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Mixture of Exponentiated Generalized Weibull-Gompertz Distribution and its Applications in Reliability

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Abstract: This paper introduces studies on mixture of two exponentiated generalized Weibull-Gompertz distribution (MEGWGD) by using mixing parameter p where $0 < p < 1$, which generalizes a lot of distributions. Several properties of the MEGWGD such as reversed (hazard) function, moments, maximum likelihood estimation, mean residual (past) lifetime, MTTF, MTTR, MTBF, maintainability, availability and order statistics are studied in this paper. It is observed that the present distribution can provide a better fit than some other very well-known distributions.

Keywords: Reversed (hazard) rate function, generalized Weibull-Gompertz distribution, mean residual (past) lifetime, maintainability.

1 Introduction

Mixture distributions are one of the most important distributions which play an important role in many fields. For example, life testing and reliability, engineering, zoology, medicine, botany, economics, agriculture, among others. Also, mixture distribution is even more useful because multiple causes of failure can be simultaneously modeled. In the literature the mixture model has been discussed by many authors such that, [1,3,4,5,7,8,11]. The mixture distribution produced from the combination of two or more distributions has a number of parameters. These parameters include; the shape parameters, scale parameters in addition to the mixing parameter p where $0 < p < 1$. On the other hand, by using mixing parameter p we get new properties of the new distribution.

EL-Damcese et al. [2] introduced exponentiated generalized Weibull-Gompertz distribution which generalizes a lot of distributions such that generalized Gompertz distribution, generalized Weibull-Gompertz distribution, exponential power distribution, generalized exponential distribution, Weibull extension model [12], among others.

2 Exponentiated Generalized Weibull-Gompertz Distribution

The random variable X is said to be has EGWGD if it has the following CDF for $a,b,c,d,\theta > 0$ as follows.

$$F_X(x;a,b,c,d,\theta) = \left[1 - e^{-ax^b\left(e^{cx^d}-1\right)}\right]^{\theta}, \quad x > 0, \tag{1}$$

where b,θ and d are shape parameters, a is scale parameter and c is an acceleration parameter.

Exponentiated generalized Weibull-Gompertz distribution with five parameters will denoted by EGWGD(a,b,c,d,θ). The probability density function $f_X(x;a,b,c,d,\theta)$ of EGWGD(a,b,c,d,θ) is

$$f_X(x;a,b,c,d,\theta) = ab\theta x^{b-1} e^{-ax^b\left(e^{cx^d}-1\right)+cx^d} \left(1 + \frac{cd}{b}x^d - e^{-cx^d}\right) \left[1 - e^{-ax^b\left(e^{cx^d}-1\right)}\right]^{\theta-1}. \tag{2}$$

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The survival function can be obtained as follows

$$R(x;a,b,c,d,\theta) = 1 - \left[1 - e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}; \quad x > 0. \tag{3}$$

The hazard function $h(x)$ is

$$h(x;a,b,c,d,\theta) = \frac{ab\theta x^{b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}}{1-\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}}. \tag{4}$$

Recently, it is observed that the reversed hazard function plays an important role in the reliability analysis; see [9]. The reversed hazard function $r(x)$ of the EGWGD(a,b,c,d,θ) is

$$r(x;a,b,c,d,\theta) = \frac{ab\theta x^{b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right)}{\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}}. \tag{5}$$

3 Mixture of Exponentiated Generalized Weibull-Gompertz Distribution

The random variable X is said to be has mixture of exponentiated generalized Weibull-Gompertz distribution (MEGWGD) if it has the following CDF for each element in the vector $\Theta = \Theta(a,b,c,d,\theta,\alpha,\beta,\lambda,\rho,\omega)$ greater than zero and $x > 0$ as follows.

$$\begin{aligned} F_X(x;\Theta) &= pF_1(x;a,b,c,d,\theta) + (1-p)F_2(x;\alpha,\beta,\lambda,\rho,\omega) \\ &= p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta} + (1-p)\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega}. \end{aligned} \tag{6}$$

The probability density function $f_X(x;\Theta)$ of MEGWGD is

$$\begin{aligned} f_X(x;\Theta) &= ab\theta p x^{b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1} \\ &\quad + \alpha\beta\omega(1-p)x^{\beta-1}e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)+\lambda x^{\rho}}\left(1+\frac{\lambda\rho}{\beta}x^{\rho} - e^{-\lambda x^{\rho}}\right)\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega-1}. \end{aligned} \tag{7}$$

Also; the survival function can be obtained as follows.

$$R_X(x;\Theta) = 1 - p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta} - (1-p)\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega}. \tag{8}$$

On the other hand, the hazard rate function is

$$\begin{aligned} h(x;\Theta) &= \frac{f_X(x;\Theta)}{R_X(x;\Theta)} \\ &= \frac{ab\theta p x^{b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}}{1-p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta} - (1-p)\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega}} \\ &\quad + \frac{\alpha\beta\omega(1-p)x^{\beta-1}e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)+\lambda x^{\rho}}\left(1+\frac{\lambda\rho}{\beta}x^{\rho} - e^{-\lambda x^{\rho}}\right)\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega-1}}{1-p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta} - (1-p)\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega}}. \end{aligned} \tag{9}$$

and the reversed hazard rate function is

$$r(x;\Theta)=\frac{f_X(x;\Theta)}{F_X(x;\Theta)}$$
$$=\frac{ab\theta px^{b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d-e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}}{p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}+(1-p)\left[1-e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega}}$$
$$+\frac{\alpha\beta\omega(1-p)x^{\beta-1}e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)+\lambda x^\rho}\left(1+\frac{\lambda\rho}{\beta}x^\rho-e^{-\lambda x^\rho}\right)\left[1-e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega-1}}{p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}+(1-p)\left[1-e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega}}. \quad (10)$$

Figures 1 and 3 provide the PDFs and hazard rates functions of two EGWGD for different parameter values, also Figures 2 and 4 provide the PDFs and hazard rates functions of MEGWGD for different mixing parameter values.

