Journal of Statistics Applications & Probability
*An International Journal*

http://dx.doi.org/10.18576/jsap/050109

# Inferential Problem about Homogeneity of Several Systems under Frechet Distribution

*D. P. Raykundaliya**

Department of Statistics, Sardar Patel University, Vallbh Vidyangar - 388120, Gujarat - India

Received: 6 Jul. 2015, Revised: 4 Jan. 2016, Accepted: 8 Jan. 2016
Published online: 1 Mar. 2016

**Abstract:** In this paper, we consider estimation of unknown parameters of the $'m'$ Frechet distributions using the generalized Type II censoring scheme. We obtain maximum likelihood estimators of the unknown parameters. As Likelihood equations are not mathematical tractable we use iterative procedure to obtain estimate of parameters and demonstrate their performance using Monte-Carlo simulation. Further, likelihood ratio test is discussed to test homogeneity of several scale parameters

**Keywords:** Generalized Type II censoring scheme, Frechet distribution, Newton-Raphson method, ML estimation, Monte-Carlo Simulation technique, likelihood ratio test

## 1 Introduction

In reliability and life testing experiments, items are kept simultaneously on experiment and observed the specified number of failures, such scheme is known as Type II censoring scheme. The purpose of this scheme is to study the performance of any non living product through their survival time, consistency with respect to operating condition etc in minimum experimental time and budget. In literature, several authors [1,3,4,5,6,7] have studied Type II censoring with various lifetime distributions such as normal, exponential, Webull, gamma and Frechet. In the era of globalization, several manufacturing industries are producing same kind of product for specific operations. Therefore, it is necessary to produce and deliver reliable product to customers for remain in business. For establishing such high standards, the problem of comparing effectiveness of products is important. In this situation, after placing several independent samples of units manufactured by the several processes, the reliability engineer would like to make early and efficient decision on the effectiveness of the products under the life test in terms of standard hazard rate function. The extensive study for the problem of comparing two populations in terms of stochastic ordering is discussed in [2]. The distribution free test for comparison of hazard rates of two distributions under Type II censoring is given in [11]. The study of inferential problem about homogeneity of several systems under generalized inverted family of distributions and generalized exponential distribution respectively when observations are subject to the generalized Type II censoring discussed in [9,10]. Further, they have studied the cost effectiveness of experiments through simulation.

Due to the extreme events happening in manufacturing industries and nature, recently, researchers have focused on study of extreme value distributions for better planning purpose. The Frechet (extreme value type II) distribution is one of the probability distributions used to model extreme events. It has many applications like in earthquakes, flood, queues in supermarkets; wind speeds etc. For more detail One can refer [8].

In this paper, we discuss inferential problem about homogeneity of several systems under Frechet distribution when observations are subject to generalized Type II censored and further, study the reliability characteristics of distributions. We now consider a design, where we put '$m$' types of systems simultaneously on test in which for each type of systems we start with '$u$' units and continue the experiment till $G^*$ failures are observed i.e. the total numbers of units put on test are '$mu$' and the total number of failures we observe at the end of experiment are "$G = mG^*$." Assuming that the lifetime distribution of unit for each type of systems to be Frechet with shape parameter $\alpha$ and scale parameters $\beta_i; i = 1,2,...,m$. In the experiment after each failure the failure time is observed, and denoted it by $t_{gi}; g = 1,2,,G^*; i = 1,2,...,m$. At the end of experiments, we have data $(u,G,t_{gi}; g = 1,2,...,G^*; i = 1,2,...,m)$. The organization of whole paper is as below.

* Corresponding author e-mail: dp_raykundaliya@spuvvn.edu

In Section 2 we give the probability density function, the survival function and the hazard rate of the Frechet distribution and develop the likelihood for generalized Type II censored sampling design under generalized exponential distribution. In Section 3 we derive the expressions for maximum likelihood estimators of parameters and their asymptotic variance-covariance matrix when shape parameter of the distribution is known and when it is unknown. Section 4 discusses iterative procedure for estimation of the parameters through Newton-Raphson method. Further, the tables of ML estimates and their asymptotic standard errors, estimate of reliability and hazard rates and their mean square error at fixed time point are given which are simulated through Monte-Carlo simulation technique for the case of known shape parameter. In Section 5 we discuss likelihood ratio test for simultaneous testing of homogeneity of scale parameters when the shape parameter is known. The cut-off points for the test statistics are obtained through Monte-Carlo simulation. Some concluding remarks are given in Section 6.

