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On Fractional Model of an HIV/AIDS with Treatment and Time Delay

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Abstract: In this article, a fractional model of HIV/AIDS that includes treatment and a time delay is investigated. The global dynamics of the spread of the disease are discussed using the reproduction number. There is no infected equilibrium if $R_0 \leq 1$. We also show that the equilibrium point E_1 is globally asymptotically stable (the disease disappear). When $R_0 > 1$, there is a unique infected point E_2. We introduce sufficient conditions for the stability of E_2. Sufficient conditions are given to guarantee the asymptotic stability of the equilibria independent of time delay. We present threshold values of the time delay that the treatment will be succeeded if its positive effects appear before this values. A finite difference method for a general fractional system is presented and is used in the numerical simulations of the model.

Keywords: Epidemic model, fractional differential equations, stability, non standard finite, difference method and time delay.

1 Introduction

Fractional calculus is an important tool to formulate many physical problems and recently, a large number of fractional order models are appeared. These models represent various applications in fluid mechanics, viscoelasticity, biology and engineering [1-10]. In what concerns application of fractional order derivatives to epidemiological models some relevant works start to appear [11-15]. The treatment and the existence of time delays in treatment have a great effect in the dynamical behavior of HIV/AIDS [9,16-19].

Recently Yan et al. [9] considered an HIV/AIDS model including fractional differentiation with time delay.

In this article, we study the stability behavior of a fractional model for HIV/AIDS dynamics that includes treatment and a time delay. This article is organized as follows: We display the mathematical model of our system in Section 1. Stability analysis of the fractional model for HIV/AIDS is presented in Section 2 while the stability behavior of the model including existence of time delay in treatment is considered in Section 3. In Section 4, we introduce a non-standard finite difference scheme of a general system. Illustrative examples are discussed in Section 5. Finally, our conclusion is given in Section 6.

2 Description of the Model

The total population in our model is divided into a susceptible class of size S, the infection population is classified into two groups, asymptomatic phase of size I, symptomatic phase of size J and the group of AIDS patients with size A. The fractional order system takes the form:

$$\begin{aligned}
&D^{\alpha} S(t) = \mu k - c\beta\,(I(t)+bJ(t))\,S(t) - \mu\,S(t), \qquad \alpha \in (0,1],\\
&D^{\alpha} I(t) = c\beta\,(I(t)+bJ(t))\,S(t) - (\mu+k_1)\,I(t) + \delta\,J(t),\\
&D^{\alpha} J(t) = k_1\,I(t) - (\mu+k_2+\delta)\,J(t),\\
&D^{\alpha} A(t) = k_2 J(t) - (\mu+d)\,A(t),\\
&S(0)=S_0\,,\ I(0)=I_0\,,\ J(0)=J_0 \quad \text{and} \quad A(0)=A_0
\end{aligned} \tag{1}$$

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with initial conditions:

$$S(0)=S_0\ ,\ I(0)=I_0\ ,\ J(0)=J_0 \quad \text{and} \quad A(0)=A_0,$$

where μk is the recruitment rate of the population, c is the average number of contacts of an individual per unit time. β , $b\beta$ are probability of disease transmission per contact by an infective in stages *I and J* respectively. The parameter $b>1$ captures the fact that the individuals in the symptomatic phase stage (*J)* are more infectious than the asymptomatic phase stage (*I)*. μ is the death rate, k_1 , k_2 are transfer rates from stage *I*to stage *J* and from stage *J* to AIDS cases respectively. δ is the treatment rate from stage *J* to stage *I* and *d* is the death rate for AIDS.

Model (1) is the generalization, to fractional order, of the model proposed by Cai et al [16]. Their model is equivalent to system (1) with the fractional-order derivative $\alpha=1$.

A new technique based on the non standard finite difference method (NSFDM) is developed to solve system (1). Before we study to the stability analysis of the fractional order system (1); we first give a definition of fractional order differentiation.

Definition 1 [20]: Caputo fractional derivative of order $\alpha\in(n-1,n)$ of a function $f:R^{+}\rightarrow R$ is given by:

$$D^{\alpha} f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\,dx,$$

where $\Gamma(.)$ is the Gamma function. Also we need the following Lemma.

Lemma 1[21]: Let $X^{*}=(x_1^{*},\, x_2^{*},\, ...,\, x_n^{*})^{T}$ be an equilibrium point of the fractional differential equations:

$$D^{\alpha}X(t)=F(X)\ ,\ \alpha\in(0,1] \quad \text{and} \quad X(0)=X_0, \tag{2}$$

where $X=(x_1,\, x_2\,,\, ...,\, x_n)^{T}$ and $F=(f_1\,,\, f_2\,,\, \dots,\, f_n)^{T}$. Then, X^{*} is locally asymptotically stable if all the eigenvalues of the Jacobin matrix $B(X^{*})$ of system (2) satisfies:

$$|\arg(eig\,B(X^{*}))|>\frac{\alpha\pi}{2}, \tag{3}$$

where $B(X^{*})=[b_{ij}]_{X=X^{*}}$, $i,\ j=1,2,...,n$ and $b_{ij}=\partial f_i/\partial x_j$.