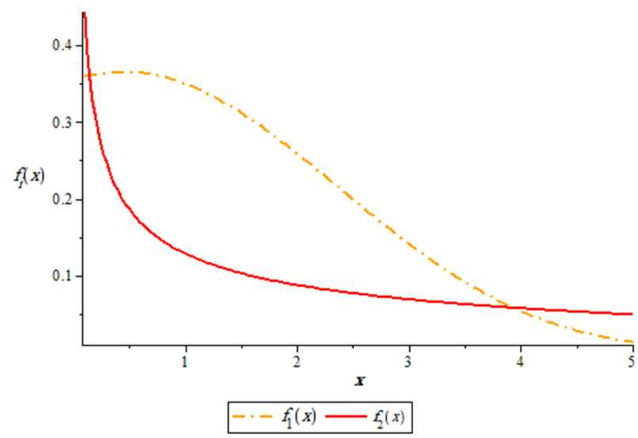

Fig. 1: Probability density function of two EGWGD with parameters: $a=0.4, b=0.5, c=0.6, d=0.7, \theta=0.8$; $\alpha=0.2, \beta=0.3, \rho=0.4, \lambda=0.5, \omega=0.6$

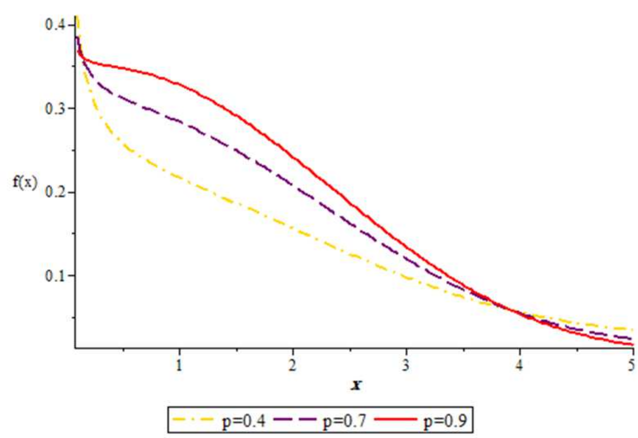

Fig. 2: Probability density function of two MEGWGD with parameters: $a=0.4, b=0.5, c=0.6, d=0.7, \theta=0.8$; $\alpha=0.2, \beta=0.3, \rho=0.4, \lambda=0.5, \omega=0.6$ and different parameter p.

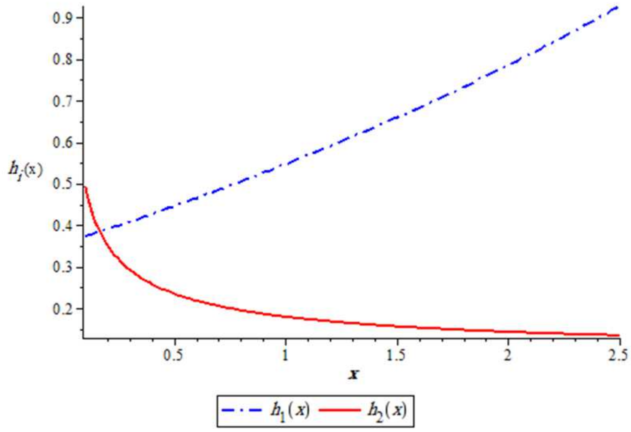

Fig. 3: Hazard rate function of two EGWGD with parameters:$a = 0.4, b = 0.5, c = 0.6, d = 0.7, \theta = 0.8$; $\alpha = 0.2, \beta = 0.3, \rho = 0.4, \lambda = 0.5, \omega = 0.6$.

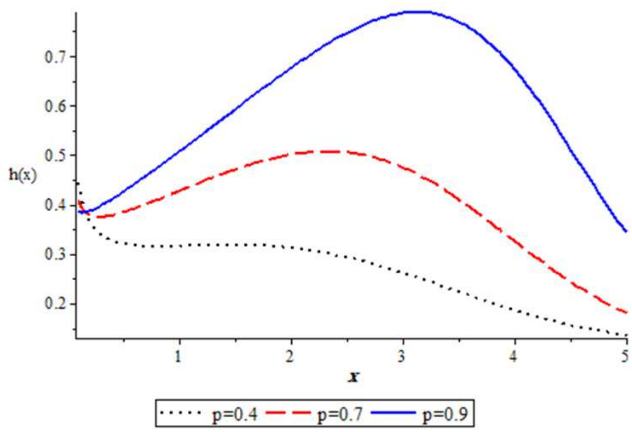

Fig. 4: Hazard rate function of two MEGWGD with parameters:$a = 0.4, b = 0.5, c = 0.6, d = 0.7, \theta = 0.8$; $\alpha = 0.2, \beta = 0.3, \rho = 0.4, \lambda = 0.5, \omega = 0.6$ and different parameter p.

Remark 1. From MEGWGD the following special cases can be derived:

1. Generalized Weibull - Gompertz distribution GWGD(a, b, c, d), when $\theta = 1$ and $p = 1$.
2. Exponentiated generalized Weibull - Gompertz distribution EGWGD(a, b, c, d, θ), when $p = 1$.
3. Gompertz distribution GD (a, c), when $\theta = 1,\ b = 0, p = 1$ and $d = 1$.
4. Generalized Gompertz distribution GGD (a, c, θ), when $b = 0, p = 1$ and $d = 1$.
5. Exponentiated modified Weibull extension, when $b = 0, p = 1$ and $c = \left(\frac{1}{\alpha}\right)^d,\ \alpha > 0$.
6. Exponential power distribution EPD(a, d, c), when $\theta = 1, p = 1$ and $b = 0$.
7. Generalized exponential power distribution GEPD(a, d, c, θ), when $b = 0$ and $p = 1$.
8. Weibull extension model of [12], when $b = 0$ and $p = 1$.
9. Exponential distribution ED(a), when c tends to zero, $d = 1, \theta = 1, p = 1$ and $b = 0$.
10. Generalized exponential distribution GED(a, θ), when c tends to zero, $d = 1, p = 1$ and $b = 0$.