## 2 Frechet Distribution and Likelihood Function for the Generalized Type II Censoring Design

Consider an item whose life time is denoted by $T$. The random variable $T$ is assumed to have Frechet distribution with distribution function

$$F(t;\alpha,\beta)=\exp[-(\frac{\beta}{t})^{\alpha}],(t>0,\alpha>0,\beta>0). \tag{1}$$

The corresponding density function is given by

$$f(t;\alpha,\beta)=\frac{\alpha}{\beta}(\frac{\beta}{t})^{\alpha+1}\exp[-(\frac{\beta}{t})^{\alpha}]. \tag{2}$$

Here $\alpha$ is a shape parameter, $\beta$ is a scale parameter. Then the reliability function is

$$\hat{F}(t)=P(T>t)=1-\exp[-(\frac{\beta}{t})^{\alpha}] \tag{3}$$

and the hazard function is

$$h(t)=\frac{f(t)}{F(t)}$$

$$h(t)=\frac{\frac{\alpha}{\beta}(\frac{\beta}{t})^{\alpha+1}\exp[-(\frac{\beta}{t})^{\alpha}]}{1-\exp[-(\frac{\beta}{t})^{\alpha}]}. \tag{4}$$

If $Z$ follows $Frechet(\alpha,1)$, then the corresponding $k$-th raw moment, is given by

$$\mu_k=\Gamma(1-\frac{k}{\alpha}) \tag{5}$$

Therefore, the mean and variance of $Z$ is

$$E(Z)=\Gamma(1-\frac{1}{\alpha})\text{ and }var(Z)=\Gamma(1-2/\alpha)-(\Gamma(1-\frac{1}{\alpha}))^{2}. \tag{6}$$

If $Z$ follows $Frechet(\alpha,1)$ and $T=\frac{1}{\beta}Z$ then $T$ follows $Frechet(\alpha,\beta)$. Therefore, the mean and variance of $T$ is given by

$$E(T)=\beta\Gamma(1-\frac{1}{\alpha})\text{ and }var(T)=\beta^2\{\Gamma(1-2/\alpha)-(\Gamma(1-\frac{1}{\alpha}))^{2}\}. \tag{7}$$

The likelihood function for Type II censoring design for $i$-th type of systems observing $G^*$ failures from $u$ units given in literature as

$$L_i=\frac{u!}{(u-G^*)!}\prod_{g=1}^{G^*}f_i(t_g)[\bar{F_i}(t_{G^*})]^{(u-G^*)} \tag{8}$$

Therefore, the likelihood for whole experiments

$$\begin{aligned}L&=\prod_{i=1}^{m}L_i\\&=\prod_{i=1}^{m}\{\frac{u!}{(u-G^*)!}\prod_{g=1}^{G^*}f_i(t_g)[\bar{F_i}(t_{G^*})]^{(u-G^*)}\}.\end{aligned} \tag{9}$$

Substitute the equations (2),(3) in equations (9) we have, the likelihood function

$$L=\left[\frac{u!}{(u-G^*)!}\right]^m [\prod_{i=1}^m (\frac{\alpha}{\beta_i})^{G^*}][\prod_{i=1}^m \prod_{g=1}^{G^*} (\frac{\beta_i}{t_{gi}})^{\alpha+1} \exp[-(\frac{\beta_i}{t_{gi}})^{\alpha}]]$$
$$\prod_{i=1}^m [1-\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]]^{u-G^*}. \tag{10}$$

## 3 Maximum Likelihood Estimation

In this section we obtain maximum likelihood estimates of $\alpha, \beta_i (i=1,2,...,m)$, reliability function, hazard rate and observed information matrix under the design. The log likelihood equation of (10) would be

$$l=mln(\frac{u!}{(u-G^*)!})+mG^* ln\alpha+G^*\alpha\sum_{i=1}^m ln\beta_i-(\alpha+1)\sum_{i=1}^m\sum_{g=1}^{G^*} ln(t_{gi})$$
$$-\sum_{i=1}^m\sum_{g=1}^{G^*}(\frac{\beta_i}{t_{gi}})^{\alpha}+(u-G^*)\sum_{i=1}^m ln[1-\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]]. \tag{11}$$

Differentiate (11) with respect to $\alpha$ and $\beta_i (i=1,2,...,m)$ we have

$$\frac{\partial l}{\partial\alpha}=\frac{mG^*}{\alpha}+G^*\sum_{i=1}^m ln\beta_i-\sum_{i=1}^m\sum_{g=1}^{G^*} ln(t_{gi})$$
$$-\sum_{i=1}^m\sum_{g=1}^{G^*}(\frac{\beta_i}{t_{gi}})^{\alpha} ln(\frac{\beta_i}{t_{gi}})+(u-G^*)\sum_{i=1}^m \frac{(\frac{\beta_i}{t_{G^*i}})^{\alpha} ln(\frac{\beta_i}{t_{G^*i}})\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]}{1-\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]} \tag{12}$$

$$\frac{\partial l}{\partial\beta_i}=\frac{G^*\alpha}{\beta_i}-\alpha\sum_{g=1}^{G^*}(\frac{\beta_i}{t_{gi}})^{\alpha-1}(\frac{1}{t_{gi}})+\frac{\alpha(u-G^*)}{t_{G^*i}}\frac{(\frac{\beta_i}{t_{G^*i}})^{\alpha-1}\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]}{1-\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]}. \tag{13}$$