Return to the system (1), then we only analyze the following subsystem:

$$\begin{aligned}
&D^{\alpha}S(t)=\mu k-c\beta\,(I(t)+bJ(t))S(t)-\mu S(t)\\
&D^{\alpha}I(t)=c\beta\,(I(t)+bJ(t))S(t)-(\mu+k_1)I(t)+\delta J(t)\\
&D^{\alpha}J(t)=k_1 I(t)-(\mu+k_2+\delta)J(t).
\end{aligned} \tag{4}$$

It follows from system (4) that

$$D^{\alpha}\,(S(t)+I(t)+J(t))=\mu k-\mu\,(S(t)+I(t)+J(t))-k_2 J(t)$$

. Since $k_2 J(t)\geq 0$ and consider $\Phi(t)=S(t)+I(t)+J(t)-k$, then we have

$$D^{\alpha}\,\Phi(t)+\mu\,\Phi(t)\leq 0. \tag{5}$$

Using Laplace transform, we get the solution of Eq.(5)as:

$$\Phi(t)\leq ct^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}) \text{ where } c=D^{\alpha-1}\,\Phi(t)\big|_{t=0}$$

. Then $\lim_{t\rightarrow\infty}\Phi(t)\leq 0$ and the feasible region for system (4) is

$$\Omega=\{\,(S,I,J):S+I+J\leq k\,,\,S>0,\,I\geq 0,\,J\geq 0\}$$

.

To evaluate the equilibrium points of system (4), we solve the nonlinear algebraic equations

$$D^{\alpha}S(t)=D^{\alpha}I(t)=D^{\alpha}J(t)=0$$

.

Applying the next generation method, we calculate the basic reproduction number. it is proved in [16] that for $R_0\leq 1$ there exists only the disease free equilibrium $E_1(k,0,0)$, and for $R_0>1$ there exists only the endemic equilibrium $E_2(S_2,I_2,J_2)$ (in addition to E_1), where:

$$R_0=\frac{c\beta\,k(\mu+k_2+\delta+bk_1)}{(\mu+k_1)\,(\mu+k_2)+\mu\,\delta}$$

, $S_2=\frac{k}{R_0},\ I_2=\frac{\mu\,k\,(\mu+k_2+\delta)}{(\mu+k_1)\,(\mu+k_2)+\mu\,\delta}\,(1-\frac{1}{R_0})$ and $J_2=\frac{k_1}{\mu+k_2+\delta}\,I_2.$

3 Equilibria and Their Stability of the Fractional Order HIV/AIDS Model

In this section, we compute the equilibrium points of system (4). To achieve this target, we obtain the Jacobin matrix of system (4) in terms of $E_i,\ i=1,2$, it has the form:

$$B(E_i)=\begin{pmatrix} -[\mu+c\beta(I_i+bJ_i)] & -c\beta S_i & -c\beta S_i b \\ c\beta(I_i+bJ_i) & c\beta S_i-(\mu+k_1) & \delta+c\beta S_i b \\ 0 & k_1 & -(\mu+k_2+\delta) \end{pmatrix}. \tag{6}$$

Hence the associated transcendental equation of Eq. (4) is:

$$|B(E_i)-\lambda I|=0, \tag{7}$$

where I is the identity matrix. We summarize the stability behavior of the disease free equilibrium point $E_1(k,0,0)$ in the following theorem:

Theorem 2.1: Consider the disease free equilibrium point E_1 of system (4) with $\alpha\in(0,1]$, then E_1 is locally asymptotically stable if $R_0<1$ and unstable for $R_0>1$.

`Proof:`

The transcendental characteristic equation of system (4) at E_1 is given by

$$(\mu+\lambda)(\lambda^2+a_1\lambda+a_2)=0, \tag{8}$$

where

$$\begin{aligned} a_1 &= 2\mu+k_1+k_2+\delta-c\beta k,\ a_2 \\ &= (\mu+k_1)(\mu+k_2)+\mu\delta-c\beta k(\mu+bk_1+k_2+\delta). \end{aligned} \tag{9}$$

One eigenvalue of Eq.(8) is $\lambda=-\mu$. To find the other two eigenvalues, we write a_1 and a_2 in terms of the basic reproduction number R_0, where:

$$\begin{aligned} a_1 &= (1/(\mu+bk_1+k_2+\delta)), \\ &\quad [(1-R_0)((\mu+k_1)(\mu+k_2)+\mu\delta)+k_1\delta \\ &\quad +bk_1(\mu+k_1)+(\mu+k_2+\delta)(\mu+bk_1+k_2+\delta)], \\ a_2 &= (1-R_0)[(\mu+k_1)(\mu+k_2)+\mu\delta.] \end{aligned} \tag{10}$$

Then for $R_0<1$, we have $a_i>0$, $i=1,2$ hence the other two eigenvalues are

$$\lambda_{2,3}=\frac{1}{2}[-a_1\pm\sqrt{a_1^2-4a_2}]. \tag{11}$$

It is clear that $Re\,\lambda_i<0$ for $i=1,2,3$. By lemma 1.1, E_1 is locally asymptotically stable.
For $R_0>1$ this leads to $a_2<0$ which gives a positive real eigenvalue, consequently E_1 is unstable.

We discuss the properties of the solution near the infected equilibrium point E_2, the transcendental characteristic equation of $B(E_2)$ is:

$$p(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0, \tag{12}$$

where

$$\begin{aligned} a_1 &= \mu+k_1+\mu+k_2+\mu+\delta+c\beta(I_2+bJ_2-S_2), \\ a_2 &= (\mu+k_1)(\mu+k_2)+\mu\delta+\mu(\mu+k_1+\mu+k_2+\delta) \\ &\quad +k_1c\beta(I_2+bJ_2) \\ &\quad +(\mu+k_2+\mu+\delta)c\beta[I_2+bJ_2-S_2]-bk_1c\beta S_2, \\ a_3 &= \mu[(\mu+k_1)(\mu+k_2)+\mu\alpha-c\beta S_2(\mu+k_2+\delta+bk_1)] \\ &\quad +c\beta(I_2+bJ_2)[(\mu+k_1)(\mu+k_2)+\mu\delta]. \end{aligned} \tag{13}$$

