A novel numerical algorithm for solving optimal control of steady state diffusion equation

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Abstract: In this paper, a useful and efficient computational optimal control method is proposed where, the diffusion equation stands as the constraint. First, we use Galerkin finite element method for solving underlying PDE. second, by introducing L-curve method for solving ill-posed system of equations and using inverse optimization technique, control values as the right hand side vector of the system can be computed such that, the optimality of this control values is ensured by best approximating the desired function by corresponding state values at mesh points. Numerical experiments are presented for illustrating the theoretical results.

Keywords: Elliptic equation, optimal control problem, finite element method, Newton’s conjugate gradient method.

1 Introduction

Many applications in science and engineering give rise to problems formulated as PDE-constrained optimization, in which, partial differential equations stands as the constraints. This kind of problems arise in such diverse areas as environmental engineering [1, 2], mathematical finance [4,5,6], medicine [3, 17] and aerodynamics [23]. Generally in nature, PDE-constrained optimization problems are large, complex and infinite dimensional. Since we consider the problems that discretization does not lead non-differentiable components, our intention is to work with discrete then optimization (DO) algorithms for smooth functions (In contrast with optimize then discretize (OD) algorithms). Therefore, we only consider problems in which the DO approach can be taken. There are two strategy in solving these kind of optimization problems. First strategy attempts to find optimality by solving a large system of equations, namely, Saddle point system. This system comes from utilizing KKT condition on the Lagrangian augmented function and since the system can be so large, it is important to use a good preconditioner to solve the governed system. Second strategy attempts to utilize the Gradient and Hessian properties and find the iterative Gradient based method. Each of the above-mentioned methods uses the fix regularization parameter and try to find optimal control and corresponding state directly. In this paper, by using inverse optimization strategy, we find optimal regularization parameter and corresponding optimal control and state, which is more accurate than the two aforementioned strategies. The organization of this paper is as follows. In Section 2, problem is defined and the two last strategies in addition to the L-curve method is reviewed. In Section 3, novel method is discussed. In Section 4, some numerical results are reported.

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2 Problem Definition

We consider optimal control problems of the form

$$\min_{y \in Y, u \in U} J(y, u) \quad \text{subject to } e(y, u) = 0, \quad (y, u) \in W_{ad}$$

(1)

where $J: Y \times U \to R$ is the cost or objective function, $e: Y \times U \to Z$ is an operator between Banach spaces, and $W_{ad} \subset W := Y \times U$ is a nonempty closed set. Existence and uniqueness of the solution to these problems are ensured by implicit function theorem [24].

It is assumed that $J$ and $e$ are continuously F-differentiable and that for each $u \in U$ the state equation $e(y, u) = 0$ possesses a unique corresponding solution $y(u) \in Y$. Thus, we have a solution operator $u \in U \to y(u) \in Y$. In addition, it is assumed that $e_y(y(u), u) \in L(Y, Z)$ is continuously invertible. Then, the continuously differentiability of $y(u)$ is ensured by implicit function theorem. Inserting $y(u)$ in (1) yields the following reduced problem:

$$\min_{u \in U} \tilde{J}(u) := J(y(u), u) \quad \text{subject to } u \in \bar{U}_{ad} := \{u \in U: (y(u), u) \in W_{ad}\}$$

(2)

2.1 Gradient based iterative method

The derivative $\tilde{J}'(u)$ can be computed via the adjoint approach by taking as follows [15, 16]:

$$L(y, u, p) = J(y, u) + \langle p, e(y, u) \rangle_{U^*, U}$$

(3)

Algorithm 1 (Derivative computation via adjoints)
1. Solve $e(y, u) = 0$ for $y = y(u)$.
2. Solve $e_y(y(u), u)^* p = -J_y(y(u), u)$ for $p = p(u)$.
3. Compute $\tilde{J}'(u) = e_u(y(u), u)^* p + J_u(y(u), u)$

Algorithm 2 (Hessian times a vector computation)
1. Solve $e(y, u) = 0$ for $y = y(u)$.
2. Solve $e_y(y(u), u)^* p = -J_y(y(u), u)$ for $p = p(u)$.
3. Solve $e_y(y(u), u)w = e_u(y(u), u)^* v$ for $w = w(z, v)$.
4. Solve $e_y(y(u), u)^* q = L_{yy}(y(u), u, p(u)) = v$ for $q = q(z, v)$.
5. Compute $\tilde{J}''(u)v = e_u(y(u), u)^* q - L_{yy}(y(u), u, p(u))w + L_{uu}(y(u), u, p(u))v$

