

Double-Layer Potentials for a Generalized Bi-Axially Symmetric Helmholtz Equation

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Abstract: The double-layer potential plays an important rôle in solving boundary value problems for elliptic equations. Here, in this paper, we aim at introducing and investigating double-layer potentials for a generalized bi-axially symmetric Helmholtz equation. By using some properties of one of Appell's hypergeometric functions in two variables, we prove limiting theorems and derive integral equations concerning a denseness of double-layer potentials.

Keywords: Singular partial differential equations; Appell's hypergeometric functions in two variables; Generalized bi-axially symmetric Helmholtz equation; Degenerated elliptic equations; Generalized axially-symmetric potentials; Double-layer potentials.

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1 Introduction

Potential theory has played a paramount rôle in both analysis and computation for boundary value problems for elliptic partial differential equations. Numerous applications can be found in fracture mechanics, fluid mechanics, elastodynamics, electromagnetics, and acoustics. Results from potential theory allow us to represent boundary value problems in integral equation form. For problems with known Green's functions, an integral equation formulation leads to powerful numerical approximation schemes.

The double-layer potential plays an important rôle in solving boundary value problems of elliptic equations. The representation of the solution of the (first) boundary value problem is sought as a double-layer potential with unknown density and an application of certain property leads to a Fredholm equation of the second kind for determining the function (see [18] and [29]).

By applying a method of complex analysis (based upon analytic functions), Gilbert [16] constructed an integral representation of solutions of the following generalized bi-axially Helmholtz equation:

$$\begin{aligned}
 H_{\alpha,\beta}^{\lambda}(u) &\equiv u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y - \lambda^2 u \\
 &= 0 \qquad \qquad \qquad \left(H_{\alpha,\beta}^{\lambda} \right) \\
 &\qquad \qquad \qquad \left(0 < \alpha < \frac{1}{2}; 0 < \beta < \frac{1}{2} \right),
 \end{aligned}$$

where α , β and λ are constants. When $\lambda = 0$, this equation is known as the equation of the generalized axially symmetric potential theory whose name is due to Weinstein who first considered fractional dimensional space in potential theory (see [35] and [36]). The special case where $\lambda = 0$ was also investigated by (among others) Erdélyi (see [6] and [7]), Gilbert (see [10], [12], [13], [14] and [15]), Gilbert and Howard [17], Ranger [31] and Henrici (see [20] and [22]). Various interesting problems associated with the equation $\left(H_{\alpha,\beta}^{\lambda} \right)$ were studied by many authors (see, for example, [1], [2], [3], [9], [23], [24], [25], [26], [27], [28], [30], [32] and [34]).

Fundamental solutions of the equation $\left(H_{\alpha,\beta}^{\lambda} \right)$ were constructed recently (see [19]). In fact, the fundamental solutions of the equation $\left(H_{\alpha,\beta}^{\lambda} \right)$ when $\lambda = 0$ can be expressed in terms of Appell's hypergeometric function in

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two variables of the second kind, that is, the Appell function

$$F_2(a, b_1, b_2; c_1, c_2; x, y)$$

defined by (see [8, p. 224, Eq. 5.7.1 (7)]; see also [4, p. 14, Eq. (12)] and [33, p. 23, Eq. 1.3 (3)])

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad (1.1)$$

where $(\kappa)_v$ denotes the *general* Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{N} := \{1, 2, 3, \dots\}),$$

which is defined (for $\kappa, v \in \mathbb{C}$), in terms of the familiar Gamma function, by

$$(\kappa)_v := \frac{\Gamma(\kappa + v)}{\Gamma(\kappa)} = \begin{cases} 1 & (v = 0; \kappa \in \mathbb{C} \setminus \{0\}) \\ \kappa(\kappa + 1) \cdots (\kappa + v - 1) & (v = n \in \mathbb{N}; \kappa \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ -quotient exists. We thus obtain the following results:

$$q_1(x, y; x_0, y_0) = k_1 (r^2)^{-\alpha-\beta} \cdot F_2(\alpha + \beta, \alpha, \beta; 2\alpha, 2\beta; \xi, \eta), \quad (1.2)$$

$$q_2(x, y; x_0, y_0) = k_2 (r^2)^{\alpha-\beta-1} x^{1-2\alpha} x_0^{1-2\alpha} \cdot F_2(1 - \alpha + \beta, 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta), \quad (1.3)$$

$$q_3(x, y; x_0, y_0) = k_3 (r^2)^{-\alpha+\beta-1} y^{1-2\beta} y_0^{1-2\beta} \cdot F_2(1 + \alpha - \beta, \alpha, 1 - \beta; 2\alpha, 2 - 2\beta; \xi, \eta) \quad (1.4)$$

and

$$q_4(x, y; x_0, y_0) = k_4 (r^2)^{\alpha+\beta-2} x^{1-2\alpha} y^{1-2\beta} x_0^{1-2\alpha} y_0^{1-2\beta} \cdot F_2(2 - \alpha - \beta, 1 - \alpha, 1 - \beta; 2 - 2\alpha, 2 - 2\beta; \xi, \eta), \quad (1.5)$$

