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Properties of a New Fractional Derivative without Singular Kernel

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Abstract: We introduce the fractional integral corresponding to the new concept of fractional derivative recently introduced by Caputo and Fabrizio and we study some related fractional differential equations.

Keywords: Fractional calculus, fractional derivative, fractional integral, Caputo.

1 Introduction

Let us recall the well known definition of Caputo fractional derivative [1]. Given $b > 0$, $f \in H^1(0,b)$ and $0 < \alpha < 1$, the Caputo fractional derivative of f of order α is given by

$$ {}^{\mathrm{C}}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds, \qquad t > 0. $$

Fractional calculus and, in particular, Caputo fractional derivative, finds numerous applications in different areas of science [2,3,4,5].

By changing the kernel $(t-s)^{-\alpha}$ by the function $exp(-\alpha(t-s)/(1-\alpha))$ and $1/\Gamma(1-\alpha)$ by $1/\sqrt{2\pi(1-\alpha^2)}$, one obtains the new Caputo-Fabrizio fractional derivative of order $0 < \alpha < 1$, which has been recently introduced by Caputo and Fabrizio in [6]. That is,

$$ {}^{\mathrm{CF}}D^{\alpha} f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'(s)\, ds, \qquad t \geq 0, $$

where $M(\alpha)$ is a normalization constant depending on α.

According to the new definition, it is clear that if f is a constant function, then ${}^{\mathrm{CF}}D^{\alpha} f = 0$ as in the usual Caputo derivative. The main difference between old and new definition is that, contrary to the old definition, the new kernel has no singularity for $t = s$.

It is well known that Laplace Transform plays an important role in the study of ordinary differential equations. In the case of this new fractional definition, it is also known (see [6]) that, for $0 < \alpha < 1$,

$$ \mathscr{L}\left[{}^{\mathrm{CF}}D^{\alpha} f(t)\right](s) = \frac{(2-\alpha)M(\alpha)}{2\big(s+\alpha(1-s)\big)} \big(s\mathscr{L}[f(t)](s) - f(0)\big), \qquad s > 0. \tag{1} $$

where $\mathscr{L}[g(t)]$ denotes the Laplace Transform of function g. So, it is clear that if we work with Caputo-Fabrizio derivative, Laplace Transform will also be a very useful tool.

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2 The associated *fractional* integral

After the notion of fractional derivative of order $0<\alpha<1$, that of fractional integral of order $0<\alpha<1$ becomes a natural requirement. In this section we obtain the fractional integral associated to the Caputo-Fabrizio fractional derivative previously introduced.

Let $0<\alpha<1$. Consider now the following fractional differential equation,

$$ {}^{\mathrm{CF}}D^{\alpha} f(t) = u(t), \qquad t \geq 0. \tag{2}$$

using Laplace transform, we obtain:

$$\mathscr{L}\left[{}^{\mathrm{CF}}D^{\alpha} f(t)\right](s) = \mathscr{L}[u(t)](s), \qquad s>0.$$

That is, using (1), we have that

$$\frac{(2-\alpha)M(\alpha)}{2\big(s+\alpha(1-s)\big)}\big(s\mathscr{L}[f(t)](s)-f(0)\big) = \mathscr{L}[u(t)](s), \qquad s>0,$$

or equivalently,

$$\mathscr{L}[f(t)](s) = \frac{1}{s}f(0) + \frac{2\alpha}{s(2-\alpha)M(\alpha)}\mathscr{L}[u(t)](s) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathscr{L}[u(t)](s), \qquad s>0.$$

Hence, using now well known properties of inverse Laplace transform, we deduce that

$$f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)\,ds + f(0), \qquad t\geq 0. \tag{3}$$

In other words, the function defined as

$$f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)\,ds + c, \qquad t\geq 0,$$

where $c\in\mathbb{R}$ is a constant, is also a solution of (2).

We can also rewrite fractional differential equation (2) as

$$\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'(s)\,ds = u(t), \qquad t\geq 0,$$

or equivalently,

$$\int_0^t \exp\left(\frac{\alpha}{1-\alpha}s\right) f'(s)\,ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\exp\left(\frac{\alpha}{1-\alpha}t\right)u(t), \qquad t\geq 0.$$

Differentiating both sides of the latter equation, we obtain that,

$$f'(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(u'(t)+\frac{\alpha}{1-\alpha}u(t)\right), \qquad t\geq 0.$$

Hence, integrating now from 0 to t, we deduce as in (3), that

$$f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}[u(t)-u(0)] + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)\,ds + f(0), \qquad t\geq 0.$$

Thus, as consequence, we expect that the fractional integral of Caputo-Fabrizio type must be defined as follows.