These algorithms may be used to investigate iterative solvers to the following Newton’s equation

$$\tilde{J}''(u_k)s_k = -\tilde{J}'(u_k).$$

(4)

Now, using the conjugate gradient (CG) method, the Newton equation (4) is solved approximately. When we access to the sufficiently small residual of Newton's system, the CG method is truncated. In real world computation, some globalization technique for Newton's method should be employed. Considering the case of implementation and relatively low computational cost, line search techniques are popular choices in this way. Finding an optimal step size $\alpha^k$ and using this step size to generate the $u^{k+1} = u^k + \alpha^k S_k$ is the mission of a line search algorithm. The Armijo condition (or sufficient decrease condition), that the step size is required to satisfy, is
\[ J(u^k + \alpha^k S^k) \leq J(u^k) + c\alpha^k J'(u^k) S^k \]

Which \( c \in (0,1) \) and is typically quite small, e.g. \( c = 10^{-4} \) [17, 20].

### 2.2 Saddle-point system based method

Differentiation of \( L \) in (3) with respect to \( y, u \) and \( p \) yields

\[
\begin{bmatrix}
L_u(y, u, p) \\
L_y(y, u, p) \\
L_p(y, u, p)
\end{bmatrix} = \begin{bmatrix}
J_u(y, u, p) + pe_u \\
J_y(y, u, p) + pe_y \\
e(y, u, p)
\end{bmatrix} = 0,
\]

(5)

In practice, discretization of system (5) performs a large saddle point system of equations, that is not easy to solve. To overcome this difficulty, several preconditioner were presented in the literature to solve this large system of equations [21, 22].

### 2.3 L-curve method

Consider the following discrete ill-posed linear systems of equations

\[ Ax = b, \ A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n, b \in \mathbb{R}^m \]

(6)

In many such problems, a right-hand side vector \( b \) is contaminated by an error \( e \in \mathbb{R}^m \) which may come from measurement inaccuracies or discretization. Let

\[ b = \bar{b} + e \]

(7)

Where, \( \bar{b} \) is the unavailable error-free representative of \( b \). Our interest is to determine a solution \( \bar{x} \) for the following error-free linear system of equation

\[ A\bar{x} = \bar{b} \]

(8)

Since \( \bar{b} \) is not available, by computing an approximation \( \bar{x} \) of the available linear system (6), an approximate solution of \( \bar{x} \) can be determined. Whenever we could replace the linear system (6) by an equivalent system that is less sensitive to perturbations of the right hand side vector, then, the solution of such system can be considered as a meaningful approximation of \( \bar{x} \). This replacement usually referred to regularization. Here, we use Tikhonov regularization as the most popular regularization method [7, 8, 10, 11]. In Tikhonov regularization, the solution of the linear system (6) is replaced by the following minimization problem:

\[ \min_{x \in \mathbb{R}^n} \left( \|Ax - b\|^2 + \lambda \|x\|^2 \right) \]

(9)

For a suitable positive value of the regularization parameter \( \lambda \). Here \( \| \| \) refers to the Frobenius norm of a vector or matrix. For any \( \lambda > 0 \), from the norm definition, problem (9) has the unique solution [11]

\[ x_\lambda := (A^T A + \lambda^2 I)^{-1} A^T b \]

(10)

Denoting the singular value decomposition of \( A \) by \( A = U \Sigma V^T \), the regularized solution \( x_\lambda \) is given by

\[ x_\lambda = \sum_{i=1}^{n} f_i \frac{(u_i^T b)}{\sigma_i} v_i, \quad f_i = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \]

Here, the left and right singular vectors \( u_i \) and \( v_i \) are orthonormal, the singular values \( \sigma_i \) are non-increasing and nonnegative numbers and the norms of the solution and residual for \( x_\lambda \) are given respectively by [4]

\[ \|x_\lambda\|^2 = \sum_{i=1}^{n} f_i^2 \left( \frac{(u_i^T b)^2}{\sigma_i^2} \right) \]