where the constants k_1 to k_4 are determined as follows upon solving boundary value problems for the equation $(H_{\alpha, \beta}^\lambda)$:

$$k_1 = \frac{2^{2\alpha+2\beta} \Gamma(\alpha) \Gamma(\beta) \Gamma(\alpha + \beta)}{4\pi \Gamma(2\alpha) \Gamma(2\beta)}, \quad (1.6)$$

$$k_2 = \frac{2^{2-2\alpha+2\beta} \Gamma(1 - \alpha) \Gamma(\beta) \Gamma(1 - \alpha + \beta)}{4\pi \Gamma(2 - 2\alpha) \Gamma(2\beta)}, \quad (1.7)$$

$$k_3 = \frac{2^{2+2\alpha-2\beta} \Gamma(\alpha) \Gamma(1 - \beta) \Gamma(1 + \alpha - \beta)}{4\pi \Gamma(2\alpha) \Gamma(2 - 2\beta)}, \quad (1.8)$$

$$k_4 = \frac{2^{4-2\alpha-2\beta} \Gamma(1 - \alpha) \Gamma(1 - \beta) \Gamma(2 - \alpha - \beta)}{4\pi \Gamma(2 - 2\alpha) \Gamma(2 - 2\beta)}, \quad (1.9)$$

$$\begin{pmatrix} r^2 \\ r_1^2 \\ r_2^2 \end{pmatrix} = \begin{pmatrix} (x - x_0)^2 + (y - y_0)^2 \\ (x + x_0)^2 + (y - y_0)^2 \\ (x - x_0)^2 + (y + y_0)^2 \end{pmatrix},$$

$$\xi = \frac{r^2 - r_1^2}{r^2} \quad \text{and} \quad \eta = \frac{r^2 - r_2^2}{r^2}. \quad (1.10)$$

The fundamental solutions given by (1.2) to (1.5) possess the following properties:

$$x^{2\alpha} \frac{\partial}{\partial x} \{q_1(x, y; x_0, y_0)\} \Big|_{x=0} = 0, \quad (1.11a)$$

$$y^{2\beta} \frac{\partial}{\partial y} \{q_1(x, y; x_0, y_0)\} \Big|_{y=0} = 0, \quad (1.11b)$$

$$q_2(x, y; x_0, y_0) \Big|_{x=0} = 0, \quad (1.12a)$$

$$y^{2\beta} \frac{\partial}{\partial y} \{q_2(x, y; x_0, y_0)\} \Big|_{y=0} = 0, \quad (1.12b)$$

$$x^{2\alpha} \frac{\partial}{\partial x} \{q_3(x, y; x_0, y_0)\} \Big|_{x=0} = 0, \quad (1.13a)$$

$$q_3(x, y; x_0, y_0) \Big|_{y=0} = 0, \quad (1.13b)$$

$$q_4(x, y; x_0, y_0) \Big|_{x=0} = 0 \quad (1.14a)$$

and

$$q_4(x, y; x_0, y_0) \Big|_{y=0} = 0. \quad (1.14b)$$

Here, by making use of the fundamental solutions given by (1.2) to (1.5) in the domain Ω defined by

$$\Omega \subset \mathbb{R}_+^2 := \{(x, y) : x > 0 \text{ and } y > 0\}, \quad (1.15)$$

we aim at investigating a double-layer potential for the equation $(H_{\alpha, \beta}^0)$. Furthermore, we prove some results (see Lemmas 1 to 3 and Theorem 3) on limiting values of the double-layer potential in (3.2). These results are (potentially) useful for future works in which boundary value problems for the equation $(H_{\alpha, \beta}^0)$ are investigated in more general domains.

2 Green's Formula

We begin by considering the following identity:

$$\begin{aligned} & x^{2\alpha}y^{2\beta} \left[uH_{\alpha,\beta}^0(v) - vH_{\alpha,\beta}^0(u) \right] \\ &= \frac{\partial}{\partial x} \left[x^{2\alpha}y^{2\beta} (v_x u - v u_x) \right] \\ & \quad + \frac{\partial}{\partial y} \left[x^{2\alpha}y^{2\beta} (v_y u - v u_y) \right]. \end{aligned} \quad (2.1)$$

Integrating both parts of the identity (2.1) on a domain Ω in (1.15), and using Green's formula, we find that

$$\begin{aligned} & \int \int_{\Omega} x^{2\alpha}y^{2\beta} \left[uH_{\alpha,\beta}^0(v) - vH_{\alpha,\beta}^0(u) \right] dx dy \\ &= \int_S x^{2\alpha}y^{2\beta} u (v_x dy - v_y dx) \\ & \quad - x^{2\alpha}y^{2\beta} v (u_x dy - u_y dx), \end{aligned} \quad (2.2)$$

where $S = \partial\Omega$ is the boundary of the domain Ω .