Definition 2 [22]: The discriminate $D(p)$ of a real cubic polynomial, $p(\lambda)$ defined by Eq. (12) is:

$$D(p)=18\,a_1a_2a_3+(a_1a_2)^2-4a_3(a_1)^3-4(a_2)^3-27(a_3)^2. \tag{14}$$

The following theorem describes the required conditions for local stability of E_2 according to the sign of $D(p)$.
Theorem 2.2 Let $R_0 > 1$, then the unique infected equilibrium point E_2 of the system (4) is locally asymptotically stable if either

$$D(p) > 0 \quad \text{and} \quad \alpha \in (0,1] \tag{15}$$

or

$$D(p) < 0 \quad \text{and} \quad \alpha \in [0,2/3), \tag{16}$$

where $D(p)$ is defined by Eq. (14) with the coefficients in (13).
Proof:
The coefficients in (13) of the transcendental characteristic equation (12) can be written in the following form:

$$\begin{aligned}
&a_1 = \mu\,(R_0-1)+\mu+k_2+\mu+\delta+\eta,\\
&a_2 = \mu\,[\,(R_0-1)\,(\mu+k_1+\mu+k_2+\delta)+\mu+k_2+\delta+\eta\,],\\
&a_3 = \mu\,(R_0-1)\,[\,(\mu+k_1)\,(\mu+k_2)+\mu\,\delta\,]\\
&\quad \text{where} \quad \eta = k_1\,[\,\delta+b\,(\mu+k_1)\,]/\,(\mu+k_2+\delta+b\,k_1).
\end{aligned} \tag{17}$$

If $D(p) > 0$, then $p(\lambda)$ has three distinct real roots. From Eq. (17), $R_0 > 1$ (For the existence of E_2), and $a_i > 0$, $i = 1, 2, 3$. Then by the aid of Descartes' rule of signs, we can conclude that the three roots are all negative when $a_1\,a_2 - a_3 > 0$. And hence all the eigenvalues of Eq.(12) satisfy condition (3) for all $\alpha \in (0,1]$. Simplifying the value of $a_1\,a_2 - a_3$, we have:

$$\begin{aligned}
a_1\,a_2 - a_3 = {}& \mu\,(R_0-1)\,(k_2\,\delta+(\mu+k_2)^2)+\mu\,(\mu+k_2)\\
&[\mu+k_2+\delta+\eta\,]+\mu\,[\,(R_0-1)\,\mu+\mu+\delta+\eta\,]\\
&\times\,[\,\mu+k_2+\delta+(R_0-1)\,(\mu+k_1+\mu+k_2+\delta)+\eta\,\delta\,] > 0,\\
&\text{for all} \quad R_0 > 1, \quad \alpha \in [0,1].
\end{aligned} \tag{18}$$

Now if $D(p) < 0$, and using Descartes' rule of signs, $p(\lambda)$ has one negative real root $\lambda_1 = b$ and two conjugate complex roots$\lambda_{2,3} = x \pm iy$. Eq. (12) can be factorized as:

$$(\lambda-b)\,(\lambda-x-iy)\,(\lambda-x+iy) = 0. \tag{19}$$

Equating the coefficients of Eq. (12) and (19), considering $b < 0$ and $a_i > 0$, $i = 1, 2, 3$, we have:

$$-2x-b > 0\,,\; x^2+y^2+2bx > 0 \quad \text{and} \quad -(x^2+y^2)\,b > 0. \tag{20}$$

Then if $x < 0$, all the eigenvalues of Eq.(12) satisfy condition (3), while if $x > 0$ we should have $(y/x)^2 > 3$ to satisfy the relations in (20). So the fraction of derivatives α must belong to the interval $\alpha \in [0, 2/3)$.

4 Stability Behavior of the System with Delay Time

To investigate the effect of time delay on the stability behavior of system (4). Let τ represents the time interval from starting of treatment in the symptomatic stage (J) using different techniques up to the effect of this treatment exists. So we rewrite system (4) to be:

$$\begin{aligned}
&D^{\alpha}\,S(t) = \mu\,k - c\,\beta\,(I(t)+b\,J(t))\,S(t) - \mu\,S(t),\\
&D^{\alpha}\,I(t) = c\,\beta\,(I(t)+b\,J(t))\,S(t) - (\mu+k_1)\,I(t)+\delta\,J(t-\tau),\\
&D^{\alpha}\,J(t) = k_1\,I(t) - (\mu+k_2)\,J(t) - \delta\,J(t-\tau)\\
&S(0) = S_0\,,\; I(0) = I_0 \quad \text{and} \quad J(t) = J_0\,,\; t \in [0,\tau].
\end{aligned} \tag{21}$$

We investigate the behavior of the disease free equilibrium point E_1 when $R_0 < 1$in the following theorem.
Theorem 3.1 The disease free equilibrium point E_1 of system (21) with $\alpha \in (0, 1]$ is asymptotically stable when $R_0 < 1$ for any time delay $\tau \geq 0$ if:

$$\gamma = \delta^2 - \min\{A_1\,, A_2\} < 0, \tag{22}$$

where

$$\begin{aligned}
&A_1 = [\delta\,k_1(1+R_0)\,(k_2+\mu\,(1-b)\,)/(1-R_0)+(\mu+k_2+b\,k_1)\,(\mu+k_1)\,(\mu+k_2)\,]/\mu,\\
&A_2 = (2\mu+k_1+k_2-c\beta\,k)^2+2\,c\,\beta\,k(\mu+k_2+b\,k_1)-2\,(\mu+k_1)\,(\mu+k_2).
\end{aligned}$$

Proof:

The Jacobin matrix of the differential equations (21) at E_1 is:

$$B(E_1)=\begin{pmatrix} -\mu & -c\,\beta\,k & -c\,\beta\,k \\ 0 & c\,\beta\,k-(\mu+k_1) & c\,\beta\,k+\delta\,e^{-\lambda\,\tau} \\ 0 & k_1 & -(\mu+k_2+\delta\,e^{-\lambda\,\tau}) \end{pmatrix},.$$