(11)

\[ \|r_\lambda\|^2 = \|b - Ax_\lambda\|^2 = \sum_{i=1}^{n} (1 - f_i)^2 \left( \frac{(u_i^T b)^2}{\sigma_i^2} \right) \]

(12)

Tikhonov regularization, in fact, looks for some \( x_\lambda \), so that provides a moderate value of the penalty term \( \|x_\lambda\| \) as well as a small residual \( \|r_\lambda\| \) simultaneously. It is easily seen that a suitable regularization parameter...
should properly balance between $\|r_1\|$ and $\|x_2\|$. So, for a tradeoff between these two quantities, the $(\|r_1\|, \|x_2\|)$ curve, that is named as the L-curve is naturally used, where the name L-curve comes from the characteristic shape of this curve. If $\lambda$ becomes too large, then $\|r_1\|$ is sensitive and has large rate of growth and $\|x_2\|$ is relatively constant, thus, essentially the curve is a horizontal line. Conversely, if the regularization parameter is chosen too small, then $\|r_1\|$ is small too, but $\|x_2\|$ can be very sensitive and fast changing and essentially the curve is a vertical line. Such a curve has a characteristic “L” shape. The corner of the L-curve is the transition between these two region, while, the associated value of $\lambda$ at this corner is the optimal value of the regularization parameter [12]. Using such curves goes back to Lawson and Hanson [13,14,18] and Miller [19].

Algorithm 3 (Find corner)
1. Determine a few points $(\hat{P}_i, \hat{x}_i)$ on each side of the corner.
2. By replacing each point with a new point which is obtained by fitting a low-degree polynomial to a few neighboring points, Perform a local smoothing of the L-curve points, in which
3. Utilize the new smoothed points in step 2 to construct the cubic spline curve $S$ for the points $(\hat{P}_i, \hat{x}_i, \lambda_i)$, $i = 1, ..., N + 4$, where $N$ is the number of L-curve points and where $\lambda_i$ is the regularization parameter that corresponds to $(\hat{P}_i, \hat{x}_i)$.
4. Let $S_2$ be the approximation of the L-curve with the first two coordinates of $S$.
5. Compute the point that has the maximum curvature on $S_2$ and find the corresponding $\lambda_0$.
6. By solving the regularization problem for $\lambda_0$, add the new point $(\hat{P}_d, \hat{x}_d)$ to the L-curve.
7. Go to Step 2 until convergence.

By choosing largest and smallest regularization parameters, initial points for step 1 can be generated. For example, $\lambda$ equal to $\sigma_1, 0.1 \sigma_1, 10 \sigma_n$ and $\sigma_n$. These initial points may be far from the corner, thus, it could be suitable to introduce a temporary point $\left(\min_{i} \hat{P}_i, \min_{i} \hat{x}_i \right)$, between the points corresponding to largest and smallest $\lambda$. After computation of first iteration, this temporary point is replaced by the first L-curve point $(\hat{P}_d, \hat{x}_d)$.

3 Novel Method

For the linear state equation $e(y,u)$ and quadratic objective function $J(y,u)$, optimization problems in the form of Eq. (1) can be converted to a linear-quadratic optimization problem of the form [9, 21]:

$$\begin{align*}
\min_{(y,u) \in Y \times U} J(y,u) := & \frac{1}{2} \|y - y_d\|_H^2 + \frac{a}{2} \|u\|_{V_d}^2 \\
\text{s.t.} \quad & By = Nu + l, \quad u \in U_{ad}, \quad y \in Y_{ad}
\end{align*}$$

Here $H,U$ are Hilbert spaces, $Y,Z$ are Banach spaces and

$$y_d \in H, \quad l \in Z, \quad B \in \mathcal{L}(Y,Z), \quad N \in \mathcal{L}(U,Z).$$

3.1 Discretization

Now, we start to discretize the linear state equations in the form

$$-\nabla(a\nabla y) = u \quad \text{on} \quad \Omega \quad (14)$$

$$y = g \quad \text{on} \quad \partial \Omega$$

4
which $\Omega \in R^2$ is the domain and $H = U = L^2(\Omega)$.