If $u(x,y)$ and $v(x,y)$ are solutions of the equation $(H_{\alpha,\beta}^0)$, we find from (2.2) that

$$\int_S x^{2\alpha}y^{2\beta} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds = 0, \quad (2.3)$$

where

$$\begin{aligned} \frac{\partial}{\partial n} &= \frac{dy}{ds} \frac{\partial}{\partial x} - \frac{dx}{ds} \frac{\partial}{\partial y}, \\ \frac{dy}{ds} &= \cos(n,x) \quad \text{and} \quad \frac{dx}{ds} = -\cos(n,y), \end{aligned} \quad (2.4)$$

n being the exterior normal to the curve S . Here $\cos(n,x)$ denotes the cosine of the angle between the *positive* x -direction and the *outward* normal to the curve S at the point (x,y) on S . Similarly, $\cos(n,y)$ denotes the cosine of the angle between the *positive* y -direction and the *outward* normal to the curve S at the point (x,y) on S . We also obtain the following identity:

$$\int \int_{\Omega} x^{2\alpha}y^{2\beta} (u_x^2 + u_y^2) dx dy = \int_S x^{2\alpha}y^{2\beta} u \frac{\partial u}{\partial n} ds, \quad (2.5)$$

where $u(x,y)$ is a solution of the equation $(H_{\alpha,\beta}^0)$. The special case of (2.3) when $v = 1$ reduces to the following form:

$$\int_S x^{2\alpha}y^{2\beta} \frac{\partial u}{\partial n} ds = 0. \quad (2.6)$$

We note from (2.6) that the integral of the normal derivative of a solution of the equation $(H_{\alpha,\beta}^0)$ with a weight $x^{2\alpha}y^{2\beta}$ along the boundary S of the domain Ω in (1.15) is equal to zero.

3 A Double-Layer Potential $w^{(1)}(x_0, y_0)$

Let Ω in (1.15) be a domain bounded by intervals $(0,a)$ and $(0,b)$ of the x - and y -axes, respectively, and a curve Γ with the extremities at points $A(a,0)$ and $B(0,b)$. The parametric equations of the curve Γ are given by

$$x = x(s) \quad \text{and} \quad y = y(s) \quad (s \in [0,l]),$$

where l denotes the length of Γ . We assume the following properties of the curve Γ :

- (i) The functions $x = x(s)$ and $y = y(s)$ have continuous derivatives $x'(s)$ and $y'(s)$ on a segment $[0,l]$ and do not vanish simultaneously;
- (ii) The second derivatives $x''(s)$ and $y''(s)$ satisfy the Hölder condition on $[0,l]$, where l denotes the length of the curve Γ ;
- (iii) In some neighborhoods of points $A(a,0)$ and $B(0,b)$, the following conditions are satisfied:

$$\left| \frac{dx}{ds} \right| \leq cy^{1+\varepsilon}(s) \quad \text{and} \quad \left| \frac{dy}{ds} \right| \leq cx^{1+\varepsilon}(s) \quad (3.1)$$

$$(0 < \varepsilon < 1; c = \text{a constant}),$$

(x,y) being the coordinates of a variable point on the curve Γ .

Consider the following integral

$$\begin{aligned} & w^{(1)}(x_0, y_0) \\ &= \int_0^l x^{2\alpha}y^{2\beta} \mu_1(s) \frac{\partial}{\partial n} \{q_1(x,y;x_0,y_0)\} ds, \end{aligned} \quad (3.2)$$

where the density $\mu_1(s) \in C[0,l]$ and q_1 is given in (1.2). We call the integral (3.2) a *double-layer potential with denseness* $\mu_1(s)$.

We now investigate some properties of a double-layer potential $w^{(1)}(x_0, y_0)$ with denseness $\mu_1(s)$.

Lemma 1. *The following formula holds true:*

$$w_1^{(1)}(x_0, y_0) = \begin{cases} -1 & ((x_0, y_0) \in \Omega) \\ -\frac{1}{2} & ((x_0, y_0) \in \Gamma) \\ 0 & ((x_0, y_0) \notin \bar{\Omega}), \end{cases} \quad (3.3)$$

where a domain Ω and the curve Γ are described as in this section and $\bar{\Omega} := \Omega \cup \Gamma$.

Proof.

Case 1. When $(x_0, y_0) \in \Omega$, we cut a circle centered at (x_0, y_0) with a small radius ρ off the domain Ω and denote the remaining part by Ω^ρ and the circuit of the cut-off-circle by C_ρ . The function $q_1(x, y; x_0, y_0)$ in (1.2)

is a regular solution of the equation $(H_{\alpha,\beta}^0)$ in the domain Ω^p . Using the following derivative formula of Appell's hypergeometric function F_2 (see [4, p. 19, Eq. (20)]):

$$\begin{aligned} & \frac{\partial^{m+n}}{\partial x^m \partial y^n} \{F_2(a, b_1, b_2; c_1, c_2; x, y)\} \\ &= \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \\ & \cdot F_2(a+m+n, b_1+m, b_2+n; c_1+m, c_2+n; x, y), \end{aligned} \quad (3.4)$$