The transcendental characteristic equation of $B(E_1)$ is

$$(\lambda+\mu)\,[\lambda^2+(b_1+c_1\,e^{-\lambda\,\tau})\,\lambda+(b_2+c_2\,e^{-\lambda\,\tau})\,]=0. \tag{23}$$

with the coefficients

$$\begin{aligned} &b_1=2\mu+k_1+k_2-c\,\beta\,k, \\ &c_1=\delta,\; b_2=\;(\mu+k_2)\,(\mu+k_1-c\beta\,k)-c\beta\,k\,k_1 \\ &\textit{and} \\ &c_2=\delta(\mu-c\,\beta\,k). \end{aligned}$$

The eigenvalues of Eq.(23) are $\lambda_1=-\mu$ and $\lambda_{2\,,3}$ are the roots of the equation:

$$\lambda^2+(b_1+c_1\,e^{-\lambda\,\tau})\,\lambda+(b_2+c_2\,e^{-\lambda\,\tau})=0. \tag{24}$$

By Theorem 2.1, $Re\,\lambda_i<0,\; i=1,2,3$ for $\tau=0$. By increasing τ we seek about if $Re\,\lambda_{2\,,3}$ change its sign to be positive. This can occur if we get pure imaginary eigenvalues ($\lambda_{2\,,3}=\pm i\,\omega$).
Substitute by the value of $\lambda=i\;\omega$ in Eq. (24), then we have:

$$\begin{aligned} \omega\,c_1\,\sin\,\omega\tau+c_2\,\cos\,\omega\tau&=\omega^2-b_2 \\ \omega\,c_1\,\cos\,\omega\tau-c_2\,\sin\,\omega\tau&=-\omega\,b_1. \end{aligned} \tag{25}$$

Eliminating τ from the two equations in (25), we get

$$y^2+(b_1^2-2\,b_2-c_1^2)\,y+\;(b_2^2-c_2^2)\;=0, \tag{26}$$

where $y=\omega^2$, hence there is no positive roots for Eq.(26) if $b_1^2-2\,b_2-c_1^2>0$ and $b_2^2-c_2^2>0$. In this case the values of $Re\,\lambda_{2\,,3}$ cannot change their sign to be positive. Since

$$\begin{aligned} &b_1^2-2\,b_2-c_1^2= \\ &\quad(2\mu+k_1+k_2-c\beta\,k)^2-2\,[\,(\mu+k_2)\,(\mu+k_1-c\,\beta\,k) \\ &\qquad-c\beta\,k\,b\,k_1\,]-\delta^2>0, \end{aligned} \tag{27}$$

by applying the condition (22) we have: $b_2^2-c_2^2=(b_2+c_2)\,(b_2-c_2)$.
Since we can write $b_2+c_2=(1-R_0)\,[\,(\mu+k_1)\,(\mu+k_2)+\delta\mu\,]>0$ where $R_0<1$, and

$$\begin{aligned} b_2-c_2&=(\mu+k_2)\,(\mu+k_1-c\beta\,k)-c\beta\,k\,bk_1-\delta(\mu-c\beta\,k) \\ &=[\delta\,k_1(1+R_0)\,(k_2+\mu\,(1-b)\,) \\ &\quad+(1-R_0)\;(\mu+k_2+b\,k_1)\,(\mu+k_1)\,(\mu+k_2)-\mu\,\delta^2\,]/(\mu+k_2+\delta+b\,k_1)>0 \end{aligned}$$

by condition (22). Hence the proof is completed.
If the parameters of system (21) do not satisfy condition (22), we have the following theorem:
Theorem 3.2 The disease free equilibrium point E_1 of system (21) is asymptotically stable when $R_0<1$ and $\alpha\in[0,1)$ for any time delay $\tau<\tau^*$, where:

$$\tau^*=1/(x^*\,\sin(\alpha\pi/2)\,)\;\tan^{-1}f(x^*) \tag{28}$$

and x^* is the smallest positive value of x which satisfies:

$$f(x)=\frac{-x[c_1\sin(\alpha\pi/2)\,x^2+c_2\,\sin(\alpha\pi)\,x+(b_1c_2-b_2c_1)\,\sin(\alpha\pi/2)\,]}{\cos(\alpha\pi/2)\,[c_1x^3+(b_1c_2+b_2c_1)\,x]+c_2\,\cos(\alpha\pi)\,x^2+b_1c_1\,x^2+b_2c_2)}\;>0. \tag{29}$$

Proof:
The eigenvalues of the characteristic equation (23) are $\lambda_1 = -\mu$ and $\lambda_{2,3}$ roots of Eq. (24) and we have $Re\,\lambda_{2,3} < 0$ for $\tau = 0$. By increasing τ we seek about if $\lambda_{2,3}$ become out of the stability region. Let $\lambda = r e^{i\alpha\pi/2}$, in Eq. (24), we have:

$$r^2\cos(\alpha\pi) + rb_1\cos(\alpha\pi 2) + rc_1\cos(\alpha\pi/2 - r\tau\sin(\alpha\pi/2))\, e^{-r\tau\cos(\alpha\pi/2)} + c_2\cos(r\tau\,\sin(\alpha\pi/2))\, e^{-r\tau\cos(\alpha\pi/2)} + b_2 = 0. \quad (30)$$

$$r^2\sin(\alpha\pi) + r\,b_1\sin(\alpha\pi/2) + rc_1\sin(\alpha\pi/2 - r\tau\sin(\alpha\pi/2))\, e^{-r\tau\cos(\alpha\pi/2)} - c_2\,\sin(r\tau\,\sin(\alpha\pi/2))\, e^{-r\tau\cos(\alpha\pi/2)} + b_2 = 0. \quad (31)$$

Trying to solve the non linear set of equations (30) and (4), we can get the value of τ in terms of r in the following formula:

$$\tau = (1/r\,\sin(\alpha\pi/2))\,\tan^{-1} f(r).$$

Hence there exists a critical time delay transfer system (21) from its stability region into unstable region if there is a minimum positive value of $r = x^*$ satisfying that $f(x^*) > 0$. This complete the proof.