We require the weak formulation of eq. (14) in order to use the finite element method. So, the weak problem can be introduced as: find $y \in H^1_0(\Omega) = \{ u : u \in H^1(\Omega), \ u = g \text{ on } \partial \Omega \}$ such that

$$\int_{\Omega} a \nabla y \cdot \nabla v = \int_{\Omega} v u \quad \forall v \in H^1_0(\Omega)$$  \hspace{1cm} (15)

It is assumed that $V^h_0 \in H^1_0$ is an n-dimensional vector space of test functions, where $\{ \phi_1, ..., \phi_n \}$ stands for the basis. The basis is extended by defining functions $\phi_{n+1}, ..., \phi_{n+\partial n}$ and coefficients $Y_j$ to satisfy the boundary conditions, such that $\sum_{j=n+1}^{n+\partial n} Y_j \phi_j$ interpolates the boundary data. Thus, if $y_h \in V^h_0 \subset H^1_0(\Omega)$, then, $y_h$ is uniquely determined by $y = (Y_1, ..., Y_n)^T$ by

$$y_h = \sum_{j=1}^{n} Y_j \phi_j + \sum_{j=n+1}^{n+\partial n} Y_j \phi_j$$

In the above equations, functions $\phi_i$, $i = 1, ..., n$, define a set of shape functions. Also, it is assumed that this approximation is conforming, i.e. $V^h_0 = span \{ \phi_1, ..., \phi_{n+\partial n} \} \subset H^1_0(\Omega)$. So, the finite-dimensional analogue of (15) can be written as: Find $y_h \in V^h_0$ such that

$$\int_{\Omega} a \nabla y_h \cdot \nabla v = \int_{\Omega} v_h u \quad \forall v_h \in V^h_0.$$

Now, as it appears in (14), we need to discretize $u$. We do this by using the same basis used for $y$, hence,

$$u_h = \sum_{j=1}^{n} U_j \phi_j$$

Without loss of generality, it is usual to take $u_h = 0$ on $\partial \Omega$. Thus the discrete analogue of minimization problem can be written as

$$\min_{y_h \in V^h_0} \frac{1}{2} \| y_h - \tilde{y} \|_2^2 + \alpha \| y_h \|_2^2$$  \hspace{1cm} (16)

s.t. $\int_{\Omega} a \nabla y_h \cdot \nabla v_h = \int_{\Omega} v_h u_h \quad \forall v_h \in V^h_0$  \hspace{1cm} (17)

The discrete cost functional can be written as

$$\min_{y_h, u_h} \frac{1}{2} \| y_h - \tilde{y} \|_2^2 + \alpha \| y_h \|_2^2 = \min_{y,u} \frac{1}{2} y^T My - y^T b + \beta + \alpha u^T M u$$  \hspace{1cm} (18)

where $y = (Y_1, ..., Y_n)^T$, $u = (U_1, ..., U_n)^T$, $b = \{ \int \phi_i \}_{i=1,...,n}$, $\beta = \| \tilde{y} \|_2^2$ and $M = \{ \int \phi_i \phi_j \}_{i,j=1,...,n}$ is the mass matrix.

Now, we turn our attention to the constraint equation (17) that is equivalent to finding $y$ such that
\[ \int_\Omega a \nabla ( \sum_{i=1}^{n} Y_i \phi_i ) \cdot \nabla \phi_j = \int_\Omega \nabla ( \sum_{i=n+1}^{n+n} Y_i \phi_i ) \cdot \nabla \phi_j = \int_\Omega ( \sum_{i=1}^{n} U_i \phi_i ) \phi_j \quad j = 1, ..., n \]

which is

\[ \sum_{i=1}^{n} Y_i \int_\Omega \nabla \phi_i \cdot \nabla \phi_j = \sum_{i=1}^{n} U_i \int_\Omega \phi_i \phi_j - \sum_{i=n+1}^{n+n} Y_i \int_\Omega \nabla \phi_i \cdot \nabla \phi_j \quad j = 1, ..., n \]

or

\[ Ay = Mu + d \quad (19) \]

Here, \( A = \{ \int \nabla \phi_i \cdot \nabla \phi_j \}_{i,j=1,...,n} \) is the discrete Laplacian matrix (the stiffness matrix) and \( d \) contains information about the boundary values of \( y_h \). Thus, solving (18) and (19) together is equivalent to solving (16) and (17). So, the final problem may be represented as

\[ \min_{y,u} \quad \frac{1}{2} \| y - \hat{y} \|^2 + \alpha \| u \|^2 \]