we have

$$\begin{aligned} \frac{\partial}{\partial x} \{q_1(x, y; x_0, y_0)\} &= -2(\alpha + \beta) k_1 (r^2)^{-\alpha-\beta-1} (x-x_0) \\ & \cdot F_2(\alpha + \beta, \alpha, \beta; 2\alpha, 2\beta; \xi, \eta) \\ & - 2(\alpha + \beta) k_1 (r^2)^{-\alpha-\beta-1} \\ & \cdot x_0 F_2(\alpha + \beta + 1, \alpha + 1, \beta; 2\alpha + 1, 2\beta; \xi, \eta) \\ & - 2k_1 (r^2)^{-\alpha-\beta-1} (x-x_0) \\ & \cdot \left[\frac{(\alpha + \beta)\alpha}{2\alpha} \xi F_2(\alpha + \beta + 1, \alpha + 1, \beta; 2\alpha + 1, 2\beta; \xi, \eta) \right. \\ & \left. + \frac{(\alpha + \beta)\beta}{2\beta} \eta F_2(\alpha + \beta + 1, \alpha, \beta + 1; 2\alpha, 2\beta + 1; \xi, \eta) \right]. \end{aligned} \quad (3.5)$$

By applying the following known contiguous relation (see [4, p. 21]):

$$\begin{aligned} & \frac{b_1}{c_1} x F_2(a+1, b_1+1, b_2; c_1+1, c_2; x, y) \\ & + \frac{b_2}{c_2} y F_2(a+1, b_1, b_2+1; c_1, c_2+1; x, y) \\ & = F_2(a+1, b_1, b_2; c_1, c_2; x, y) - F_2(a, b_1, b_2; c_1, c_2; x, y) \end{aligned} \quad (3.6)$$

to (3.5), we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \{q_1(x, y; x_0, y_0)\} &= -2(\alpha + \beta) k_1 x_0 (r^2)^{-\alpha-\beta-1} \\ & \cdot F_2(\alpha + \beta + 1, \alpha + 1, \beta; 2\alpha + 1, 2\beta; \xi, \eta) \\ & - 2(\alpha + \beta) k_1 (x-x_0) (r^2)^{-\alpha-\beta-1} \\ & \cdot F_2(\alpha + \beta + 1, \alpha, \beta; 2\alpha, 2\beta; \xi, \eta). \end{aligned} \quad (3.7)$$

Similarly, we find that

$$\begin{aligned} \frac{\partial}{\partial y} \{q_1(x, y; x_0, y_0)\} &= -2(\alpha + \beta) k_1 y_0 (r^2)^{-\alpha-\beta-1} \\ & \cdot F_2(\alpha + \beta + 1, \alpha, 1 + \beta; 2\alpha, 1 + 2\beta; \xi, \eta) \\ & - 2(\alpha + \beta) k_1 (y-y_0) (r^2)^{-\alpha-\beta-1} \\ & \cdot F_2(\alpha + \beta + 1, \alpha, \beta; 2\alpha, 2\beta; \xi, \eta). \end{aligned} \quad (3.8)$$

Thus, with the help of (3.7) and (3.8), it follows from (1.2) and (2.4) that

$$\begin{aligned} \frac{\partial}{\partial n} \{q_1(x, y; x_0, y_0)\} &= -(\alpha + \beta) k_1 (r^2)^{-\alpha-\beta} \\ & \cdot F_2(\alpha + \beta + 1, \alpha, \beta; 2\alpha, 2\beta; \xi, \eta) \frac{\partial}{\partial n} \{\ln r^2\} \\ & + 2(\alpha + \beta) k_1 y_0 (r^2)^{-\alpha-\beta-1} \\ & \cdot F_2(\alpha + \beta + 1, \alpha, 1 + \beta; 2\alpha, 1 + 2\beta; \xi, \eta) \frac{d}{ds} \{x(s)\} \\ & - 2(\alpha + \beta) k_1 x_0 (r^2)^{-\alpha-\beta-1} \\ & \cdot F_2(\alpha + \beta + 1, \alpha + 1, \beta; 2\alpha + 1, 2\beta; \xi, \eta) \frac{d}{ds} \{y(s)\}. \end{aligned} \quad (3.9)$$

Applying (2.6) and considering the identity (1.11), we get the following formula:

$$w_1^{(1)}(x_0, y_0) = \lim_{\rho \rightarrow 0} \int_{C_\rho} x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \{q_1(x, y; x_0, y_0)\} ds. \quad (3.10)$$

Substituting from (3.9) into (3.10), we find that

$$\begin{aligned} w_1^{(1)}(x_0, y_0) &= -(\alpha + \beta) k_1 \lim_{\rho \rightarrow 0} \int_{C_\rho} x^{2\alpha} y^{2\beta} (r^2)^{-\alpha-\beta} \\ & - 2(\alpha + \beta) k_1 x_0 \lim_{\rho \rightarrow 0} \int_{C_\rho} x^{2\alpha} y^{2\beta} (r^2)^{-\alpha-\beta-1} \\ & \cdot F_2(1 + \alpha + \beta, 1 + \alpha, \beta; 1 + 2\alpha, 2\beta; \xi, \eta) \frac{d}{ds} \{y(s)\} ds \\ & + 2(\alpha + \beta) k_1 y_0 \lim_{\rho \rightarrow 0} \int_{C_\rho} x^{2\alpha} y^{2\beta} (r^2)^{-\alpha-\beta-1} \\ & \cdot F_2(1 + \alpha + \beta, \alpha, 1 + \beta; 2\alpha, 1 + 2\beta; \xi, \eta) \frac{d}{ds} \{x(s)\} ds \\ & = -(\alpha + \beta) k_1 \lim_{\rho \rightarrow 0} J_1(x_0, y_0) - 2(\alpha + \beta) k_1 x_0 \lim_{\rho \rightarrow 0} J_2(x_0, y_0) \\ & + 2(\alpha + \beta) k_1 y_0 \lim_{\rho \rightarrow 0} J_3(x_0, y_0), \end{aligned} \quad (3.11)$$