The estimates of parameters $\underline{\beta}$ are obtained in two cases when (i) shape parameter $\alpha$ is known and (ii) shape parameter $\alpha$ is unknown.

### *3.1 Maximum Likelihood Estimation When Shape Parameter is Known*

The solution of equations (13) can be evaluated numerically by some suitable iterative procedure such as Newton-Raphson method, for given values of $(u,G,t_{gi};g=1,2,...,G^*;i=1,2,...,m)$. The MLE of $\underline{\beta}=(\beta_1,\beta_2,...,\beta_m)$ are obtained as $\hat{\underline{\beta}}$ from equations (13). The MLEs of reliability $(\bar{F}(t_i);i=1,2,...,m)$ and hazard rate $(h_i(t_i);i=1,2,...,m)$ can be evaluated using invariance property of MLEs as

$$\hat{\bar{F}}_i(t_i)=1-\exp[-(\frac{\hat{\beta}_i}{t_i})^{\alpha}] \tag{14}$$

and the hazard function is

$$\hat{h}_i(t_i)=\frac{\frac{\alpha}{\hat{\beta}_i}(\frac{\hat{\beta}_i}{t_i})^{\alpha+1}\exp[-(\frac{\hat{\beta}_i}{t_i})^{\alpha}]}{1-\exp[-(\frac{\hat{\beta}_i}{t_i})^{\alpha}]}. \tag{15}$$

#### 3.1.1 Observed Fisher Information Matrix Under Design

To obtain Fisher information matrix we take derivatives of equations (13) with respect to $\beta_i; i=1,2,....,m$. Therefore, we have,

$$\frac{\partial^2 l}{\partial\beta_i^2}=-\frac{G^*\alpha}{\beta_i^2}-\alpha(\alpha-1)\sum_{g=1}^{G^*}\frac{1}{t_{gi}{}^2}(\frac{\beta_i}{t_{gi}})^{\alpha-2}+\frac{\alpha(u-G^*)(\frac{\beta_i}{t_{G^*i}})^{\alpha-2}\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]}{t_{G^*i}{}^2}$$
$$\times\{\frac{(\alpha-1)[1-\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]]-\alpha(\frac{\beta_i}{t_{G^*i}})^{\alpha}}{[1-\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]]^2}\}. \tag{16}$$

As rate of failures of systems are independent of each type of systems, derivatives of equations (13) with respect to $\beta_j; j \neq i = 1,2,...,m$ are

$$\frac{\partial^2 l}{\partial\beta_i\partial\beta_j} = 0.\ \forall j \neq i = 1,2,...,m. \quad (17)$$

We know that variance of Frechet distribution exists when $\alpha > 2$.Therefore, we have the following results.

**Theorem 3.1.** For $\alpha > 2$ and $\frac{G^*}{u}$ kept constant the maximum likelihood estimators, $\underline{\hat{\beta}}$ of $\underline{\beta}$ are consistent estimators, and $\sqrt{u}(\underline{\hat{\beta}} - \underline{\beta})$ is asymptotically $m$-variate normal with mean $\underline{0}$ and variance covariance matrix $\mathbf{V^{-1}}$, where $\mathbf{V}$ is expected value of negative of second derivative matrix of log likelihood with respect to $\underline{\beta}$.

**Note:** Since evaluation of expected value is cumbersome we will use sample information matrix $\hat{\mathbf{V}}$which, under usual regularity conditions, converges asymptotically to Fisher information matrix.