S.t. \( Ay = Mu + d \)

\( \hat{y} \) is a vector of the desired function values correspond to discretization points. Now, the aim is to find the optimal values of control vector \( u \), in which, the corresponding state vector \( y \) be as close as possible to desired vector \( \hat{y} \). It is possible to find optimality condition by inverse technique. Consider the constraints as:

\[ Ay = Mu + d \Rightarrow y = A^{-1} (Mu + d) \]

Our interest is to find \( y \) to be as close as possible to \( \hat{y} \), so

\[ A^{-1} (Mu + d) \equiv \hat{y} \]

In fact, we have \( y = \hat{y} + e \), in which \( e \) is the vector of error noise. Now, taking \( V = Mu + d \) and \( R = A^{-1} \) we have

\[ RV = \hat{y} + e \]

which is equals to

\[ (R + \alpha I)V = \hat{y} \]

Similar to Eq. (9), solving this system of equations is equivalent to solving the following regularization problem (9)

\[ \min_{V} \quad \| RV - \hat{y} \|^2 + \alpha \| V \|^2 \quad (20) \]

Since \( V = Mu + d \), thus the above minimization problem equals to

\[ \min_{u} \quad \| RV - \hat{y} \|^2 + \alpha \| u \|^2 \]

As soon as we find the regularization parameter and its corresponding vector \( V \) in (20), in fact we find the optimal control \( u = M^{-1} (V - d) \) and \( y = RV \) which simultaneously found with regularization parameter. Therefore, finding optimal regularization parameter, as well as, optimal control values for the original problem can be provided by solving the above Tikhonov regularization problem. By using the L-curve method in section 2.3, because of its computational advantages [14], these quantities can be computed.
4 Numerical results

In the following examples the, the accuracy of the novel method in comparison with other two methods is illustrated. Table 1, 2, 3 illustrate the regularization parameter computed by L-curve method and corresponding state is compared with desired value via different norms. Two other methods with different regularization parameters are solved numerically. Comparisons show the accuracy of the novel method. Figures 1, 2, 3 illustrates the numerical solutions of state via the proposed method, saddle point system and iterative methods for different values of regularization parameters in comparison with desired function. In tables 1, 2, 3, the accuracy of the proposed method is compared with other two method, where $\|y - y_d\|_r$ shows the average error norm in each mesh point and computed by $\|y - y_d\|_r = \frac{\|y - y_d\|_r}{Number \ of \ mesh \ points}$.

Example 1. Let $\Omega = [-1, 1]^2$, and consider the problem

$$
\min_{y,u} \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \alpha \|u\|_{L^2(\Omega)}^2
$$

$$
s.t. \quad -\Delta y = u \quad \text{in} \quad \Omega
$$

$$
y = y_d \quad \text{on} \quad \partial \Omega
$$

where $y_d = \sin(\pi x) \sin(\pi y)$. The problem is discretized by 50 ×50 mesh points. In table 1, the accuracy of the New method in comparison with other two methods is illustrated. Taking $\alpha = 0.000005$, and using 50 discretization point in each direction, computed optimal state are shown in Fig. 1.

Table 1. The accuracy of the proposed method in comparison with other methods in Example 1.

<table>
<thead>
<tr>
<th>Method</th>
<th>Regularization Parameter</th>
<th>$|y - y_d|_1$</th>
<th>$|y - y_d|_2$</th>
<th>$|y - y_d|_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current method</td>
<td>0.375604</td>
<td>0.000055</td>
<td>0.000001</td>
<td>0.000000</td>
</tr>
<tr>
<td>Saddle point</td>
<td>0.05</td>
<td>0.237115</td>
<td>0.007508</td>
<td>0.000384</td>
</tr>
<tr>
<td>Iterative Method</td>
<td>0.05</td>
<td>0.236884</td>
<td>0.007508</td>
<td>0.000384</td>
</tr>
<tr>
<td>Saddle point</td>
<td>0.005</td>
<td>0.171040</td>
<td>0.005416</td>
<td>0.000277</td>
</tr>
<tr>
<td>Iterative Method</td>
<td>0.005</td>
<td>0.175828</td>
<td>0.005868</td>
<td>0.000286</td>
</tr>
<tr>
<td>Saddle point</td>
<td>0.0000005</td>
<td>0.125967</td>
<td>0.003988</td>
<td>0.000204</td>
</tr>
<tr>
<td>Iterative Method</td>
<td>0.0000005</td>
<td>0.126298</td>
<td>0.003996</td>
<td>0.000210</td>
</tr>
</tbody>
</table>
Fig. 1. Numerical solution with $y_d = \sin(\pi x) \sin(\pi y)$ and $\alpha = 0.000005$. 50 × 50 mesh points.

