx, Tx)) - \phi(0, p(gx, Tx)).
\]

which is possible only when \( p(gx, Tx) = 0 \), which implies that \( Tx = gx \), that is, \( x \) is a coincidence point of \( T \) and \( g \).

**Theorem 3.** Adding to the hypotheses of Theorem 2 the following condition:
if \( T \) and \( g \) commute at their coincidence points, we obtain the uniqueness of the common fixed point of \( T \) and \( g \).

**Proof.** Suppose that \( T \) and \( g \) commute at \( x \). Set \( y = Tx = gx \). Then
\[
Ty = T(gx) = g(Tx) = gy,
\]
from (4) we get
\[
\psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \phi(p(gx, gy), p(gx, Tx))
\]
where
\[
M(x, y) = \max\{\phi(p(gx, gy)), \phi(p(Tx, Ty)), \phi(p(gy, Ty)), \phi(p(Tx, Ty))\}
\]
\[
\phi\left(\frac{p(gx, Ty) + p(gy, Tx)}{2}\right)
\]
\[
= \max\{\phi(p(Tx, Ty)), \phi(p(Tx, Ty)), \phi(p(Tx, Ty))\}
\]
\[
\phi\left(\frac{p(Tx, Ty) + p(Ty, Tx)}{2}\right)
\]
\[
= \phi(p(Tx, Ty)).
\]

Suppose that \( p(Tx, Ty) > 0 \), from (29), we get
\[
\psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \phi(p(gx, gy), p(gx, Tx))
\]
\[
= \psi(\phi(p(Tx, Ty))) - \phi(p(gx, gy), p(gx, Tx))
\]
\[
\leq \psi(\phi(p(Tx, Ty))),
\]
by property of \( \psi \) and \( \phi \), we have
\[
p(Tx, Ty) \leq \phi(p(Tx, Ty)) < p(Tx, Ty),
\]
which is a contradiction. Hence $p(Tx, Ty) = 0$, that is, $p(y, Ty) = 0$. Therefore,

$$Ty = gy = y. \quad (30)$$

Thus we proved that $T$ and $g$ have a common fixed point. Uniqueness: Let $v$ and $w$ be two common fixed points of $T$ and $g$. (i.e.) $v = Tv = gw$ and $w = Tw = gw$. Using inequality (4), we have

$$\psi(p(Tw, Tv)) \leq \psi(M(w, v)) - \phi(p(gw, gw), p(gw, Tw)),$$

where

$$M(w, v) = \max\{\phi(p(gw, gw)), \phi(p(gw, Tw)),$$

$$\phi(p(gw, Tv)), \phi(p(gw, Tw)) + p(gw, Tw)\}$$

$$= \phi(p(w, v)).$$

Therefore,

$$\psi(p(w, v)) = \psi(p(Tw, Tv)) \leq \psi(\phi(p(w, v))) - \phi(p(gw, gw), p(gw, Tw))$$

which is possible only when $w = v$. Hence $T$ and $g$ have an unique common fixed point.

**Example 4.** Let $X = [0, 1]$ be endowed with usual order and let be the complete partial metric on $X$ defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $T, g : X \rightarrow X$ and $\psi, \phi : [0, \infty) \rightarrow [0, \infty]$ and $\phi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ be given by $Tx = x^3 + 9$, $gx = x^2$, $\psi(t) = t$, $\phi(s, t) = s + t$ and $\phi(t) = t$. Clearly $\psi$ is continuous and nondecreasing, $\psi(t) = 0$ if and only if $t = 0$. $\phi$ and $\psi$ are continuous and $\phi(s, t) = 0$ if and only if $t = 0$. We show that condition (4) is satisfied.

If $x, y \in X$ with $x \leq y$, then we have

$$\psi(p(Tx, Ty)) = \psi(\max\{\frac{x^3}{3 + 9}, \frac{y^3}{3 + 9}\})$$

$$= \max\{\frac{x^3}{3 + 9}, \frac{y^3}{3 + 9}\}$$

$$\leq \frac{1}{3} \max\{\frac{x^2}{3 + 9}, \frac{y^2}{3 + 9}\}$$

$$= \frac{1}{3} p(gx, gy) = \frac{2}{3} \phi(p(gx, gy))$$

$$\leq \psi(M(x, y)) = \phi(p(gx, gy), p(gx, Tx)).$$

Note that, $T$ and $g$ satisfy all the conditions given in Theorem 2. Moreover, 0 is a unique common fixed point of $T$ and $g$.

If we replace $p$ by $p^*$ in (4) of Theorem 2, then $T$ and $g$ do not satisfy (4) of Theorem 2, because

$$p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y) = 2\max\{x, y\} - x - y = |x - y|,$$

and

$$\psi(p^*(T1, T0)) = \psi(p^*(\frac{1}{12}, 0)) = \frac{1}{12},$$

$$M(1, 0) = \max\{\phi(p^*(g1, g0)), \phi(p^*(g1, T1)),$$

$$\phi(p^*(g0, T0)), \phi(p^*(g1, T0) + p^*(g0, T1))\}$$

$$= \max\{\phi(p^*(\frac{1}{4}, 0)), \phi(p^*(\frac{1}{4}, \frac{1}{4}),$$

$$\phi(p^*(0, 0)), \phi(p^*(0, 0) + p^*(0, \frac{1}{4})),$$

$$= \frac{1}{8},$$

$$\phi(p^*(g1, g0), p^*(g1, T1)) = \phi(p^*(\frac{1}{4}, 0), p^*(\frac{1}{4}, \frac{1}{12}))$$

$$= \phi(\frac{1}{4}, \frac{1}{6}) = \frac{5}{72},$$

$$\psi(M(1, 0)) - \phi(p^*(g1, g0), p^*(g1, T1)) = \frac{4}{72}.$$

Now, we will show that many results can be deduced from our previous obtained results. An immediate consequence of Theorem 2 are the following results.