4 Statistical Properties

4.1 The median and the mode

It is observed as expected that the mean of MEGWGD cannot be obtained in explicit forms. It can be obtained as infinite series expansion so, in general different moments of MEGWGD. Also, we cannot get the quartile x_q of MEGWGD in a closed form by using the equation $F_X(x;\Theta)-q=0$. Thus, by using Equation (6), we get

$$\left[1-e^{-ax_q^b\left(e^{cx_q^d}-1\right)}\right]^{\theta}+\left(\frac{1-p}{p}\right)\left[1-e^{-\alpha x_q^{\beta}\left(e^{\lambda x_q^{\rho}}-1\right)}\right]^{\omega}=\frac{q}{p}. \tag{11}$$

The median $m(X)$ of MEGWGD can be obtained from Equation (11), when $q=0.5$.
Moreover, the mode of MEGWGD can be obtained as a solution of the following nonlinear equation.

$$\begin{aligned}
&\frac{d}{dx}f_X(x;\Theta)=0,\\
&\frac{d}{dx}\left[x^{b-1}e^{-ax^b\left(e^{cx^d}-1\right)+cx^d}\left(1+\frac{cd}{b}x^d-e^{-cx^d}\right)\left(1-e^{-ax^b\left(e^{cx^d}-1\right)}\right)^{\theta-1}+\right.\\
&\left.\frac{\alpha\beta\omega(1-p)}{ab\theta p}x^{\beta-1}e^{-\alpha x^{\beta}\left(e^{\lambda x^{\rho}}-1\right)+\lambda x^{\rho}}\left(1+\frac{\lambda\rho}{\beta}x^{\rho}-e^{-\lambda x^{\rho}}\right)\left(1-e^{-\alpha x^{\beta}\left(e^{\lambda x^{\rho}}-1\right)}\right)^{\omega-1}\right]=0.
\end{aligned} \tag{12}$$

It has to be obtained numerically.

4.2 The moments

The following Lemma 4.1 gives the r-th moment of MEGWGD.

Lemma 4.1. If X has MEGWED(x;Θ), the r-th moment of X, say $\mu_r^{'}$, is given as follows for $\omega,\theta>1$ and positive integer and the other components in $\Theta>0, x>0$

$$\begin{aligned}
\mu_r^{'}=&\frac{ab\theta p}{d}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}c^{\ell}(a(1+i))^j}{j!\ell!d(ck)^{\frac{r+b(j+1)+\ell d}{d}}}\binom{j}{k}\binom{\theta-1}{i}\left[((1+j)^{\ell}-j^{\ell})\Gamma\left(\frac{r+b(j+1)+d(\ell-1)}{d}+1\right)+\right.\\
&\left.\frac{d(1+j)^{\ell}}{kb}\Gamma\left(\frac{r+b(j+1)+\ell d}{d}+1\right)\right]+\frac{\alpha\beta\omega(1-p)}{\rho}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}\lambda^{\ell}(\alpha(1+i))^j}{j!\ell!\rho(\lambda k)^{\frac{(r+\beta(j+1)+\ell\rho)}{\rho}}}\binom{j}{k}\binom{\omega-1}{i}\times\\
&\left[((1+j)^{\ell}-j^{\ell})\Gamma\left(\frac{r+\beta(j+1)+\rho(\ell-1)}{\rho}+1\right)+\frac{\rho(1+j)^{\ell}}{k\beta}\Gamma\left(\frac{r+\beta(j+1)+\ell\rho}{\rho}+1\right)\right].
\end{aligned} \tag{13}$$

Proof. Since

$$\mu_r^{'}=\int_0^{\infty}x^r f_X(x;\Theta)dx, \tag{14}$$

substituting from (7) into (14), we have

$$\mu_r^{'}=p\int_0^{\infty}x^r f_1(x;a,b,c,d,\theta)dx+(1-p)\int_0^{\infty}x^r f_2(x;\alpha,\beta,\lambda,\rho,\omega)dx=pI_A+(1-p)I_B \tag{15}$$

where

$$\begin{aligned}
I_A&=ab\theta\int_0^{\infty}x^{r+b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d-e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}dx\\
&=ab\theta I_{A1}-ab\theta I_{A2}+acd\theta I_{A3},
\end{aligned} \tag{16}$$

$$\begin{aligned}
I_B&=\alpha\beta\omega\int_0^{\infty}x^{r+\beta-1}e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)+\lambda x^{\rho}}\left(1+\frac{\lambda\rho}{\beta}x^{\rho}-e^{-\lambda x^{\rho}}\right)\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega-1}dx\\
&=\alpha\beta\omega I_{B1}-\alpha\beta\omega I_{B2}+\alpha\omega\lambda\rho I_{B3},
\end{aligned} \tag{17}$$

and

$$I_{A1} = \int_0^\infty x^{r+b-1} e^{-ax^b(e^{cx^d}-1)+cx^d} \left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1} dx,$$

$$I_{A2} = \int_0^\infty x^{r+b-1} e^{-ax^b(e^{cx^d}-1)} \left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1} dx,$$

$$I_{A3} = \int_0^\infty x^{r+b+d-1} e^{-ax^b(e^{cx^d}-1)+cx^d} \left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1} dx,$$

$$I_{B1} = \int_0^\infty x^{r+\beta-1} e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)+\lambda x^\rho} \left[1-e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega-1} dx,$$

$$I_{B2} = \int_0^\infty x^{r+\beta-1} e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)} \left[1-e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega-1} dx,$$

$$I_{B3} = \int_0^\infty x^{r+\beta+\rho-1} e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)+\lambda x^\rho} \left[1-e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega-1} dx.$$

Now, we get I_{A1}, since $0 < [1-e^{-ax^b(e^{cx^d}-1)}]^{\theta-1} < 1$, for $x > 0$, then by using the binomial series expansion we have

$$[1-e^{-ax^b(e^{cx^d}-1)}]^{\theta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\theta-1}{i} e^{-iax^b(e^{cx^d}-1)},$$

then

$$I_{A1} = \int_0^\infty x^{r+b-1} e^{cx^d} \sum_{i=0}^{\infty} (-1)^i \binom{\theta-1}{i} e^{-(i+1)ax^b(e^{cx^d}-1)} dx,$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k}(c(1+j))^\ell (a(1+i))^j}{j!\ell!} \binom{j}{k}\binom{\theta-1}{i} \int_0^\infty x^{r+b+bj+\ell d-1} e^{-ckx^d} dx$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k}(c(1+j))^\ell (a(1+i))^j}{j!\ell! d (ck)^{\frac{r+b(j+1)+\ell d}{d}}} \binom{j}{k}\binom{\theta-1}{i} \Gamma\left(\frac{r+b(j+1)+d(\ell-1)}{d}+1\right).$$