**Example 2.** Let $\Omega = [-1, 1]^2$, and consider the optimal control problem for different values of $\alpha$ and $y_d = 1.1 \text{sign}(\sin(\pi x) \sin(\pi y))$. The problem is discretized by 30 × 30 mesh points.

**Table 2.** The accuracy of the proposed method in comparison with other methods in Example 2.

| Method        | Regularization Parameter | $A v ||_1$ | $A v ||_2$ | $A v ||_\infty$ |
|---------------|--------------------------|------------|------------|-----------------|
| Current method| 0.376147                 | 0.052246   | 0.002196   | 0.000255        |
| Saddle point  | 0.05                     | 0.491088   | 0.020582   | 0.001218        |
| Iterative Method | 0.05                  | 0.490672   | 0.020570   | 0.001218        |
| Saddle point  | 0.005                    | 0.371512   | 0.017413   | 0.001212        |
| Iterative Method | 0.005                | 0.370485   | 0.017387   | 0.001220        |
| Saddle point  | 0.000005                 | 0.501938   | 0.017890   | 0.001010        |
| Iterative Method | 0.000005              | 0.483347   | 0.018080   | 0.001101        |

Fig. 2. Numerical solution with $y_d = 1.1 \text{sign}(\sin(\pi x) \sin(\pi y))$ and $\alpha = 0.000005$. 30 × 30 mesh points.
Example 3. Let $\Omega = [-1,1]^2$, and consider the optimal control problem for different values of $\alpha$ and $y_d = \exp[-64*\left(\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2\right)]$. The problem is discretized by 50 $\times$ 50 mesh points.

Table 3. The accuracy of the proposed method in comparison with other methods in Example 3.

<table>
<thead>
<tr>
<th>Method</th>
<th>Regularization Parameter</th>
<th>$|A|\cdot|1|$</th>
<th>$|A|\cdot|2|$</th>
<th>$|A|\cdot|\infty|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current method</td>
<td>0.375383</td>
<td>0.010112</td>
<td>0.000411</td>
<td>0.000026</td>
</tr>
<tr>
<td>Saddle point</td>
<td>0.05</td>
<td>0.014543</td>
<td>0.001501</td>
<td>0.000389</td>
</tr>
<tr>
<td>Iterative Method</td>
<td>0.05</td>
<td>0.015969</td>
<td>0.001497</td>
<td>0.000388</td>
</tr>
<tr>
<td>Saddle point</td>
<td>0.005</td>
<td>0.018534</td>
<td>0.001380</td>
<td>0.000365</td>
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<tr>
<td>Iterative Method</td>
<td>0.005</td>
<td>0.016485</td>
<td>0.001407</td>
<td>0.000372</td>
</tr>
<tr>
<td>Saddle point</td>
<td>0.0000005</td>
<td>0.007965</td>
<td>0.000635</td>
<td>0.000080</td>
</tr>
<tr>
<td>Iterative Method</td>
<td>0.000005</td>
<td>0.029347</td>
<td>0.001156</td>
<td>0.000232</td>
</tr>
</tbody>
</table>

Fig. 3. Numerical solution with $y_d = \exp[-64*\left(\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2\right)]$ and $\alpha = 0.00005$. 50 $\times$ 50 mesh points.

5 Conclusion

Optimization problems with constraints which require the solution of the partial differential equation arise widely in many areas of the science and engineering, in particular in problems of design. The solution of such PDE-constrained optimization problems is usually a major computational task. Here, by considering distributed control problems, where 2D steady state diffusion equation is the PDE, based on Tikhonov regularization and L-curve methods, a useful and efficient computational optimal control is proposed, which, in contrast with the other methods, computes the regularization parameter as a part of the problem. This optimal value calculation for the regularization parameter in conjunction with the use of finite element method and L-curve technique advantages to gain more efficient solution as it is shown by the two numerical examples.
6 references