**Corollary 1.** Let $(X, \leq)$ be a partially ordered set and suppose there is a partial metric $p$ on $X$ such that $(X, p)$ is a complete partial metric space. Assume there is a continuous function $\psi : [0, \infty) \rightarrow [0, \infty]$ with $\phi(t) < t$ for each $t > 0$ and suppose $T : X \rightarrow X$ be a non-decreasing function for all comparable $x, y \in X$, we have

$$\psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \theta(\max\{p(x, y), p(x, Tx)\}),$$

where

$$M(x, y) = \max\{\phi(p(x, y)), \phi(p(x, Tx)),$$

$$\phi(p(y, Ty)), \phi(p(x, Ty) + p(y, Ty))\},$$

$$\psi \in \Psi([0, \infty)) \text{ and } \theta : [0, \infty) \rightarrow [0, \infty) \text{ is continuous mapping such that } \theta(t) = 0 \text{ if and only if } t = 0. \text{ Also suppose either}$$

(i) $T$ is continuous or
(ii) $X$ has the following property:
if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x \forall n$.

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$ then have a unique fixed point $x \in X$. Moreover, $p(x, x) = 0$.

**Proof.** In Theorem 2, taking $\phi(x, y) = \theta(\max\{x, y\})$ for all $x, y \in [0, \infty)$, we get Corollary 1.

**Corollary 2.** Let $(X, \leq)$ be a partially ordered set and suppose there is a partial metric $p$ on $X$ such that $(X, p)$ is a complete partial metric space. Assume there is a continuous function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(t) < t$ for
each \( t > 0 \) and suppose \( T : X \rightarrow X \) be a non-decreasing function for all comparable \( x, y \in X \), we have

\[
\psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \theta(p(x, y) + p(x, Tx)),
\]

where

\[
M(x, y) = \max\{\varphi(p(x, y)), \varphi(p(x, Tx)), \varphi(p(y, Ty)), \\
\varphi\left(\frac{p(x, Ty) + p(y, Tx)}{2}\right)\},
\]

\( \psi \in \Psi(0, \infty) \) and \( \theta : [0, \infty) \rightarrow [0, \infty) \) is continuous mapping such that \( \theta(t) = 0 \) if and only if \( t = 0 \). Also suppose either

(i) \( T \) is continuous or

(ii) \( X \) has the following property :

if a non-decreasing sequence \( x_n \rightarrow x \), then \( x_n \leq x \ \forall n \).

If there exists an \( x_0 \in X \) with \( x_0 \leq T(x_0) \) then have a unique fixed point \( x \in X \). Moreover, \( p(x, x) = 0 \).

Proof. In Theorem 2, taking \( \phi(x, y) = \theta(x + y) \) for all \( x, y \in [0, \infty) \), we get Corollary 2.

Corollary 3. Let \( (X, \leq) \) be a partially ordered set and suppose there is a total ordering \( p \) on \( X \) such that \( (X, p) \) is a complete partial metric space. Assume there is a continuous function \( \varphi : [0, \infty) \rightarrow [0, \infty) \) with \( \varphi(t) < t \) for each \( t > 0 \) and suppose \( T : X \rightarrow X \) be a non-decreasing function for all comparable \( x, y \in X \), we have

\[
\psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(p(x, y), p(x, Tx)),
\]

where

\[
M(x, y) = \max\{\varphi(p(x, y)), \varphi(p(x, Tx)), \varphi(p(y, Ty)), \\
\varphi\left(\frac{p(x, Ty) + p(y, Tx)}{2}\right)\},
\]

\( \psi \in \Psi(0, \infty) \) and \( \varphi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \) is continuous mapping such that \( \varphi(x, y) = 0 \) if and only if \( x = y = 0 \). Also suppose either

(i) \( T \) is continuous or

(ii) \( X \) has the following property :

if a non-decreasing sequence \( x_n \rightarrow x \), then \( x_n \leq x \ \forall n \).

If there exists an \( x_0 \in X \) with \( x_0 \leq T(x_0) \) then have a unique fixed point \( x \in X \). Moreover, \( p(x, x) = 0 \).

Remark. The following condition

\[
\psi(p(Tx, Ty)) \leq \psi\left(\varphi\left(\frac{\max\{p(gx, gy), p(gx, Tx), p(gy, Ty), \\
p(gx, Ty) + p(gy, Tx)/2\}}{2} \right)\right) \\
- \varphi(p(gx, gy), p(gx, Tx)),
\]

implies condition (4). We observe also that condition (31) is equivalent to condition (4) if we suppose that \( \varphi \) is a non-decreasing function.

References


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