Definition 1. *Let $0<\alpha<1$. The fractional integral of order α of a function f is defined by,*

$$ {}^{\mathrm{CF}}I^{\alpha} f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)\,ds, \qquad t\geq 0.$$

Remark. Note that, according to the previous definition, the fractional integral of Caputo-Fabrizio type of a function of order $0 < \alpha < 1$ is an average between function f and its integral of order one.

Imposing

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1,$$

we obtain an explicit formula for $M(\alpha)$,

$$M(\alpha) = \frac{2}{2-\alpha}, \qquad 0 \leq \alpha \leq 1.$$

Due to this, we propose the following definition of fractional derivative of order $0 < \alpha < 1$.

Definition 2. *Let $0 < \alpha < 1$. The fractional Caputo-Fabrizio derivative of order α of a function f is given by,*

$$^{\mathrm{CF}}D_{*}^{\alpha} f(t) = \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'(s)\, ds, \qquad t \geq 0.$$

3 Some fractional differential equations

In this section we study some simple but useful fractional differential equations.

Lemma 1. *Let $0 < \alpha < 1$ and f be a solution of the following fractional differential equation,*

$$^{\mathrm{CF}}D^{\alpha} f(t) = 0, \qquad t \geq 0. \tag{4}$$

Then, f is a constant function. The converse, as indicated in the Introduction, is also true.

Proof. From (3), we obtain that the solution of (4) must satisfy $f(t) = f(0)$ for all $t \geq 0$. Hence, it is clear that f must be a constant function. □

Proposition 1. *Let $0 < \alpha < 1$. Then, the unique solution of the following initial value problem*

$$^{\mathrm{CF}}D^{\alpha} f(t) = \sigma(t), \qquad t \geq 0, \tag{5}$$

$$f(0) = f_0 \in \mathbb{R}; \tag{6}$$

is given by

$$f(t) = f_0 + a_\alpha\big(\sigma(t) - \sigma(0)\big) + b_\alpha I^1 \sigma(t), \qquad t \geq 0, \tag{7}$$

where $I^1\sigma$ denotes a primitive of σ and

$$a_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}, \qquad b_\alpha = \frac{2\alpha}{(2-\alpha)M(\alpha)}. \tag{8}$$

Proof. Suppose that the initial value problem (5)-(6) has two solutions, f_1 and f_2. In that case, we have that

$$^{\mathrm{CF}}D^{\alpha} f_1(t) - {}^{\mathrm{CF}}D^{\alpha} f_2(t) = \left[{}^{\mathrm{CF}}D^{\alpha} f_1 - f_2\right](t) = 0 \qquad \text{and} \qquad \big(f_1 - f_2\big)(0) = 0.$$

So, by Lemma 1, we have that $f_1 - f_2 = 0$. That is $f_1(t) = f_2(t)$ for all $t \geq 0$.

By (3), it is clear that the function defined by (7) is a solution of the fractional differential equation (5). Moreover, if we substitute t by 0 in (7), we obtain f_0.

Hence, the function defined by (7) is the unique solution of initial value problem (5)-(6). □

Remark. For $\alpha = 1$, we have that the solution of (5) is the usual primitive of σ.

Now, we consider the following linear fractional differential equation

$$^{\mathrm{CF}}D^{\alpha} f(t) = \lambda f(t) + u(t), \qquad t \geq 0, \tag{9}$$

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$ ($\lambda = 0$ corresponds to the case previously studied).

From Proposition 1, we have that solving equation (9) is equivalent to find a function f such that

$$f(t) = f_0 + a_\alpha \left[\lambda \left(f(t) - f_0\right) + u(t) - u(0)\right] + b_\alpha \int_0^t [\lambda f + u](s)\, ds, \qquad t \geq 0$$

where a_α, b_α are given by (8). Equivalently, we must find f such that

$$(1 - \lambda\, a_\alpha)\; f(t) - \lambda\, b_\alpha I^1 f(t) = (1 - \lambda\, a_\alpha)\; f_0 + a_\alpha \left(u(t) - u(0)\right) + b_\alpha I^1 u(t), \qquad t \geq 0.$$

If $\lambda\, a_\alpha = 1$, we obtain:

$$f(t) = -\frac{a_\alpha}{\lambda\, b_\alpha} u'(t) - \frac{b_\alpha}{\lambda} u(t), \qquad t \geq 0.$$

In the other case, i. e., $\lambda\, a_\alpha \neq 1$, we have that:

$$f(t) - \frac{\lambda\, b_\alpha}{1 - \lambda\, a_\alpha} I^1 f(t) = \tilde{\sigma}(t), \qquad t \geq 0, \tag{10}$$

where

$$\tilde{\sigma}(t) = f_0 + \frac{a_\alpha}{1 - \lambda\, a_\alpha} \left(u(t) - u(0)\right) + \frac{b_\alpha}{1 - \lambda\, a_\alpha} I^1 u(t), \qquad t \geq 0.$$

The case $\lambda = 0$ is trivial, and we obtain $f = \tilde{\sigma}$. If $\lambda \neq 0$, we see that (10) can be rewritten as

$$f(t) - \tilde{\lambda}\, I^1 f(t) = \tilde{\sigma}(t), \qquad t \geq 0,$$

where

$$\tilde{\lambda} = \frac{\lambda\, b_\alpha}{1 - \lambda\, a_\alpha}.$$

Hence,

$$f'(t) = \tilde{\lambda}\, f(t) + \tilde{\sigma}(t), \qquad t \geq 0.$$

Thus, we have obtained an ordinary differential equation, which has a unique solution if we consider an initial condition.