where J_1 , J_2 and J_3 are the corresponding integrals in the first equality. Now, by introducing the polar coordinates:

$$x = x_0 + \rho \cos \varphi \quad \text{and} \quad y = y_0 + \rho \sin \varphi,$$

we get

$$J_1(x_0, y_0) = \lim_{\rho \rightarrow 0} \int_0^{2\pi} (x_0 + \rho \cos \varphi)^{2\alpha} (y_0 + \rho \sin \varphi)^{2\beta} \cdot (\rho^2)^{-\alpha-\beta} F_2(\alpha + \beta + 1, \alpha, \beta; 2\alpha, 2\beta; \xi, \eta) d\varphi. \quad (3.12)$$

By using the following known formulas (see [5, p. 253, Eq. (26)]; see also [8, p. 113, Eq. (4)]):

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{j=0}^{\infty} \frac{(a)_j (b_1)_j (b_2)_j (xy)^j}{(c_1)_j (c_2)_j j!} \cdot {}_2F_1(a + j, b_1 + j; c_1 + j; x) \cdot {}_2F_1(a + j, b_2 + j; c_2 + j; y) \quad (3.13)$$

and

$${}_2F_1(a, b; c; x) = (1-x)^{-b} {}_2F_1\left(c-a, b; c, \frac{x}{x-1}\right), \quad (3.14)$$

we obtain

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = (1-x)^{-b_1} (1-y)^{-b_2} \cdot \sum_{j=0}^{\infty} \frac{(a)_j (b_1)_j (b_2)_j}{(c_1)_j (c_2)_j j!} \left(\frac{x}{1-x}\right)^j \left(\frac{y}{1-y}\right)^j \cdot {}_2F_1\left(c_1-a, b_1+j; c_1+j; \frac{x}{x-1}\right) \cdot {}_2F_1\left(c_2-a, b_2+j; c_2+j; \frac{y}{y-1}\right), \quad (3.15)$$

where

$${}_2F_1(a, b; c; x)$$

is Gauss's hypergeometric function (see [8, p. 69, Eq. (2)]). Hence we have

$$F_2(1 + \alpha + \beta; \alpha, \beta; 2\alpha, 2\beta; \xi, \eta) = (\rho^2)^{\alpha+\beta} (\rho^2 + 4x_0^2 + 4x_0\rho \cos \varphi)^{-\alpha} \cdot (\rho^2 + 4y_0^2 + 4y_0\rho \sin \varphi)^{-\beta} P_{11}, \quad (3.16)$$

where

$$P_{11} = \sum_{j=0}^{\infty} \frac{(1 + \alpha + \beta)_j (\alpha)_j (\beta)_j}{(2\alpha)_j (2\beta)_j (j!)^2} \cdot \left(\frac{4x_0^2 + 4x_0\rho \cos \varphi}{\rho^2 + 4x_0^2 + 4x_0\rho \cos \varphi}\right)^j \cdot \left(\frac{4y_0^2 + 4y_0\rho \sin \varphi}{\rho^2 + 4y_0^2 + 4y_0\rho \sin \varphi}\right)^j \cdot {}_2F_1\left(\alpha - \beta - 1, \alpha + j; 2\alpha + j; \frac{4x_0^2 + 4x_0\rho \cos \varphi}{\rho^2 + 4x_0^2 + 4x_0\rho \cos \varphi}\right) \cdot {}_2F_1\left(\beta - \alpha - 1, \beta + j; 2\beta + j; \frac{4y_0^2 + 4y_0\rho \sin \varphi}{\rho^2 + 4y_0^2 + 4y_0\rho \sin \varphi}\right).$$

Using the well-known Gauss's summation formula for ${}_2F_1$ (see [8, p. 112, Eq. (46)])

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\Re(c-a-b) > 0; c \neq 0, -1, -2, \dots),$$

we obtain

$$\lim_{\rho \rightarrow 0} P_{11} = \frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(1+\alpha+\beta)}. \quad (3.17)$$

Thus, by virtue of the identities (3.12), (3.16), and (3.17), we get

$$(\alpha + \beta) k_1 \lim_{\rho \rightarrow 0} J_1(x_0, y_0) = 1. \quad (3.18)$$

Similarly, by considering the corresponding identities and the fact that

$$\lim_{\rho \rightarrow 0} \rho \ln \rho = 0,$$

we find that

$$2(\alpha + \beta) k_1 x_0 \lim_{\rho \rightarrow 0} J_2(x_0, y_0) = 2(\alpha + \beta) k_1 y_0 \lim_{\rho \rightarrow 0} J_3(x_0, y_0) = 0. \quad (3.19)$$