## *3.2 Maximum Likelihood Estimation When Shape Parameter is Unknown*

The solution of equations (12),(13) can be evaluated numerically by some suitable iterative procedure such as Newton-Raphson method, for given values of $(u, G, t_{gi}; g = 1,2,..,G^*; i = 1,2,...,m)$. The MLE of $(\alpha, \underline{\beta})$ are obtained as $(\hat{\alpha}, \underline{\hat{\beta}})$ from equations (12),(13). The MLEs of reliability $(\bar{F}(t_i); i = 1,2,...,m)$ and hazard rate $(h_i(t_i); i = 1,2,...,m)$ can be evaluated using invariance property of MLEs as

$$\hat{\bar{F_i}}(t_i) = 1 - \exp[-(\frac{\hat{\beta}_i}{t_i})^{\hat{\alpha}}] \quad (18)$$

and the hazard function is

$$\hat{h}_i(t_i) = \frac{\frac{\hat{\alpha}}{\hat{\beta}_i}(\frac{\hat{\beta}_i}{t_i})^{\hat{\alpha}+1}\exp[-(\frac{\hat{\beta}_i}{t_i})^{\hat{\alpha}}]}{1-\exp[-(\frac{\hat{\beta}_i}{t_i})^{\hat{\alpha}}]}. \quad (19)$$

### 3.2.1 Observed Fisher Information Matrix Under Design

To obtain Fisher information matrix we take derivatives of equations (12) and (13) with respect to $\alpha, \beta_i; i = 1,2,....,m$. Therefore, we have,

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha^2} &= -\frac{mG^*}{\alpha^2} - \sum_{i=1}^m\sum_{g=1}^{G^*}(\frac{\beta_i}{t_{G^*i}})^{\alpha} ln(\frac{\beta_i}{t_{G^*i}}) + (u - G^*) \\ &\times \{\sum_{i=1}^m [ln(\frac{\beta_i}{t_{G^*i}})]^2(\frac{\beta_i}{t_{G^*i}})^{\alpha}\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]\frac{1-(\frac{\beta_i}{t_{G^*i}})^{\alpha} - \exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]}{[1-\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha})]^2}\}.\end{aligned} \quad (20)$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha\partial\beta} &= \frac{G^*}{\beta_i} - \sum_{g=1}^{G^*}\frac{1}{t_{gi}}(\frac{\beta_i}{t_{gi}})^{\alpha-1}[1+\alpha(\frac{\beta_i}{t_{gi}})] + \frac{(u-G^*)}{t_{G^*i}}(\frac{\beta_i}{t_{G^*i}})^{\alpha-1}\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}] \\ &\times \{\frac{[1-\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]][1+\alpha ln(\frac{\beta_i}{t_{G^*i}})] - \alpha(\frac{\beta_i}{t_{G^*i}})^{\alpha} ln(\frac{\beta_i}{t_{G^*i}})}{[1-\exp[-(\frac{\beta_i}{t_{G^*i}})^{\alpha}]]^2}\}.\end{aligned} \quad (21)$$

Derivatives of equation (13) with respect to $\beta_i; i = 1,2,...,m$ and $\beta_j; j \neq i = 1,2,...,m$ are given in equations (16) and (17) respectively. Therefore, we have the following result:

**Theorem 3.2.** For $\alpha > 2$ and $\frac{G^*}{u}$ kept constant the maximum likelihood estimators,$(\alpha, \underline{\hat{\beta}})$ of $(\alpha, \underline{\beta})$ are consistent estimators, and $\sqrt{u}(\hat{\alpha} - \alpha, \underline{\hat{\beta}} - \underline{\beta})$ is asymptotically $(m+1)$-variate normal with mean $(0, \underline{0})$ and variance covariance matrix $\mathbf{W^{-1}}$, where $\mathbf{W}$ is expected value of negative of second derivative matrix of log likelihood with respect to $(\alpha, \underline{\beta})$.

## 4 Algorithm, Numerical Exploration and Conclusions

In this Section, a Monte-Carlo simulation study is conducted to compare the performance of the estimates developed in the previous sections. Maximum likelihood estimates are obtained for observations generated through the generalized Type II censoring design when numbers of systems to be compared 2 and 3 for known shape parameter having failure distribution is the Frechet$(\alpha,\beta_i); i = 1,2,...,m$. All calculations are performed on the $R$-language version $R$.3.1.2. The simulation study is conducted for only known shape parameter. For simulation purpose we consider two sets of parameter values $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3$ and for $m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4$ to carry out simulation study. Further, the simulation is carried out for different values of $u$ and $G^*$. Here we kept total number of failures in whole experiment $G = uG^*$ fixed. We simulate 1000 samples for each case using the algorithm discussed in [9,10] by considering Frechet distribution. The simulated results are summarized in Table 1 and Table 2.