Similarly, we find that

$$I_{A2} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k}(cj)^\ell (a(1+i))^j}{j!\ell! d (ck)^{\frac{r+b(j+1)+\ell d}{d}}} \binom{j}{k}\binom{\theta-1}{i} \Gamma\left(\frac{r+b(j+1)+d(\ell-1)}{d}+1\right).$$

$$I_{A3} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k}(c(1+j))^\ell (a(1+i))^j}{j!\ell! d (ck)^{\frac{r+b(j+1)+(\ell+1)d}{d}}} \binom{j}{k}\binom{\theta-1}{i} \Gamma\left(\frac{r+b(j+1)+\ell d}{d}+1\right).$$

$$I_{B1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k}(\lambda(1+j))^\ell (\alpha(\ell+i))^j}{j!\ell! \rho (\lambda k)^{\frac{r+\beta(j+1)+\ell\rho}{\rho}}} \binom{j}{k}\binom{\omega-1}{i} \Gamma\left(\frac{r+\beta(j+1)+\rho(\ell-1)}{\rho}+1\right).$$

$$I_{B2} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k}(\lambda j)^\ell (\alpha(1+i))^j}{j!\ell! \rho (\lambda k)^{\frac{r+\beta(j+1)+\ell\rho}{\rho}}} \binom{j}{k}\binom{\omega-1}{i} \Gamma\left(\frac{r+\beta(j+1)+\rho(\ell-1)}{\rho}+1\right).$$

$$I_{B3} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k}(\lambda(1+j))^\ell (\alpha(1+i))^j}{j!\ell! \rho (\lambda k)^{\frac{r+\beta(j+1)+(\ell+1)\rho}{\rho}}} \binom{j}{k}\binom{\omega-1}{i} \Gamma\left(\frac{r+\beta(j+1)+\ell\rho}{\rho}+1\right).$$

Substituting from I_{Ai} and $I_{Bi},\ i = 1,2,3$ into Equations (16) and (17) then into Equation (15), we get Equation (13). This completes the proof.

5 Order Statistics

Let $X_{(1:n)} \leq X_{(2:n)} \leq \cdots \leq X_{(n:n)}$ denote the order statistics from this sample which $X_{(i:n)}$ denote the lifetime of an $(n-i+1)$-out of-n system which consists of n independent and identically components (iid). Then the pdf of $X_{(i:n)},\ i =$

$1,2,3,\cdots,n$ is given by

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}. \tag{18}$$

We defined the first order statistics $X_{(1)}=\min(X_1,X_2,\cdots,X_n)$, the last order statistics as $X_{(n)}=\max(X_1,X_2,\cdots,X_n)$. Let $X_1,X_2,\cdots,X_n$ be a random sample from MEGWGD with CDF and PDF in Equations (6) and (7) respectively. Then the PDF of the first order statistics $X_{(1)}=X_{1:n}$ is given from equation (18) as follows.

$$\begin{aligned}f_{1:n}(x,\Theta)=&\,n\left[1-p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}-(1-p)\left[1-e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega}\right]^{n-1}\left[ab\theta p x^{b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\times\right.\\&\left(1+\frac{cd}{b}x^d-e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}+\alpha\beta\omega(1-p)x^{\beta-1}e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)+\lambda x^\rho}\left(1+\frac{\lambda\rho}{\beta}x^\rho-e^{-\lambda x^\rho}\right)\times\\&\left.\left[1-e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega-1}\right].\end{aligned}\tag{19}$$

Also, the PDF of the last order statistics $X_{(n)}=X_{n:n}$ is given from Equation (18) as follows.

$$\begin{aligned}f_{n:n}(x,\Theta)=&\,n\left[p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}+(1-p)\left[1-e^{-\alpha^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega}\right]^{n-1}\left[ab\theta p x^{b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\times\right.\\&\left(1+\frac{cd}{b}x^d-e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}+\alpha\beta\omega(1-p)x^{\beta-1}e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)+\lambda x^\rho}\left(1+\frac{\lambda\rho}{\beta}x^\rho-e^{-\lambda x^\rho}\right)\times\\&\left.\left[1-e^{-\alpha x^\beta(e^{\lambda x^\rho}-1)}\right]^{\omega-1}\right].\end{aligned}\tag{20}$$

6 Reliability Analysis

6.1 Mean time to failure (repair)

In order to design and manufacture a maintainable system, it is necessary to predict the mean time to failure (repair), say MTTF(MTTR), for various fault conditions that could occur in the system.

Lemma 6.1. If T is a random variable has MEGWGD, then the mean time to failure (repair) is given as follows for $\omega,\theta>1$ and positive integer, the other components in $\Theta>0$, $x>0$

$$\begin{aligned}MTTF(MTTR)=&\frac{ab\theta p}{d}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}c^{\ell}(a(1+i))^j}{j!\ell!d(ck)^{\frac{1+b(j+1)+\ell d}{d}}}\binom{j}{k}\binom{\theta-1}{i}\times\\&\left[((1+j)^{\ell}-j^{\ell})\Gamma\left(\frac{1+b(j+1)+d(l\ell-1)}{d}+1\right)+\frac{d(1+j)^{\ell}}{kb}\Gamma\left(\frac{1+b(j+1)+\ell d}{d}+1\right)\right]+\\&\frac{\alpha\beta\omega(1-p)}{\rho}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}\lambda^{\ell}(\alpha(1+i))^j}{j!\ell!\rho(\lambda k)^{\frac{1+\beta(j+1)+\ell\rho}{\rho}}}\binom{j}{k}\binom{\omega-1}{i}\times\\&\left[((1+j)^{\ell}-j^{l})\Gamma\left(\frac{1+\beta(j+1)+\rho(\ell-1)}{\rho}+1\right)+\frac{\rho(1+j)^{\ell}}{k\beta}\Gamma\left(\frac{1+\beta(j+1)+\ell\rho}{\rho}+1\right)\right].\end{aligned}\tag{21}$$

Proof. We have

$$MTTF(MTTR)=\int_0^{\infty}R(t;\Theta)dt=\int_0^{\infty}t\,f(t;\Theta)dt=\mu_1^{'},$$

then from Equation (13), when $r=1$, it is easy to prove the lemma.