Hence, by view of (3.18) and (3.19), the formula (3.11) in the case of $(x_0, y_0) \in \Omega$ becomes

$$w_1^{(1)}(x_0, y_0) = -1. \quad (3.20)$$

Case 2. When $(x_0, y_0) \in \Gamma$, we cut a circle C_ρ centered at (x_0, y_0) with a small radius ρ off the domain Ω and denote the remaining part of the curve by $\Gamma - \Gamma_\rho$. Let C'_ρ denote a part of the circle C_ρ lying inside the domain Ω . We consider the domain Ω_ρ which is bounded by a curve

$\Gamma - \Gamma_\rho$, C'_ρ and the segments $[0, a]$ and $[0, b]$ along the x - and y -axes, respectively. Then we have

$$\begin{aligned} w_1^{(1)}(x_0, y_0) &= \int_0^l x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \{q_1(x, y; x_0, y_0)\} ds \\ &= \lim_{\rho \rightarrow 0} \int_{\Gamma - \Gamma_\rho} x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \{q_1(x, y; x_0, y_0)\} ds. \end{aligned} \quad (3.21)$$

When the point (x_0, y_0) lies outside the domain Ω_ρ , it is found that, in this domain, $q_1(x, y; x_0, y_0)$ is a regular solution of the equation $(H_{\alpha, \beta}^0)$. Therefore, by virtue of (2.6), we have

$$\begin{aligned} \int_{\Gamma - \Gamma_\rho} x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \{q_1(x, y; x_0, y_0)\} ds \\ = \int_{C'_\rho} x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \{q_1(x, y; x_0, y_0)\} ds. \end{aligned} \quad (3.22)$$

Substituting from (3.22) into (3.21), we get

$$\begin{aligned} w_1^{(1)}(x_0, y_0) &= \int_0^l x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \{q_1(x, y; x_0, y_0)\} ds \\ &= \lim_{\rho \rightarrow 0} \int_{C'_\rho} x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \{q_1(x, y; x_0, y_0)\} ds. \end{aligned} \quad (3.23)$$

Similarly, by again introducing the polar coordinates centered at the point (x_0, y_0) , we find that

$$w_1^{(1)}(x_0, y_0) = -\frac{1}{2}. \quad (3.24)$$

Case 3. When $(x_0, y_0) \notin \bar{\Omega}$, it is noted that the function $q_1(x, y; x_0, y_0)$ is a regular solution of the equation $(H_{\alpha, \beta}^0)$. Hence, in view of the formula (2.6), we have

$$\begin{aligned} w_1^{(1)}(x_0, y_0) &= \int_0^l x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \{q_1(x, y; x_0, y_0)\} ds \\ &= 0. \end{aligned} \quad (3.25)$$

The proof of Lemma 1 is thus completed.

Lemma 2. The following formula holds true:

$$w_1^{(1)}(x_0, 0) = \begin{cases} -1 & (x_0 \in (0, a)) \\ -\frac{1}{2} & (x_0 = 0 \text{ or } x_0 = a) \\ 0 & (a < x_0). \end{cases} \quad (3.26)$$

Proof. For considering the first case when $x_0 \in (0, a)$, we introduce a straight line $y = h$ for a sufficiently small positive real number h and consider a domain Ω_h which is the part of the domain Ω lying above the straight line $y = h$. Applying the formula (2.6), we obtain

$$\begin{aligned} w_1^{(1)}(x_0, 0) &= \lim_{h \rightarrow 0} \int_0^{x_1} x^{2\alpha} y^{2\beta} \frac{\partial}{\partial y} \{q_1(x, y; x_0, 0)\} \Big|_{y=h} dx, \end{aligned} \quad (3.27)$$

where $x_1(\varepsilon)$ is an abscissa of a point at which the straight line $y = h$ intersects the curve Γ . It follows from (3.8) and (3.27) that

$$\begin{aligned} w_1^{(1)}(x_0, 0) &= -2(\alpha + \beta) k_1 \lim_{h \rightarrow 0} h^{1+2\beta} \int_0^{x_1} x^{2\alpha} \\ &\cdot \frac{{}_2F_1\left(\alpha + \beta + 1, \alpha; 2\alpha; -\frac{4xx_0}{(x-x_0)^2 + h^2}\right)}{[(x-x_0)^2 + h^2]^{\alpha+\beta+1}} dx. \end{aligned} \quad (3.28)$$