**Table 1:** Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency Measures $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \underline{t} = (1.3403, 1.1616), \bar{F}(\underline{t}) = (0.7342, 0.7342), h(\underline{t}) = (0.7994, 0.9244)$

| $u$ | $G^*$ | | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\bar{F}}_1(t_1)$ | $\hat{\bar{F}}_2(t_2)$ | $\hat{h}_1(t_1)$ | $\hat{h}_2(t_2)$ |
|---|---|---|---|---|---|---|---|---|
| 12 | 06 | EV | 1.5366 | 1.3304 | 0.7432 | 0.7424 | 0.7782 | 0.9001 |
| | | MSE | 0.0354 | 0.0262 | 0.0096 | 0.0094 | 0.0764 | 0.09836 |
| | | SE | 0.1852 | 0.1603 | - | - | - | - |
| 24 | 12 | EV | 1.516 | 1.3219 | 0.7378 | 0.7423 | 0.7912 | 0.8989 |
| | | MSE | 0.0161 | 0.0141 | 0.0048 | 0.0055 | 0.037 | 0.0564 |
| | | SE | 0.1284 | 0.112 | - | - | - | - |
| 36 | 16 | EV | 1.5085 | 1.3083 | 0.7354 | 0.7363 | 0.7973 | 0.9172 |
| | | MSE | 0.01059 | 0.0074 | 0.0034 | 0.0032 | 0.0256 | 0.0322 |
| | | SE | 0.1041 | 0.0903 | - | - | - | - |
| 48 | 24 | EV | 1.5098 | 1.3076 | 0.737 | 0.7366 | 0.7927 | 0.916 |
| | | MSE | 0.0083 | 0.0057 | 0.0026 | 0.0025 | 0.0199 | 0.0251 |
| | | SE | 0.0916 | 0.0781 | - | - | - | - |
| 60 | 30 | EV | 1.5058 | 1.3047 | 0.7353 | 0.7353 | 0.7972 | 0.92 |
| | | MSE | 0.0064 | 0.0046 | 0.0021 | 0.002 | 0.016 | 0.0205 |
| | | SE | 0.0804 | 0.0696 | - | - | - | - |
| 72 | 36 | EV | 1.5036 | 1.3032 | 0.7346 | 0.7347 | 0.7992 | 0.9217 |
| | | MSE | 0.005 | 0.0036 | 0.0016 | 0.0016 | 0.0125 | 0.0162 |
| | | SE | 0.0732 | 0.0635 | - | - | - | - |
| 84 | 42 | EV | 1.503 | 1.3033 | 0.7345 | 0.7349 | 0.7993 | 0.9208 |
| | | MSE | 0.0043 | 0.0032 | 0.0014 | 0.0014 | 0.0107 | 0.0144 |
| | | SE | 0.0677 | 0.0587 | - | - | - | - |
| 96 | 48 | EV | 1.506 | 1.3022 | 0.7361 | 0.7344 | 0.7947 | 0.9223 |
| | | MSE | 0.0042 | 0.003 | 0.0014 | 0.0013 | 0.0107 | 0.0126 |
| | | SE | 0.0635 | 0.0549 | - | - | - | - |

From Table 1 and Table 2 we observed that means of MLEs for scale parameters $\beta_i; i = 1,2,...,m$, the reliability characteristics and hazard rates are very close to their true values. At average mean square errors are relatively small. Further we observe that the estimates and standard/mean square error are decreasing functions of number $u$ of each systems put on test.

## 5 Testing of Hypotheses

In this section we test the hypothesis of homogeneity of $m$ systems. To achieve this objective we test

$$H_0 : \beta_1 = \beta_2 = ... = \beta_m = \beta \text{ against } H_1 : \beta_i \neq \beta_j \text{ for at least one pair } (i, j), i \neq j \tag{22}$$

As we are considering maximum likelihood estimation, the use of likelihood ratio test is much convenient. The test statistic is

$$\lambda_{LR} = \frac{max_{\alpha,\beta} L(\underline{t}, \beta, \alpha)}{max_{\alpha,\underline{\beta}} L(\underline{t}, \underline{\beta}, \alpha)}$$

**Table 2:** Maximum Likelihood Estimate of Parameters, Reliability and Hazard Rates and their Efficiency Measures $m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4, \underline{t} = (1.3402, 1.1616, 1.2509), \bar{F}(\underline{t}) = (0.7342, 0.7342, 0.7342), h(\underline{t}) = (0.7994, 0.9224, 0.8565)$