6.2 *Availability* $A(t)$

It is defined by the probability that the system (component) is successful at time t.

Lemma 6.2. If the reliability function for a component is given by $R_T(t;\Theta)$, where T has MEGWGD and the distribution of a repair time density is the PDF $f(t;\Theta)$ of MEGWGD then the availability $A(t)$ is given as $(t)=0.5$.

Proof. The proof is simple by using Lemma (6.1) and

$$A(t)=\frac{MTTF}{MTTF+MTTR}.$$

Moreover, the mean time between failures (MTBF) is an important measure in repairable system (component). This implies that the system (component) has failed and has been repaired. Like MTTF and MTTR, MTBF is the expected value of the random variable time between failures. Mathematically,

$$MTBF=MTTF+MTTR.$$

6.3 *Maintainability* $V(t)$

Let T denote the random variable of the time to repair or the total downtime. If the repair time T has a repair time density function $f(t;\Theta)$, then the maintainability $V(t)$ is defined as the probability of isolating and repairing a fault in a system within a given time. If the repair time T is a random variable has a repair time density function $f(t;\Theta)$ of MEGWGD, then the maintainability $V(t)$ is defined as

$$V(t)=P(T\le t)=p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}+(1-p)\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega}. \tag{22}$$

6.4 *The mean residual (past) lifetime MRL(MPL) for MEGWGD*

In reliability theory and survival analysis, to study the lifetime characteristics of a live organism there have been defined in a several measures such as the mean residual lifetime $m(t)$ and the mean past lifetime $p(t)$, assuming that each component of the system has survived up to time t.

Lemma 6.3. If T is a random variable has MEGWGD, then the mean residual lifetime (MRL), is given as follows for $\omega,\theta>1$ and the other components in $\Theta>0$, $x>0$

$$m(t)=\frac{1}{R_T(t;\Theta)}\left[\mu_1^{'}-I(t)\right] \tag{23}$$

where $R_T(t;\Theta)$ be the reliability (survival) function, Eq.(8) and $\mu_1^{'}$ the mean (MTTF), Eq. (13) (when $r=1$), and

$$I(t)=t-p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(ck)^{\ell}(ai)^{j}}{j!\ell!d(cj)^{\frac{bj+\ell d+1}{d}}}\binom{j}{k}\binom{\theta}{i}e^{cjt^d}\left[(cjt^d)^{\frac{bj+\ell d-d+1}{d}}+\sum_{m=1}^{\frac{bj+\ell d-d+1}{d}}(-1)^m m!\binom{\frac{bj+\ell d-d+1}{d}}{m}\times\right.$$
$$\left.(cjt^d)^{\frac{bj+\ell d-d+1}{d}-m}\right]-(1-p)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(\lambda k)^{\ell}(\alpha i)^{j}}{j!\ell!d(\alpha j)^{\frac{\beta j+\ell\rho+1}{\rho}}}\binom{j}{k}\binom{\omega}{i}e^{\lambda jt^{\rho}}\left[(\lambda jt^{\rho})^{\frac{\beta j+\ell\rho-\rho+1}{\rho}}+\right.$$
$$\left.\sum_{m=1}^{\frac{\beta j+\ell\rho-\rho+1}{\rho}}(-1)^m m!\binom{\frac{\beta j+\ell\rho-\rho+1}{\rho}}{m}(\lambda jt^{\rho})^{\frac{\beta j+\ell\rho-\rho+1}{\rho}-m}\right].$$

Proof. Since

$$m(t)=\frac{1}{R(t)}\int_t^{\infty}R(x;\Theta)dx=\frac{1}{R(t;\Theta)}\left(\mu_1^{'}-I(t)\right),$$

where

$$\mu_1^{'}=\int_0^{\infty}xf(x;\Theta)dx,\qquad I(t)=\int_0^{t}R(x;\Theta)dx$$

$$R_X(x;\Theta)=1-p\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}-(1-p)\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega},$$

by using Lemma (6.1) when $r=1$, we get $\mu_1^{'}$.
Now, we want to get $I(t)$ as follows.

$$\begin{aligned} I(t) &= \int_0^t R(x)dx = t - p\int_0^t \left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta} dx - (1-p)\int_0^t \left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega} dx \\ &= t - pI_1 - (1-p)I_2, \end{aligned} \tag{24}$$

where

$$I_1 = \int_0^t \left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta} dx, \qquad I_2 = \int_0^t \left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega} dx,$$

since

$$\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k+\ell}(ck)^{\ell}(ai)^j}{j!l!} \binom{j}{k}\binom{\theta}{i} x^{bj+\ell d} e^{cjx^d},$$

then

$$\begin{aligned} I_1 &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k+\ell}(ck)^{\ell}(ai)^j}{j!l!} \binom{j}{k}\binom{\theta}{i} \int_0^t x^{bj+\ell d} e^{cjx^d} dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k+\ell}(ck)^{\ell}(ai)^j}{j!\ell!d(cj)^{\frac{bj+\ell d+1}{d}}} \binom{j}{k}\binom{\theta}{i} e^{cjt^d} \times \\ &\quad \left[(cjt^d)^{\frac{bj+\ell d-d+1}{d}} + \sum_{m=1}^{\frac{bj+\ell d-d+1}{d}} (-1)^m m! \binom{\frac{bj+\ell d-d+1}{d}}{m} (cjt^d)^{\frac{bj+\ell d-d+1}{d}-m}\right]. \end{aligned} \tag{25}$$

Similarly;

$$\begin{aligned} I_2 &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty} \frac{(-1)^{i+j+k+\ell}(\lambda k)^{\ell}(\alpha i)^j}{j!\ell!\rho(\lambda j)^{\frac{\beta j+\ell\rho+1}{\rho}}} \binom{j}{k}\binom{\omega}{i} e^{\lambda jt^{\rho}} \times \\ &\quad \left[(\lambda jt^{\rho})^{\frac{\beta j+\ell\rho-\rho+1}{\rho}} + \sum_{m=1}^{\frac{\beta j+\ell\rho-\rho+1}{\rho}} (-1)^m m! \binom{\frac{\beta j+\ell\rho-\rho+1}{\rho}}{m} (\lambda jt^{\rho})^{\frac{\beta j+\ell\rho-\rho+1}{\rho}-m}\right]. \end{aligned} \tag{26}$$

Substituting from Equations (25) and (26) into Equation (24), we get Equation (23). This completes the proof.
The mean past lifetime (MPL) corresponds to the mean time elapsed since the failure of T_i given that $T_i \le t$. In this case, the random variable of interest is $[t-T_i|T_i \le t]$, $i=1,2,\cdots,n$. This conditional random variable shows the time elapsed since the failure of T_i given that it failed at or before t. The expectation of this random variable gives the mean past lifetime (MPL) $P(t)$.