Similar analysis can be done to defined a threshold value of the time delay τ^* when $R_0 > 1$. The time delay τ must not exceed τ^* to guarantee the asymptotic stability of system (21). The characteristic equation of system (21) in terms of E_2 is:

$$\lambda^3 + (b_1 + c_1\, e^{-\lambda\tau})\,\lambda^2 + (b_2 + c_2\, e^{-\lambda\tau})\,\lambda + (b_3 + c_3\, e^{-\lambda\tau}) = 0, \quad (32)$$

with the coefficients

$$\begin{aligned} b_1 &= 2\mu + k_1 + k_2 + \mu R_0 - c\beta k/R_0, \\ b_2 &= \mu R_0(2\mu + k_1 + k_2) - \mu\delta + (\delta - \mu)\, c\beta k/R_0, \\ b_3 &= \mu\,(\mu + k_1)(\mu + k_2)(R_0 - 1) + \mu\delta\,(c\beta k/R_0 - \mu), \end{aligned} \quad (33)$$

$$\begin{aligned} c_1 = \delta,\ \ c_2 &= \delta\,(\mu + \mu R_0 - c\beta k/R_0), \\ c_3 &= \delta\mu\,(\mu R_0 - c\beta k/R_0). \end{aligned} \quad (34)$$

The following theorem investigates the effect of existence of time delay on the stability of the infected equilibrium point E_2.

Theorem 3.3 Let $R_0 > 1$, the equilibrium point E_2 of (21) is asymptotically stable for any time delay $\tau \geq 0$ if the following conditions and condition (15) are satisfied:

$$\psi = (\mu + k_1)(\mu + k_2)(R_0 - 1) + 2\delta\, c\beta k/R_0 - \mu\delta(1 + R_0) > 0, \quad (35)$$

$$\zeta = b_2^2 - c_2^2 + 2c_1c_3 - 2b_1b_3 > 0, \quad (36)$$

where b_i and c_i, $i = 1, 2, 3$ are defined by (33) and (34).

Proof:

Since $R_0 > 1$ then by Theorem 2.2, the infected equilibrium point E_2 is asymptotically stable for the cases (15) or (16) when $\tau = 0$. Now for $\tau > 0$, we assume that Eq. (32) with the coefficients (33) and (34) has pure imaginary roots $\lambda = \pm i\omega$, $\omega > 0$ for certain value of $\tau > 0$. Hence we can write Eq. (32) in the form:

$$\begin{aligned} &-i\omega^3 - \omega^2(b_1 + c_1\cos\omega\tau - ic_1\sin\omega\tau) + \\ &i\omega(b_2 + c_2\cos\omega\tau - ic_2\sin\omega\tau) \\ &\quad + b_3 + c_3\cos\omega\tau - ic_2\sin\omega = 0. \end{aligned} \quad (37)$$

Equating both real and imaginary parts of Eq. (37) by zero, and eliminating τ, we get:

$$\begin{aligned} y^3 + (b_1^2 - c_1^2 - 2b_1)y^2 + (b_2^2 - c_2^2 + 2c_1c_3 - 2b_1b_3)y \\ + (b_3^2 - c_3^2) = 0, \end{aligned} \quad (38)$$

where $y=\omega^2$. It is clear that there will be no positive roots of Eq. (39) if $b_2^2-c_2^2+2c_1c_3-2b_1b_3>0$ and $b_3^2-c_3^2>0$. Consequently there will be no value of τ such that the real part of the eigenvalues change their sign from negative to positive.

Since

$$b_3^2-c_3^2=(b_3+c_3)(b_3-c_3)$$
$$=\mu(R_0-1)[(\mu+k_1)(\mu+k_2)+\mu\,\delta](b_3-c_3)$$

and by simplification we have

$$b_3-c_3=\mu[(\mu+k_1)(\mu+k_2)(R_0-1)$$
$$+2\delta\,c\,\beta\,k/R_0-\mu\,\delta(1+R_0)]>0$$

by condition (35). The proof is completed by applying the condition (36).

Theorem 3.4 Consider $R_0>1$ and assume that the parameters of (21) satisfy condition (15) or (16), then the infected point E_2 of system (21) is locally asymptotically stable for all time delay $\tau\in[0,\tau^*)$and τ^* satisfies that

$$\tau^*=(1/x^*\sin(\alpha\pi/2))\tan^{-1}[f(x^*)/g(x^*)] \tag{39}$$

and x^* is the smallest positive value of x that satisfies $[f(x)/g(x)]>0$, where

$$\begin{aligned}f(x)&=c_1\sin(\alpha\pi/2)x^4-c_2\sin(\alpha\pi)x^3\\&+[c_3\sin(3\alpha\pi/2)+(b_1c_2+b_2c_1)\sin(\alpha\pi/2)]x^2\\&+(b_1c_3-b_3c_1)\sin(\alpha\pi)x+(b_2c_3-b_3c_2)\sin(\alpha\pi/2),\end{aligned} \tag{40}$$

$$\begin{aligned}g(x)=&-c_1\cos(\alpha\pi/2)x^4+[c_2\cos(\alpha\pi)-b_1c_1]x^3\&-[c_3\cos(3\alpha\pi/2)\\&+(b_1c_2-b_2c_1)\cos(\alpha\pi/2)]x^2\\&-[b_2c_2+(b_1c_3+b_3c_1)\cos(\alpha\pi)]x+(b_2c_3+b_3c_2)\cos(\alpha\pi/2).\end{aligned} \tag{41}$$