Now, by using the hypergeometric transformation formula (3.14) inside the integrand of (3.28), we have

$$\begin{aligned} w_1^{(1)}(x_0, 0) &= -2(\alpha + \beta) k_1 \lim_{h \rightarrow 0} h^{1+2\beta} \int_0^{x_1} x^{2\alpha} \\ &\cdot \frac{{}_2F_1\left(\alpha - \beta - 1, \alpha; 2\alpha; \frac{4xx_0}{(x+x_0)^2 + h^2}\right)}{[(x-x_0)^2 + h^2]^{\beta+1} [(x+x_0)^2 + h^2]^\alpha} dx, \end{aligned} \quad (3.29)$$

which, upon setting $x = x_0 + ht$ inside the integrand, yields

$$\begin{aligned} w_1^{(1)}(x_0, 0) &= -2(\alpha + \beta) k_1 \lim_{h \rightarrow 0} \int_{l_1}^{l_2} (x_0 + ht)^{2\alpha} \\ &\cdot \frac{{}_2F_1\left(\alpha - \beta - 1, \alpha; 2\alpha; \frac{4x_0(x_0 + ht)}{(2x_0 + ht)^2 + h^2}\right)}{(1+t^2)^{\beta+1} [(2x_0 + ht)^2 + h^2]^\alpha} dt, \end{aligned} \quad (3.30)$$

where

$$l_1 = -\frac{x_0}{h} \quad \text{and} \quad l_2 = \frac{x_1 - x_0}{h}.$$

Considering

$$\begin{aligned} \lim_{h \rightarrow 0} {}_2F_1 \left(\alpha - \beta - 1, \alpha; 2\alpha; \frac{4x_0(x_0 + ht)}{(2x_0 + ht)^2 + h^2} \right) \\ = {}_2F_1(\alpha - \beta - 1, \alpha; 2\alpha; 1) \\ = \frac{\Gamma(2\alpha)\Gamma(1 + \beta)}{(\alpha + \beta)\Gamma(\alpha + \beta)\Gamma(\alpha)} \end{aligned}$$

and

$$\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{\beta+1}} = \frac{\pi\Gamma(2\beta)}{2^{2\beta-1}\beta\Gamma^2(\beta)},$$

we find from (3.30) that

$$w_1^{(1)}(x_0, 0) = -1. \tag{3.31}$$

The other *three* cases when $x_0 = 0$, $x_0 = a$ and $x_0 > a$ can be proved by using arguments similar to those detailed above in the first case.

This evidently completes our proof of Lemma 2.

Lemma 3. *The following formula holds true:*

$$w_1^{(1)}(0, y_0) = \begin{cases} -1 & (y_0 \in (0, b)) \\ -\frac{1}{2} & (y_0 = 0 \text{ or } y_0 = b) \\ 0 & (b < y_0). \end{cases} \tag{3.32}$$

Proof. The proof of Lemma 3 would run parallel to that of Lemma 2.

Theorem 1. *For any points (x, y) and $(x_0, y_0) \in \mathbb{R}_+^2$ and $x \neq x_0$ and $y \neq y_0$, the following inequality holds true:*

$$\begin{aligned} |q_1(x, y; x_0, y_0)| \leq k_1 \frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma^2(\alpha + \beta)} (r_1^2)^{-\alpha} (r_2^2)^{-\beta} \\ \cdot {}_2F_1 \left[\alpha, \beta; \alpha + \beta; \left(1 - \frac{r^2}{r_1^2}\right) \left(1 - \frac{r^2}{r_2^2}\right) \right], \end{aligned} \tag{3.33}$$

where α and β are real parameters with $(0 < \alpha, \beta < \frac{1}{2})$ as in the equation $(H_{\alpha, \beta}^\lambda)$ (with $\lambda = 0$), and r, r_1 and r_2 are as in (1.10).

Proof. It follows from (3.15) that

$$\begin{aligned} q_1(x, y; x_0, y_0) = k_1 (r_1^2)^{-\alpha} (r_2^2)^{-\beta} \\ \cdot \sum_{j=0}^{\infty} \frac{(\alpha + \beta)_j (\alpha)_j (\beta)_j}{(2\alpha)_j (2\beta)_j j!} \left(1 - \frac{r^2}{r_1^2}\right)^j \left(1 - \frac{r^2}{r_2^2}\right)^j \\ \cdot {}_2F_1 \left(\alpha - \beta, \alpha + j; 2\alpha + j; 1 - \frac{r^2}{r_1^2} \right) \\ \cdot {}_2F_1 \left(\beta - \alpha, \beta + j; 2\beta + j; 1 - \frac{r^2}{r_2^2} \right). \end{aligned} \tag{3.34}$$

Now, in view of the following inequalities:

$$\begin{aligned} {}_2F_1 \left(\alpha - \beta, \alpha + j; 2\alpha + j; 1 - \frac{r^2}{r_1^2} \right) \\ \leq \frac{\Gamma(2\alpha)\Gamma(\beta)(2\alpha)_j}{\Gamma(\alpha + \beta)\Gamma(\alpha)(\alpha + \beta)_j} \end{aligned}$$

and

$$\begin{aligned} {}_2F_1 \left(\beta - \alpha, \beta + j; 2\beta + j; 1 - \frac{r^2}{r_2^2} \right) \\ \leq \frac{\Gamma(2\beta)\Gamma(\alpha)(2\beta)_j}{\Gamma(\alpha + \beta)\Gamma(\beta)(\alpha + \beta)_j}, \end{aligned}$$

we find from (3.34) that the inequality (3.33) holds true. Hence Theorem 1 is proved.