Lemma 6.4. If T is a random variable has MEGWGD, then the mean past lifetime (MPL), is given as follows for all component of $\Theta > 0$, $x > 0$.

$$P(t) = \frac{1}{F(t;\Theta)}[pI_1 + (1-p)I_2], \tag{27}$$

where

$$F_T(t;\Theta) = p\left[1-e^{-at^b(e^{ct^d}-1)}\right]^{\theta} + (1-p)\left[1-e^{-\alpha t^{\beta}(e^{\lambda t^{\rho}}-1)}\right]^{\omega},$$

$$I_1=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(ck)^{\ell}(ai)^{j}}{(j!\ell!d(cj)^{\frac{bj+\ell d+1}{d}}}\binom{j}{k}\binom{\theta}{i}e^{cjt^{d}}\times$$
$$\left[(cjt^{d})^{\frac{bj+\ell d-d+1}{d}}+\sum_{m=1}^{\frac{bj+\ell d-d+1}{d}}(-1)^{m}m!\binom{\frac{bj+\ell d-d+1}{d}}{m}(cjt^{d})^{\frac{bj+\ell d-d+1}{d}-m}\right],$$
$$I_2=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(\lambda k)^{\ell}(\alpha i)^{j}}{(j!\ell!\rho(\lambda j)^{\frac{\beta j+\ell\rho+1}{\rho}}}\binom{j}{k}\binom{\omega}{i}e^{\lambda jt^{\rho}}\times$$
$$\left[(\lambda jt^{\rho})^{\frac{\beta j+\ell\rho-\rho+1}{\rho}}+\sum_{m=1}^{\frac{\beta j+\ell\rho-\rho+1}{\rho}}(-1)^{m}m!\binom{\frac{\beta j+\ell\rho-\rho+1}{\rho}}{m}(\lambda jt^{\rho})^{\frac{\beta j+\ell\rho-\rho+1}{\rho}-m}\right].$$

Proof.

$$\begin{aligned}P(t)&=\frac{1}{F(t;\Theta)}\int_0^t F(x;\Theta)dx\\&=\frac{1}{F(t;\Theta)}\left[p\int_0^t\left[1-e^{-ax^{b}(e^{cx^{d}}-1)}\right]^{\theta}dx+(1-p)\int_0^t\left[1-e^{-\alpha x^{\beta}(e^{\lambda x^{\rho}}-1)}\right]^{\omega}dx\right]\\&=\frac{1}{F(t;\Theta)}[pI_1+(1-p)I_2].\end{aligned}\tag{28}$$

Substituting from Equations (25) and (26) into Equation (28), we get Equation (27). This completes the proof.

7 Parameters Estimation

7.1 Maximum likelihood estimates

In this section, we derive the maximum likelihood estimates of the unknown parameters c,θ,λ,ω and p of MEGWGD based on a complete sample when the parameters a,b are and d are equal to 0.6. and α,β and ρ are equal to 0.5. Consider a random sample $X_1,X_2,\cdots,X_n$ from MEGWGD. The likelihood function of this sample is

$$\ell=\prod_{i=1}^{n}f(x_i;c,\theta,\lambda,\omega,p).\tag{29}$$

The log likelihood function becomes

$$L=\sum_{i=1}^{n}\ln f(x_i;c,\theta,\lambda,\omega,p)=\sum_{i=1}^{n}\ln\left[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)\right].\tag{30}$$

The derivative of the log likelihood function with respect to each of the five parameters was as follows.

$$\begin{aligned}\frac{\partial L}{\partial c}&=\sum_{i=1}^{n}\frac{p}{pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)}\frac{\partial f_1(x_i)}{\partial c},\\\frac{\partial L}{\partial \theta}&=\sum_{i=1}^{n}\frac{p}{pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)}\frac{\partial f_1(x_i)}{\partial \theta},\\\frac{\partial L}{\partial \lambda}&=\sum_{i=1}^{n}\frac{1-p}{pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)}\frac{\partial f_2(x_i)}{\partial \lambda},\\\frac{\partial L}{\partial \omega}&=\sum_{i=1}^{n}\frac{1-p}{pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)}\frac{\partial f_2(x_i)}{\partial \omega},\\\frac{\partial L}{\partial p}&=\sum_{i=1}^{n}\frac{f_1(x_i;c,\theta)-f_2(x_i;\lambda,\omega)}{pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)}.\end{aligned}$$

MLEs can be obtained by equating the above equations to zero and solving for all the parameters. Numerical methods and computer facilities are required to solve these equations and get the required results.

7.2 Asymptotic confidence bounds

In this section, we derive the asymptotic confidence intervals of these parameters when $c, \theta, \lambda, \omega$ and p as the MLEs of the unknown parameters $c, \theta, \lambda, \omega$ and p can not be obtained in closed forms, by using variance covariance matrix I_0^{-1} see [6], where I_0^{-1} is the inverse of the observed information matrix