Proof:

Since $R_0>1$ then by Theorem 2.2, the infected equilibrium point E_2 is asymptotically stable for $\tau=0$ when condition (15) or (16) is satisfied. This means that all the eigenvalues satisfy condition (3). Now for $\tau>0$, we assume that Eq. (32) with the coefficients (33) and (34) has an eigenvalue $\lambda=r\,e^{i\,\alpha\,\pi/2}$. Really if there is a time delay τ^* gives this eigenvalue then by increasing τ the system may be unstable. Substituting by this eigenvalue in Eq.(32), we have

$$\begin{aligned}e^{-r\tau\cos(\alpha\pi/2)}&[c_1r^2\cos(\alpha\pi)\cos(r\tau\sin(\alpha\pi/2))\\&+r^2c_1\sin(\alpha\pi)\sin(r\tau\sin(\alpha\pi/2))\\&+rc_2\cos(\alpha\pi/2)\cos(r\tau\sin(\alpha\pi/2))+\\&c_2r\sin(\alpha\pi/2)\sin(r\tau\sin(\alpha\pi/2))\\&+c_3\cos(r\tau\sin(\alpha\pi/2))=\\&-[r^3\cos(3\alpha\pi/2)+b_1r^2\cos(\alpha\pi)+b_2r\cos(\alpha\pi/2)+b_3],\end{aligned} \tag{42}$$

$$\begin{aligned}e^{-r\tau\cos(\alpha\pi/2)}&[-c_1r^2\cos(\alpha\pi)\sin(r\tau\sin(\alpha\pi/2))\\&+r^2c_1\sin(\alpha\pi)\cos(r\tau\sin(\alpha\pi/2))\\&-rc_2\cos(\alpha\pi/2)\sin(r\tau\sin(\alpha\pi/2))+\\&c_2r\sin(\alpha\pi/2)\cos(r\tau\sin(\alpha\pi/2))\\&-c_3\sin(r\tau\sin(\alpha\pi/2))\\&=-[r^3\sin(3\alpha\pi/2)+b_1r^2\sin(\alpha\pi)+b_2r\sin(\alpha\pi/2)].\end{aligned} \tag{43}$$

Simplifying Eq. (42) and (43), we can get the value of τ as a function of r satisfying that: $\tau=(1/r\sin(\alpha\pi/2))\tan^{-1}[f(r)/g(r)]$, and the proof can be completed as Theorem 3.2.

5 NSFDM of a General Fractional System

In this section, we develop a numerical method to solve our fractional system. We first introduce the NSFDM [15] for a single fractional differential equation

$$D^{\alpha} x(t) = f(x(t),t)\ ,\ \ 0 \le t \le T\ ,\ \alpha > 0 \quad \text{and} \quad x(t_0) = x_0. \tag{44}$$

We use the Grünwald-Letnikov approximation for the fractional terms $D^{\alpha}x(t)$ in order to get a numerical solution for Eq.(44). The Grünwald-Letnikov for fractional derivative [24] is:

$$D^{\alpha} x(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} x(t-jh), \tag{45}$$

where N is the integer part of $[\frac{t}{h}]$ and h is the step size. So Eq. (44) can be discretized to be:

$$\sum_{j=0}^{n+1} G_j^{\alpha}\, x(t-jh) = f(x(t_n),\, t_n)\ \ , n = 1, 2, 3, ..., \tag{46}$$

where $t_n = nh$ and G_j^{α} are the Grünwald-Letnikov coefficients

$$G_j^{\alpha} = (1-(1+\alpha)/j)\, G_{j-1}^{\alpha}, j = 1, 2, 3, \ldots, \tag{47}$$

where $G_0^{\alpha} = h^{-\alpha}$.

Mickens [25], introduced the basics of nonstandard finite difference technique, we introduce the NSFDM for ODEs then we apply it for fractional differential equation. If $\alpha = 1$ in Eq. (44), the discrete derivative is:

$$\frac{dx}{dt} = \frac{x_{k+1} - x_k}{\phi(h)}, \tag{48}$$

where φ is a function of the step size h,φ satisfies that $\phi(h) = h + o(h^2)$. The functions $h,\ \sin h,\ \sinh h$ *and* $e^h - 1$ are examples of $\varphi(h)$. For more details, we may refer to [25].

Now applying the NSFD technique with the Grünwald-Letnikov discretization method to obtain numerical solution of the systems (4) and (21), yields

$$\begin{aligned}
S(t_{n+1}) &= \frac{\mu k - \sum_{j=1}^{n+1} G_j^{\alpha}\, S(t_{n+1-j})}{G_0^{\alpha} + \mu + c\beta\, [I(t_n) + bJ(t_n)]}\ , && S(t_0) = S_0,\\
I(t_{n+1}) &= \frac{c\beta\, b\, J(t_n)\, S(t_{n+1}) + \delta\, J(t_n) - \sum_{j=1}^{n+1} G_j^{\alpha}\, I(t_{n+1-j})}{G_0^{\alpha} + \mu + k_1 - c\beta\, S(t_{n+1})}\ , && I(t_0) = I_0,\\
J(t_{n+1}) &= \frac{k_1\, I(t_{n+1}) - \sum_{j=1}^{n+1} G_j^{\alpha}\, j(t_{n+1-j})}{G_0^{\alpha} + \mu + c\beta\, [I(t_n) + bJ(t_n)]}\ , && J(t_0) = J_0,
\end{aligned}$$

where $t_n = nh,\ n = 0, 1, 2, 3, ...,\ G_0^{\alpha} = (\phi(h))^{-\alpha}$.

6 Discussion

We give two examples to illustrate the results of our article.