By virtue of the following known formula [8, p. 117, Eq. (12)]:

$$\begin{aligned} {}_2F_1(a, b; a + b; z) \\ = -\frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} {}_2F_1(a, b; 1; 1 - z) \ln(1 - z) \\ + \frac{\Gamma(a + b)}{[\Gamma(a)]^2[\Gamma(b)]^2} \sum_{j=0}^{\infty} \frac{\Gamma(a + j)\Gamma(b + j)}{(j!)^2} \\ \cdot [2\psi(1 + j) - \psi(a + j) - \psi(b + j)] (1 - z)^j \end{aligned}$$

($|\arg(1 - z)| \leq \pi - \varepsilon$ ($0 < \varepsilon < \infty$); $a, b \neq 0, -1, -2, \dots$),

where

$$\psi(z) := \frac{d}{dz} \{ \ln \Gamma(z) \} = \frac{\Gamma'(z)}{\Gamma(z)}$$

or, equivalently,

$$\ln \Gamma(z) = \int_1^z \psi(t) dt,$$

we observe from (3.33) that $q_1(x, y; x_0, y_0)$ has a logarithmic singularity at $r = 0$.

Theorem 2. *If the curve Γ satisfies conditions (3.1), then the following inequality holds true:*

$$\int_{\Gamma} x^{2\alpha} y^{2\beta} \left| \frac{\partial}{\partial n} \{ q_1(x, y; x_0, y_0) \} \right| ds \leq C_1,$$

where C_1 is a constant.

Proof. Theorem 2 follows by suitably applying Lemmas 1 to 3.

Theorem 3. *The following limiting formulas hold true for a double-layer potential (3.2):*

$$w_i^{(1)}(t) = -\frac{1}{2}\mu_1(t) + \int_0^l \mu_1(s) K_1(s, t) ds \quad (3.35)$$

and

$$w_e^{(1)}(t) = \frac{1}{2}\mu_1(t) + \int_0^l \mu_1(s) K_1(s, t) ds, \quad (3.36)$$

where, as usual, $\mu_1(t) \in [0, l]$,

$$K_1(s, t) = [x(s)]^{2\alpha} [y(s)]^{2\beta} \cdot \frac{\partial}{\partial n} \{q_1[x(s), y(s); x_0(t), y_0(t)]\} \quad (1)$$

$$\left((x(s), y(s)) \in \Gamma; (x_0(t), y_0(t)) \in \Gamma \right),$$

$w_i^{(1)}(t)$ and $w_e^{(1)}(t)$ are limiting values of the double-layer potential (3.2) at

$$(x_0(t), y_0(t)) \rightarrow \Gamma$$

from the inside and the outside, respectively.

Proof. We find from Lemma 1, in conjunction with Theorems 1 and 2, that each of the limiting formulas asserted by Theorem 3 holds true.

4 Computational and Applied Aspects

In such widely-investigated subject as Potential Theory, both single-layer potential and double-layer potential play significant rôles in solving boundary value problems involving various families of elliptic partial differential equations. In particular, a double-layer potential provides a solution of Laplace's equation corresponding to the electrostatic or magnetic potential associated with a dipole distribution on a closed surface in the three-dimensional Euclidean space or (more generally) on a hypersurface in the n -dimensional Euclidean space.

In our present investigation of the generalized bi-axially Helmholtz equation $(H_{\alpha, \beta}^\lambda)$, we have successfully developed a worthwhile alternative to the method of complex analysis (based upon analytic functions). We make use of results from potential theory in order to represent boundary value problems in integral equation form. In fact, in problems with known Green's functions, an integral equation formulation leads to powerful numerical approximation schemes. Thus, by seeking the representation of the solution of the boundary value problem as a double-layer potential with unknown density, we are eventually led to a Fredholm equation of

the second kind for the explicit determination of the solution in terms of the Appell function F_2 of the second kind in two variables, which is defined by (1.1).

Various known properties and formulas involving the Appell function F_2 such as (see, for details, [4], [8] and [33])

$$\begin{aligned} F_2(a, b, b'; c, c'; x, y) &:= \sum_{m, n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} {}_2F_1(a+m, b'; c'; y) \frac{x^m}{m!} \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b')_n}{(c')_n} {}_2F_1(a+n, b; c; x) \frac{y^n}{n!}, \end{aligned} \quad (4.1)$$

$$\begin{aligned} F_2(a, b, b'; a, a; x, y) &= (1-x)^{-b} (1-y)^{-b'} \\ &\cdot {}_2F_1 \left[\begin{matrix} b, b'; \\ a; \end{matrix} \frac{xy}{(1-x)(1-y)} \right] \end{aligned} \quad (4.2)$$

and

$$F_2(a, b, b'; c, b'; x, y) = (1-y)^{-a} {}_2F_1 \left[\begin{matrix} a, b; \\ c; \end{matrix} \frac{x}{1-y} \right], \quad (4.3)$$

which express the function F_2 in terms of the simpler Gauss hypergeometric function ${}_2F_1$ that possesses easily-accessible numerical algorithms for computational purposes, can indeed be used to numerically compute the solution presented here for many different special values of the parameters a, b, b', c and c' and the arguments x and y .

Numerous applications of several suitably specialized versions of the solutions presented in this paper can be found in fracture mechanics, fluid mechanics, elastodynamics, electromagnetics, and acoustics (see, for details, some of the citations handling special situations which were motivated by such widespread applications).

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