$$I_0^{-1} = \begin{bmatrix} -\frac{\partial^2 L}{\partial c^2} & -\frac{\partial^2 L}{\partial c \partial \theta} & -\frac{\partial^2 L}{\partial c \partial \lambda} & -\frac{\partial^2 L}{\partial c \partial \omega} & -\frac{\partial^2 L}{\partial c \partial p} \\ -\frac{\partial^2 L}{\partial \theta \partial c} & -\frac{\partial^2 L}{\partial \theta^2} & -\frac{\partial^2 L}{\partial \theta \partial \lambda} & -\frac{\partial^2 L}{\partial \theta \partial \omega} & -\frac{\partial^2 L}{\partial \theta \partial p} \\ -\frac{\partial^2 L}{\partial \lambda \partial c} & -\frac{\partial^2 L}{\partial \lambda \partial \theta} & -\frac{\partial^2 L}{\partial \lambda^2} & -\frac{\partial^2 L}{\partial \lambda \partial \omega} & -\frac{\partial^2 L}{\partial \lambda \partial p} \\ -\frac{\partial^2 L}{\partial \omega \partial c} & -\frac{\partial^2 L}{\partial \omega \partial \theta} & -\frac{\partial^2 L}{\partial \omega \partial \lambda} & -\frac{\partial^2 L}{\partial \omega^2} & -\frac{\partial^2 L}{\partial \omega \partial p} \\ -\frac{\partial^2 L}{\partial p \partial c} & -\frac{\partial^2 L}{\partial p \partial \theta} & -\frac{\partial^2 L}{\partial p \partial \lambda} & -\frac{\partial^2 L}{\partial p \partial \omega} & -\frac{\partial^2 L}{\partial p^2} \end{bmatrix}, \tag{31}$$

thus

$$I_0^{-1} = \begin{bmatrix} var(\hat{c}) & cov(\hat{c},\hat{\theta}) & cov(\hat{c},\hat{\lambda}) & cov(\hat{c},\hat{\omega}) & cov(\hat{c},\hat{p}) \\ cov(\hat{\theta},\hat{c}) & var(\hat{\theta}) & cov(\hat{\theta},\hat{\lambda}) & cov(\hat{\theta},\hat{\omega}) & cov(\hat{\theta},\hat{p}) \\ cov(\hat{\lambda},\hat{c}) & cov(\hat{\lambda},\hat{\theta}) & var(\hat{\lambda}) & cov(\hat{\lambda},\hat{\omega}) & cov(\hat{\lambda},\hat{p}) \\ cov(\hat{\omega},\hat{c}) & cov(\hat{\omega},\hat{\theta}) & cov(\hat{\omega},\hat{\lambda}) & var(\hat{\omega}) & cov(\hat{\omega},\hat{p}) \\ cov(\hat{p},\hat{c}) & cov(\hat{p},\hat{\theta}) & cov(\hat{p},\hat{\lambda}) & cov(\hat{p},\hat{\omega}) & var(\hat{p}) \end{bmatrix}, \tag{32}$$

The derivatives in I_0 are given as follows:

$$\begin{aligned}
\frac{\partial^2 L}{\partial c^2} &= -\sum_{i=1}^{n} \frac{p^2}{[p f_1(x_i;c,\theta)+(1-p) f_2(x_i;\lambda,\omega)]^2} \left(\frac{\partial f_1(x_i)}{\partial c}\right)^2 + \sum_{i=1}^{n} \frac{p}{p f_1(x_i;c,\theta)+(1-p) f_2(x_i;\lambda,\omega)} \frac{\partial^2 f_1(x_i)}{\partial c^2}, \\
\frac{\partial^2 L}{\partial c \partial \theta} &= -\sum_{i=1}^{n} \frac{p^2}{[p f_1(x_i;c,\theta)+(1-p) f_2(x_i;\lambda,\omega)]^2} \frac{\partial f_1(x_i)}{\partial c} \frac{\partial f_1(x_i)}{\partial \theta} + \sum_{i=1}^{n} \frac{p}{p f_1(x_i;c,\theta)+(1-p) f_2(x_i;\lambda,\omega)} \frac{\partial^2 f_1(x_i)}{\partial c \partial \theta}, \\
\frac{\partial^2 L}{\partial c \partial \lambda} &= -\sum_{i=1}^{n} \frac{p(1-p)}{[p f_1(x_i;c,\theta)+(1-p) f_2(x_i;\lambda,\omega)]^2} \frac{\partial f_2(x_i)}{\partial \lambda} \frac{\partial f_1(x_i)}{\partial c}, \\
\frac{\partial^2 L}{\partial c \partial \omega} &= -\sum_{i=1}^{n} \frac{p(1-p)}{[p f_1(x_i;c,\theta)+(1-p) f_2(x_i;\lambda,\omega)]^2} \frac{\partial f_2(x_i)}{\partial \omega} \frac{\partial f_1(x_i)}{\partial c}, \\
\frac{\partial^2 L}{\partial c \partial p} &= \sum_{i=1}^{n} \frac{f_2(x_i,\lambda,\omega)}{[p f_1(x_i;c,\theta)+(1-p) f_2(x_i;\lambda,\omega)]^2} \frac{\partial f_1(x_i)}{\partial c}, \\
\frac{\partial^2 L}{\partial \theta^2} &= -\sum_{i=1}^{n} \frac{p^2}{[p f_1(x_i;c,\theta)+(1-p) f_2(x_i;\lambda,\omega)]^2} \left(\frac{\partial f_1(x_i)}{\partial \theta}\right)^2 + \sum_{i=1}^{n} \frac{p}{p f_1(x_i;c,\theta)+(1-p) f_2(x_i;\lambda,\omega)} \frac{\partial^2 f_1(x_i)}{\partial \theta^2},
\end{aligned}$$