| $u$ | $G^*$ | | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\bar{F}}_1(t_1)$ | $\hat{\bar{F}}_2(t_2)$ | $\hat{\bar{F}}_3(t_3)$ | $\hat{h}_1(t_1)$ | $\hat{h}_2(t_2)$ | $\hat{h}_3(t_3)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 24 | 08 | EV | 1.5175 | 1.3178 | 1.4125 | 0.738 | 0.74 | 0.7363 | 0.7911 | 0.9061 | 0.8525 |
| | | MSE | 0.0182 | 0.0131 | 0.0142 | 0.0055 | 0.0052 | 0.0051 | 0.0426 | 0.0537 | 0.0453 |
| | | SE | 0.1273 | 0.1106 | 0.1186 | - | - | - | - | - | - |
| 36 | 12 | EV | 1.5133 | 1.3059 | 1.4101 | 0.738 | 0.7345 | 0.7367 | 0.7904 | 0.923 | 0.8507 |
| | | MSE | 0.0113 | 0.0078 | 0.0095 | 0.0035 | 0.0034 | 0.0035 | 0.0268 | 0.034 | 0.0304 |
| | | SE | 0.1034 | 0.0892 | 0.0964 | - | - | - | - | - | - |
| 48 | 16 | EV | 1.5113 | 1.3101 | 1.4084 | 0.7379 | 0.738 | 0.7364 | 0.7902 | 0.9117 | 0.8512 |
| | | MSE | 0.0018 | 0.0065 | 0.0078 | 0.0026 | 0.0027 | 0.0028 | 0.0198 | 0.0276 | 0.0243 |
| | | SE | 0.0893 | 0.0774 | 0.0832 | - | - | - | - | - | - |
| 60 | 20 | EV | 1.5065 | 1.3053 | 1.4097 | 0.7357 | 0.7354 | 0.7377 | 0.7962 | 0.9196 | 0.8472 |
| | | MSE | 0.0066 | 0.005 | 0.0064 | 0.0021 | 0.0022 | 0.0024 | 0.0162 | 0.0221 | 0.0204 |
| | | SE | 0.0795 | 0.0689 | 0.0744 | - | - | - | - | - | - |
| 72 | 24 | EV | 1.505 | 1.3072 | 1.4059 | 0.7353 | 0.7347 | 0.736 | 0.7972 | 0.9219 | 0.8519 |
| | | MSE | 0.0053 | 0.0044 | 0.0047 | 0.0018 | 0.0019 | 0.0018 | 0.0134 | 0.0193 | 0.0153 |
| | | SE | 0.0725 | 0.0628 | 0.0677 | - | - | - | - | - | - |
| 84 | 28 | EV | 1.5078 | 1.3066 | 1.4037 | 0.7371 | 0.7371 | 0.7346 | 0.7924 | 0.9143 | 0.8561 |
| | | MSE | 0.0051 | 0.0036 | 0.0047 | 0.0017 | 0.0016 | 0.0018 | 0.0126 | 0.0157 | 0.0154 |
| | | SE | 0.0672 | 0.0582 | 0.0626 | - | - | - | - | - | - |
| 96 | 32 | EV | 1.5016 | 1.3042 | 1.4055 | 0.7338 | 0.7356 | 0.7362 | 0.801 | 0.9188 | 0.8511 |
| | | MSE | 0.0038 | 0.0031 | 0.0035 | 0.0013 | 0.0014 | 0.0013 | 0.0098 | 0.014 | 0.0115 |
| | | SE | 0.0626 | 0.0543 | 0.0586 | - | - | - | - | - | - |

The test based on $-2ln(\lambda_{LR})$ rejects $H_0$ in support of $H_1$ if it is larger than upper $\zeta$-th cut of poit of chi-square distribution $(m-1)$ degrees of freedom.

## *5.1 Computation of Likelihood Under $H_0$*

The log likelihood $lnLG$ under null hypothesis from equation (11) we have,

$$l = mln(\frac{u!}{(u-G^*)!}) + mG^* ln\alpha + mG^*\alpha ln\beta - (\alpha+1)\sum_{i=1}^{m}\sum_{g=1}^{G^*} ln(t_{gi})$$
$$-\sum_{i=1}^{m}\sum_{g=1}^{G^*}(\frac{\beta}{t_{gi}})^{\alpha} + (u-G^*)\sum_{i=1}^{m} ln[1-\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]]. \tag{23}$$

Differentiate (23) with respect to $(\alpha, \beta)$ and $\beta$ respectively, we have

$$\frac{\partial l}{\partial \alpha} = \frac{mG^*}{\alpha} + G^* ln\beta - \sum_{i=1}^{m}\sum_{g=1}^{G^*} ln(t_{gi}) - \sum_{i=1}^{m}\sum_{g=1}^{G^*}(\frac{\beta}{t_{gi}})^{\alpha} ln(\frac{\beta}{t_{gi}})$$
$$+ (u-G^*)\sum_{i=1}^{m}\frac{(\frac{\beta}{t_{G^*i}})^{\alpha} ln(\frac{\beta}{t_{G^*i}})\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]}{1-\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]}. \tag{24}$$

$$\frac{\partial l}{\partial \beta} = \frac{mG^*\alpha}{\beta} - \alpha\sum_{g=1}^{G^*}(\frac{\beta}{t_{gi}})^{\alpha-1}(\frac{1}{t_{gi}})$$
$$+ \alpha(u-G^*)\sum_{i=1}^{m}\frac{(\frac{\beta}{t_{G^*i}})^{\alpha-1}\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]\frac{1}{t_{G^*i}}}{1-\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]}. \tag{25}$$