Example 1: Let the parameters of system (21) be $k = 100, \beta = 0.005, b = 1.5,\ \mu = 0.01,\ k_1 = 0.09,\ k_2 = 0.01, \delta = 0.05,\ c = 0.02$, and $\alpha = 0.9$. Hence $R_0 = 0.82$ and by Theorem 3.1, E_1 is asymptotically stable for all $\tau \ge 0$ where $\gamma = -0.0087 < 0$. Figure 1 (a, b, c) represent time response of $J(t)$ for a numerical solution of system (21) during a simulation time (5000 days) when $\tau = 0,\ 50$ and 150 (days) respectively (the large values of the time delay are just to verify the theoretical results in Theorem 3.1). For the parameters $k = 100,\ \beta = 0.003, b = 1.1,\ \mu = 0.01, k_1 = 0.007,\ k_2 = 0.001, \delta = 0.17,\ c = 0.03$ and $\alpha = 0.9$. Hence $R_0 = 0.87$ and $\gamma = 0.0286 > 0$, hence condition of Theorem 3.1 is not satisfied. We examine the conditions of Theorem 3.2. We find where the numerator of $f(x)$ in Eq. (29) is greater than zero, where:

$-x\,[c_1 \sin(\alpha\pi/2)\, x^2 + c_2\, \sin(\alpha\pi)\, x + (b_1c_2 - b_2c_1)\, \sin(\alpha\pi/2)] = -x\,(x - 0.0023)\,(x + 0.0026) > 0$ when $x \in (0,\ 0.0023)$ and also the denominator of it has positive values in the same interval. So for any time delay greater than zero the system will be unstable. In Figure 2 (a, b), the time response of $J(t)$ *for* $\tau = 0$ and $\tau = 15$ days are displayed. It is clear that the solution is unstable for $\tau = 15$.

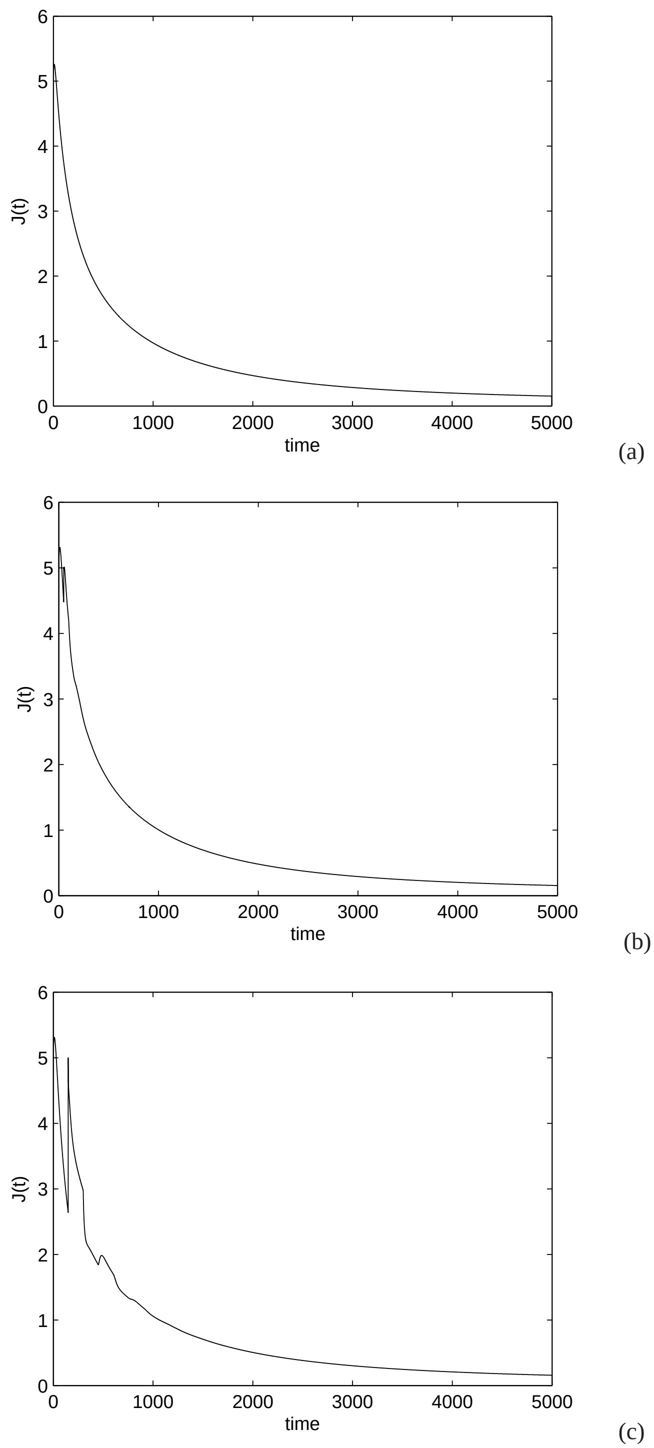
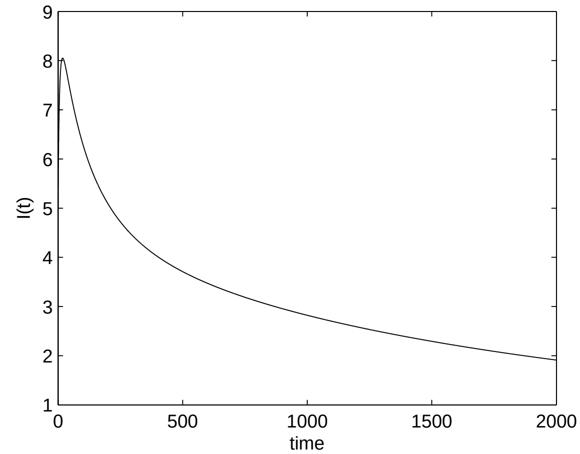
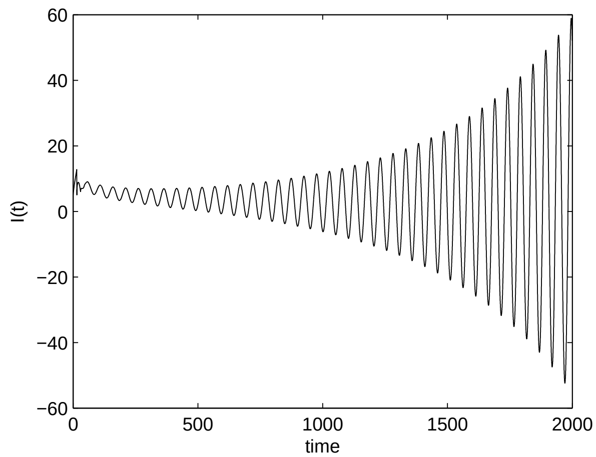