In consequence, we have proved the following result.

Proposition 2. *Let $0 < \alpha < 1$. Then, initial value problem given by*

$$\begin{aligned} {}^{\mathrm{CF}}D^\alpha f(t) &= \lambda\, f(t) + u(t), \qquad t \geq 0, \\ f(0) &= f_0 \in \mathbb{R}; \end{aligned}$$

has a unique solution for any $\lambda \in \mathbb{R}$.

4 Nonlinear fractional differential equations

Theorem 1. *Let $0 < \alpha < 1$, $T > 0$ and $\varphi\colon [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function such that there exits $L > 0$ satisfying,*

$$|\varphi(t,s_1) - \varphi(t,s_2)| \leq L\, |s_1 - s_2| \qquad \textit{for all} \quad s_1, s_2 \in \mathbb{R}.$$

If $(a_\alpha + b_\alpha\, T)\, L < 1$, then the initial value problem given by

$${}^{\mathrm{CF}}D^\alpha f(t) = \varphi(t, f(t)), \qquad t \in [0,T], \tag{11}$$
$$f(0) = f_0 \in \mathbb{R}; \tag{12}$$

has a unique solution on $\mathscr{C}[0,T]$.

Proof. Let $\mathscr{C}[0,T]$ be the space of all continuous functions defined on the interval $[0,T]$ endowed with the usual supremum norm, that is,

$$\|f\| = \sup_{t\in[0,T]} |f(t)| \qquad \text{for all} \quad f\in\mathscr{C}[0,T].$$

We consider the operator $\mathscr{N}:\mathscr{C}[0,T]\longrightarrow\mathscr{C}[0,T]$ defined by,

$$\mathscr{N} f(t) = c + a_\alpha\,\varphi(t,f(t)) + b_\alpha\int_0^t \varphi(s,f(s))\,ds, \qquad \text{for all} \quad f\in\mathscr{C}[0,T],$$

where

$$c = -a_\alpha\,\varphi(0,f_0) + f_0$$

By (3), finding a solution of (11)-(12) in $\mathscr{C}[0,T]$ is equivalent to finding a fixed point of the operator $\mathscr{N}$.

Since for all $f_1,\, f_2\in\mathscr{C}[0,T]$ and all $t\in[0,T]$ we have that

$$\begin{aligned}
|\mathscr{N} f_1(t) - \mathscr{N} f_2(t)| &= \left| a_\alpha\Big(\varphi\big(t,f_1(t)\big) - \varphi\big(t,f_2(t)\big)\Big) + b_\alpha\left(\int_0^t \varphi\big(s,f_1(s)\big)\,ds - \int_0^t \varphi\big(s,f_2(s)\big)\,ds\right)\right| \\
&\le a_\alpha\left|\varphi\big(t,f_1(t)\big) - \varphi\big(t,f_1(t)\big)\right| + b_\alpha\int_0^t \left|\varphi\big(s,f_1(s)\big) - \varphi\big(s,f_2(s)\big)\right| ds \\
&\le a_\alpha L\,|f_1(t) - f_2(t)| + b_\alpha L\int_0^t |f_1(s) - f_2(s)|\,ds \\
&\le (a_\alpha + b_\alpha T)\,L\,\|f_1 - f_2\|,
\end{aligned}$$

we conclude that operator $\mathscr{N}$ is a contraction. The statement follows now from Banach's Fixed Point Theorem. □

5 Application to fractional falling body problem

Consider a mass m falling due to gravity. The net force acting on the body is equal to the rate of change of the momentum of that body. For constant mass, applying the classical Newton second law, we have

$$mv'(t) = mg - kv(t),$$

where g is the gravitational constant, and the air resistance is proportional to the velocity with proportionality constant k. If air resistance is negligible, then $k=0$ and the equation simplifies to

$$v'(t) = g.$$

If we replace $D^1 = v'$ by D^α we have the following fractional falling body equation

$$^{\mathrm{CF}}D^\alpha v(t) = -\frac{k}{m}v(t) + g.$$

For an initial velocity $v(0) = v_0$ then, according to Proposition 2, it has a unique solution.

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