Differentiate (24) and (25) with respect to $(\alpha, \beta)$ and $\beta$ respectively, we have

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{mG^*}{\alpha^2} - \sum_{i=1}^{m}\sum_{g=1}^{G^*}(\frac{\beta}{t_{gi}})^{\alpha}[ln(\frac{\beta}{t_{gi}})]^2$$
$$+ (u-G^*)\sum_{i=1}^{m}\frac{(\frac{\beta}{t_{G^*i}})^{\alpha} ln(\frac{\beta}{t_{G^*i}}) \exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]}{[1-\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]]^2}. \quad (26)$$

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = \frac{mG^*}{\beta} - \sum_{i=1}^{m}\sum_{g=1}^{G^*}\frac{1}{t_{gi}}(\frac{\beta}{t_{gi}})^{\alpha-1}(1+\alpha ln(\frac{\beta}{t_{gi}}))$$
$$+ (u-G^*)\sum_{i=1}^{m}\frac{1}{t_{G^*i}}(\frac{\beta}{t_{G^*i}})^{\alpha-1}\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]$$
$$\times \{\frac{[1-\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]][1+\alpha ln(\frac{\beta}{t_{G^*i}})] - \alpha(\frac{\beta}{t_{G^*i}})^{\alpha} ln(\frac{\beta}{t_{G^*i}})}{[1-\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]]^2}\}. \quad (27)$$

$$\frac{\partial^2 l}{\partial \beta^2} = -\frac{mG^*\alpha}{\beta^2} - \alpha(\alpha-1)\sum_{i=1}^{m}\sum_{g=1}^{G^*}(\frac{\beta}{t_{gi}})^{\alpha-2}(\frac{1}{t_{gi}^2})$$
$$+ \alpha(u-G^*)\sum_{i=1}^{m}(\frac{1}{t_{G^*i}^2})(\frac{\beta}{t_{G^*i}})^{\alpha-2}\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]$$
$$\times \{\frac{(\alpha-1)[1-\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]] - \alpha(\frac{\beta}{t_{G^*i}})^{\alpha}}{[1-\exp[-(\frac{\beta}{t_{G^*i}})^{\alpha}]]^2}\}. \quad (28)$$

The likelihood equation (25) is not mathematically tractable for known as well as unknown shape parameter we use the Newton-Rapshon method to obtain the estimate of parameter $\beta$. Here we deal with only known shape parameter. We demonstrate the test procedure for $m = 2$ and $m = 3$. We generate data under our design for the parameter values under $H_1 : \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3$ and $H_1 : \alpha = 2.5, \beta_1 = 1.9, \beta_2 = 1.5, \beta_3 = 1$ respectively. Then carry out the test procedure as suggested above. The procedure is repeated for the different choices of $u$ and $G^*$. The results are produced in the Table 3 and Table 4 respectively.

**Table 3:** Likelihood Ratio Test for Testing $H_0 : \beta_1 = \beta_2 = \beta$ vs $H_1 : \beta_1 \neq \beta_2$ when $\alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3$

| $u$ | $G^*$ | $\hat{\beta}$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $LL_{H_0}$ | $LL_{H_1}$ | $\chi^2$ | $p-value$ |
|---|---|---|---|---|---|---|---|---|
| 12 | 6 | 1.4044 | 1.3488 | 1.4661 | 5.7962 | 5.9237 | 0.255 | 0.6135 |
| 24 | 12 | 1.5611 | 1.6222 | 1.5072 | 39.9983 | 40.1956 | 0.3945 | 0.5289 |
| 36 | 18 | 1.5061 | 1.6191 | 1.4154 | 67.5572 | 68.5469 | 1.9791 | 0.01594 |
| 48 | 24 | 1.3307 | 1.4394 | 1.2457 | 110.2069 | 111.7471 | 3.0803 | 0.0792 |
| 60 | 30 | 1.4023 | 1.5245 | 1.3081 | 149.345 | 151.48 | 4.267 | 0.0387 |
| 72 | 36 | 1.3859 | 1.5216 | 1.2834 | 194.396 | 197.5705 | 6.3489 | 0.0117 |
| 84 | 42 | 1.3947 | 1.5386 | 1.2891 | 236.7859 | 240.783 | 7.9943 | 0.0047 |
| 96 | 48 | 1.3787 | 1.5849 | 1.2414 | 290.6936 | 299.3099 | 17.2325 | 3.31E-05 |