Fig. 1: (a) Time response of $J(t)$ for $\tau = 0$ & (b) Time response of $J(t)$ for $\tau = 50$ days & (c) Time response of $J(t)$ for $\tau = 150$ days.



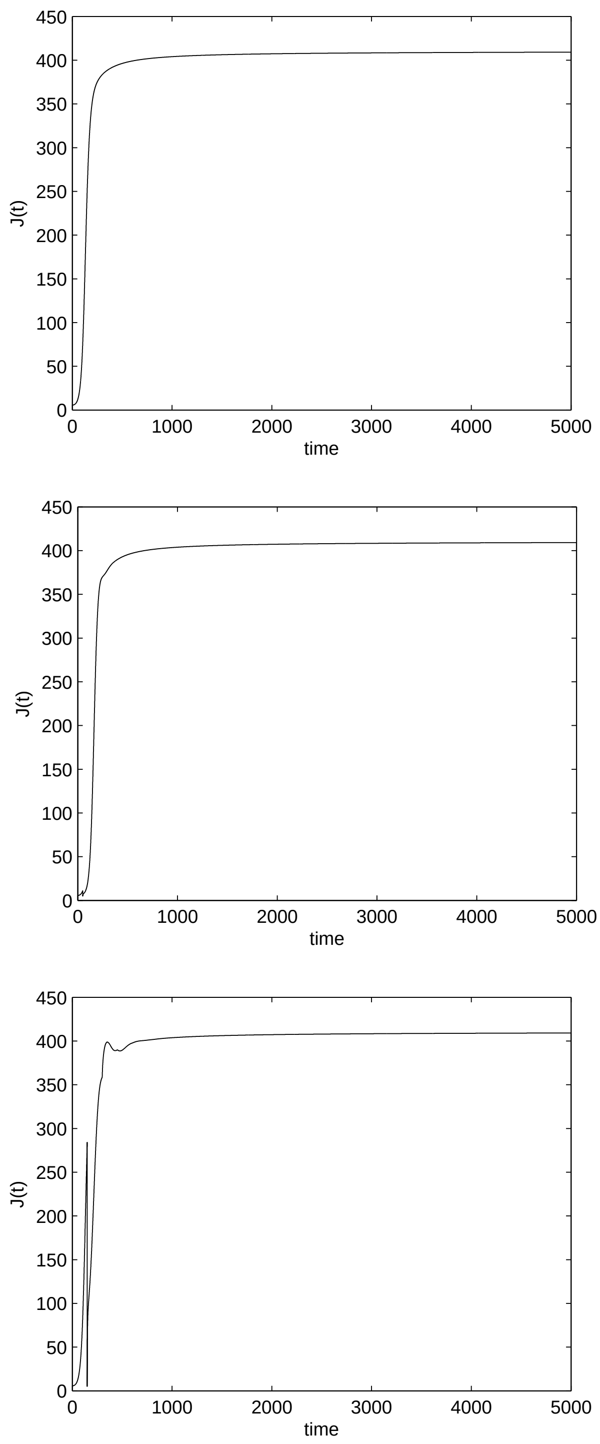
(a)



(b)

Fig. 2: (a) Time response of $I(t)$ for $\tau = 0$ & (b) Time response of $I(t)$ for $\tau = 15$ days.

Example 2: Let the parameters of system (21) as $k = 1000$, $\beta = 0.005$, $b = 1.2$, $\mu = 0.02$, $k_1 = 0.09$, $k_2 = 0.01$, $\delta = 0.02$, $c = 0.03$ and $\alpha = 0.9$. Hence $R_0 = 6.4054$ and by Theorem 3.3, E_2 is asymptotically stable for all $\tau \geq 0$ where $\psi > 0\, and\, \zeta > 0$. Figure 3 (a, b, c) represent time response of $J(t)$ for a numerical solution of system (21) during a simulation time (5000 days) when $\tau = 0\,,\ 50\ and\ 150$ (days) respectively.



(a)

(b)

(c)

Fig. 3: (a) Time response of $J(t)$ for $\tau = 0$ & (b) Time response of $J(t)$ for $\tau = 50$ days & (c) Time response of $J(t)$ for $\tau = 150$ days.

7 Conclusions

A fractional model for HIV/AIDS with nonlinear incidence and treatment is discussed. System (4) with $\alpha = 1$ is just the model considered by Cai et. al. [16], the results of Cai et. al. agree with our established results. and hence it is a special cases of our work. The equilibria of the system and corresponding stability are analyzed. Sufficient conditions for asymptotic stability of the system with time delay are given in Theorem 3.1 - Theorem 3.4. Also we give threshold values of time delay defined by (28) and (39). If the antiretroviral drugs give positive effects in patients after an interval less than τ^*, then the infected equilibrium E_2 is asymptotically stable, while if the positive effects take time duration more than τ^*, hence E_2 will be unstable (fail in treatment). Finally, illustrative examples are given with their numerical solutions carried out using Matlab7.

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