$$\frac{\partial^2 L}{\partial\theta\partial\lambda} = -\sum_{i=1}^{n}\frac{p(1-p)}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}\frac{\partial f_2(x_i)}{\partial\lambda}\frac{\partial f_1(x_i)}{\partial\theta},$$
$$\frac{\partial^2 L}{\partial\theta\partial\omega} = -\sum_{i=1}^{n}\frac{p(1-p)}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}\frac{\partial f_2(x_i)}{\partial\omega}\frac{\partial f_1(x_i)}{\partial\theta},$$
$$\frac{\partial^2 L}{\partial\theta\partial p} = \sum_{i=1}^{n}\frac{f_2(x_i,\lambda,\omega)}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}\frac{\partial f_1(x_i)}{\partial\theta},$$
$$\frac{\partial^2 L}{\partial\lambda^2} = -\sum_{i=1}^{n}\frac{(1-p)^2}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}\left(\frac{\partial f_2(x_i)}{\partial\lambda}\right)^2+\sum_{i=1}^{n}\frac{1-p}{pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)}\frac{\partial^2 f_2(x_i)}{\partial\lambda^2},$$
$$\frac{\partial^2 L}{\partial\lambda\partial\omega} = -\sum_{i=1}^{n}\frac{(1-p)^2}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}\frac{\partial f_2(x_i)}{\partial\omega}\frac{\partial f_1(x_i)}{\partial\lambda}+\sum_{i=1}^{n}\frac{1-p}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}\frac{\partial f_2(x_i)}{\partial^2\omega\partial\lambda},$$
$$\frac{\partial^2 L}{\partial\lambda\partial p} = -\sum_{i=1}^{n}\frac{f_1(x_i,\lambda,\omega)}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}\frac{\partial f_2(x_i)}{\partial\lambda},$$
$$\frac{\partial^2 L}{\partial\omega^2} = -\sum_{i=1}^{n}\frac{(1-p)^2}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}\left(\frac{\partial f_2(x_i)}{\partial\omega}\right)^2+\sum_{i=1}^{n}\frac{1-p}{pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)}\frac{\partial^2 f_2(x_i)}{\partial\omega^2},$$
$$\frac{\partial^2 L}{\partial\omega\partial p} = -\sum_{i=1}^{n}\frac{f_1(x_i,\lambda,\omega)}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}\frac{\partial f_2(x_i)}{\partial\omega},$$
$$\frac{\partial^2 L}{\partial\partial p^2} = -\sum_{i=1}^{n}\frac{f_1(x_i,\lambda,\omega)-f_2(x_i,\lambda,\omega)}{[pf_1(x_i;c,\theta)+(1-p)f_2(x_i;\lambda,\omega)]^2}.$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters c,θ,λ,ω and p by using variance covariance matrix as in the following forms

$$\hat{c}\pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{c})},\quad \hat{\theta}\pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\theta})},\quad \hat{\lambda}\pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\lambda})},\quad \hat{\omega}\pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\omega})},\quad \text{and}\quad \hat{p}\pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{p})}.$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})$th percentile of the standard normal distribution.

8 Data Analysis and Discussion

In this section, we present the analysis of a real data set using the MEGWG model. Therefore, provide a comparison between the proposed model and the other fitted models like mixture of generalized exponential distribution (MGED), mixture of exponential distribution (MED) and mixture of generalized Weibull-Gompertz distribution (MGWGD). The data have been obtained from [10]. It represents the strength of 1.5 cm glass fibres, measured at National physical laboratory, England.
The MEGWG model is used to fit this data set. The MLE(s) of the unknown parameter(s), the value of log likelihood (L), the corresponding Kolmogorove Smirnove (K-S)), Akaike information criterion (AIC), correct Akaike information criterion(CAIC) and Bayesian information criterion (BIC) test statistic and its respective p-values for four different models are given in Tables 1 and 2.

Table 1: The MLE(s) of the parameter(s), (K-S) values and P-values.

	MLEs						
The Model	$\hat{p}$	$\hat{\theta}$	$\hat{\omega}$	$\hat{c}$	$\hat{\lambda}$	$K-S$	$p-Value$
MED	0.258	-	-	0.401	0.205	0.225	0.00276
MGED	0.364	0.035	0.326	0.236	2.981	0.201	0.01045
MGWGD	0.487	-	-	0.941	1.269	0.194	0.01497
MEGWGD	0.746	0.015	0.701	0.814	1.258	0.113	0.38009

Table 2: The log likelihood, AIC, CAIC and BIC.

The Model	- L	AIC	CAIC	BIC
MED	27.358	60.71	61.12	67.14
MGED	22.107	54.21	55.26	64.92
MGWGD	14.296	34.59	34.99	41.02
MEGWGD	11.012	32.04	33.07	40.37

Table 1 and 2 show that, MEGWG model is the best among those distributions because it has the smallest value of (K-S), AIC, CAIC and BIC test. Figure 5 obtains the empirical and estimated reliability functions of the MEGWG model for data.

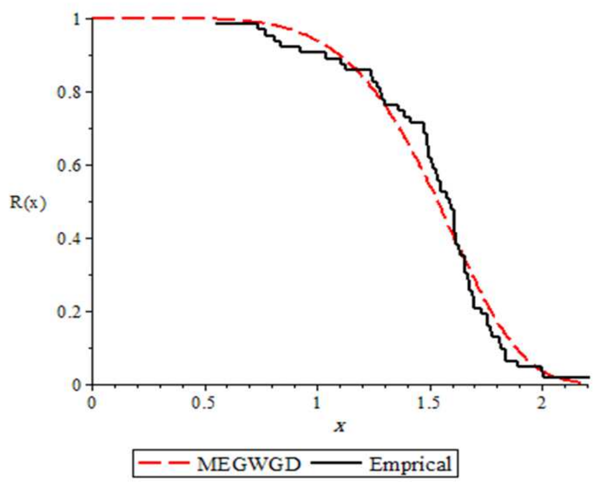

Fig. 5: The Empirical and estimated reliability functions of the MEGWG model for data.

By substituting the MLE of unknown parameters in Equation (31), we get estimation of the variance covariance matrix as

$$I_0^{-1} = \begin{bmatrix} 1.8\times10^{-8} & 9.5\times10^{-6} & 1.2\times10^{-12} & 8.2\times10^{-5} & 2.6\times10^{-7} \\ 9.5\times10^{-6} & 4.5\times10^{-2} & -1.6\times10^{-9} & 9.7\times10^{-3} & -9.9\times10^{-8} \\ 1.2\times10^{-12} & -1.6\times10^{-9} & 8.3\times10^{-5} & 1.3\times10^{-3} & 5.3\times10^{-5} \\ 8.2\times10^{-5} & 9.7\times10^{-3} & 1.3\times10^{-3} & 2.3\times10^{-4} & 1.3\times10^{-2} \\ 2.6\times10^{-7} & -9.9\times10^{-8} & 5.3\times10^{-5} & 1.3\times10^{-2} & 2.6\times10^{-3} \end{bmatrix}.$$

The approximate 90% two sided confidence interval of the parameters $c, \theta, \lambda, \omega$ and p are [0.671, 1.125], [0.0013, 0.023], [0.913, 1.514], [0.613, 0.817] and [0.511, 0.817] respectively.

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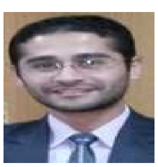

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