**Table 4:** Likelihood Ratio Test for Testing $H_0 : \beta_1 = \beta_2 = \beta_2 = \beta$ vs $H_1 : \beta_i \neq \beta_j (i \neq j = 1,2,3)$ when $\alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.4, \beta_2 = 1.3$

| $u$ | $G^*$ | $\hat{\beta}$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $LL_{H_0}$ | $LL_{H_1}$ | $\chi^2$ | $p-value$ |
|---|---|---|---|---|---|---|---|---|---|
| 24 | 8 | 1.4444 | 1.7201 | 1.3474 | 1.3519 | 28.231 | 30.8357 | 5.2094 | 0.0739 |
| 36 | 12 | 1.4008 | 1.5351 | 1.4901 | 1.2366 | 74.5913 | 77.5738 | 5.9049 | 0.0522 |
| 48 | 16 | 1.3137 | 1.4892 | 1.3113 | 1.1943 | 107.0447 | 110.4224 | 6.7554 | 0.0341 |
| 60 | 20 | 1.4301 | 1.5311 | 1.5389 | 1.2696 | 145.9008 | 150.2344 | 8.6672 | 0.0131 |
| 72 | 24 | 1.4033 | 1.5696 | 1.437 | 1.2577 | 187.0187 | 192.2813 | 10.5252 | 0.0052 |
| 84 | 28 | 1.349 | 1.4937 | 1.4007 | 1.2052 | 248.443 | 254.4792 | 12.0725 | 0.0024 |
| 96 | 32 | 1.4014 | 1.5836 | 1.4313 | 1.249 | 270.0367 | 278.0853 | 16.0973 | 0.00031 |

From the Table 3 and Table 4 we observe that as sample size increases, the Likelihood Ratio Test converges for identifying its true alternative. Therefore we can say that the test is powerful to identify heteroscedasticity of several systems.

## 6 Concluding Remarks

In this article, we have studied fitting of Frechet distribution for several systems when sample observations are drawn based on the generalized Type II censoring scheme. Further, we carried out simulation study to demonstrate the performance of the estimators in terms of their MSE and SE. Finally, we provided likelihood ratio test for homogeneity of lifetimes of several of systems.

## Acknowledgement

The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

## References

[1] Abbas, K. and Tang,Y. (2013). Estimation of Parameters for Frechet Distribution Based on Type-II Censored, *Caspian Journal of Applied Science Research*, **2(7)**, 36-43.
[2] Balakrishnan, N. and Ng, H.K.T. (2006). Precedence-type tests and applications. Hoboken, NJ:John Wiley & Sons.
[3] Cohen, A.C. Jr (1965). Maximum likelihood estimation with Weibull distribution based on complete and on censored samples, *Technometrics*, **7**, 559–588.
[4] Gupta, A.K. (1952). Estimation of mean and standard deviation of a normal population from a censored sample, *Biometrika*, **39**, 260–273.
[5] Hossain, A. and Zimmer, W. (2003). Comparison of estimation methods for Weibull parameters: Completed censored samples, *Journal of Statistical Computation and Simulation*, **73(2)**, 145–153.
[6] Lawless, J.F. (1982). Statistical Models and Methods for Lifetime Data, John Wiley & Sons, New York, NY.
[7] Mann, N., Schafer, E. and Singpurwalla, N. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley & Sons, New York, NY.
[8] Nadarajah,S. and Kotz,S. (2000). Extreme Value Distributions: Theory and Applications, Imperial College Press, London, UK.
[9] Shanubhouge, A., Raykundaliya, D.P. (2014). Test for homogeneity of lifetimes of several systems under generalized inverted scale family of distributions based on Type II censored sampling design,*InterStat*.
[10] Shanubhogue, A. and Raykundaliya, D.P. (2015). A Study of Inferential Problem about Comparison of Lifetimes of Homogeneity of Several Systems under Generalized Exponential Distribution with Type II Censored Sampling Design, *ProbStat*,**08**, 24–33.
[11] Sharafi, M.M Balkrishnan, N. And Khaledi, B.E. (2013). Distribution-free comparison of hazard rates of two distributions under Type II censoring, *Communications in Statistics-Theory and Methods*,**42**, 1889–1898

**Raykundaliya Dharmesh Pravinkumar** received the PhD degree in Statistics at Sardar Patel University, Vallabh Vidyanagar , Anand-Gujarat-India. His research interests are Reliability Oriented Designs, Applied Statistics and Big Data Analytics. He has published 22 research articles in National and International Journals. He has received five awards in which one is international.