Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.12785/pfda/010206

Large Stability Regions Method for the Two-dimensional Fractional Diffusion Equation

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Received: 18 Dec. 2014, Revised: 6 Jan. 2015, Accepted: 9 Jan. 2015
Published online: 1 Apr. 2015

Abstract: In this paper, the non-standard finite difference method (NSFDM) is presented for solving numerically the two-dimensional space fractional diffusion equation (SFDE), where the fractional derivative is defined in the sense of the right-shifted Grünwald. The stability and the error analysis for the proposed method are given. Two numerical test examples are presented. It is concluded that the NSFDM scheme preserves numerical stability in larger regions than the standard finite difference method (SFDM).

Keywords: Two-dimensional fractional diffusion equation, The right-shifted Grünwald derivative, Non-standard finite difference method, von Neumann stability analysis.

1 Introduction

It is well known that fractional order differential equations have been the focus of many studies due to their frequent appearance in various applications especially in the fields of fluid mechanics, viscoelasticity, biology, physics and engineering, see ([1], [2], [3], [4], [5], [6], [7] and the references cited therein). Analytic closed-form solutions for such models are in general difficult to evaluate. Difference methods and, in particular, explicit finite difference methods, are a simple and important class of numerical methods for solving fractional differential equations ([8], [9], [10], [11], [12], [13], [14]). The usefulness of the explicit method and the reason why they are widely employed is based on their particularly attractive features ([15], [16]). The most attractive feature is that there is no need to solve the resultant system of equations, especially for large scale problems. The main disadvantage of these methods is that the stability condition can only be proved in a small interval of space and time ([17], [18]). The non-standard numerical method is another numerical method, which can solve the difficulties of SFDM ([19], [20], [7], [21]). In the following we used the merits of NSFDM to study numerically the fractional order differential equations. Specifically, we apply the Mickens non-standard discretization scheme [22] with the right-shifted Grünwald discretization process for the following two-dimensional fractional diffusion equation:

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\frac{\partial^{\alpha} u(x,y,t)}{\partial x^{\alpha}} + e(x,y)\frac{\partial^{\beta} u(x,y,t)}{\partial y^{\beta}} + q(x,y,t), \tag{1}$$

on a finite domain $x_L < x < x_H$ and $y_L < y < y_H$, with fractional orders $1 < \alpha \leqslant 2$ and $1 < \beta \leqslant 2$, where the diffusion coefficients $d(x,y) > 0$ and $e(x,y) > 0$. The function $q(x,y,t)$ can be used to represent sources and sinks. let us consider initial condition $u(x,y,t=0) = f(x,y)$, $x_L < x < x_H$, $y_L < y < y_H$, and Dirichlet boundary conditions $u(x,y,t) = B(x,y,t)$ on the boundary of the rectangular region $x_L \leqslant x \leqslant x_H$, $y_L \leqslant y \leqslant y_H$, with $B(x_L,y,t) = B(x,y_L,t) = 0$. The classical dispersion equation in two dimensions is given by $\alpha = \beta = 2$, for more details on the model problem, see [23].

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2 NSFDM for SFDE

NSFDM as introduced by Mickens ([22], [24], [25]), it has many advantages when compared to SFDM, for more details see ([26], [27], [28], [29], [30]). General speaking, we can say that NSFDM is more efficient and accurate than other explicit methods ([31]-[32]).

In this work, the spatial $\alpha-$order fractional derivative is discretized using the right-shifted Grünwald formula [4], for $1<\alpha\leqslant 2$:

$$\frac{\partial^{\alpha}u(x,y,t)}{\partial x^{\alpha}}=\frac{1}{\Gamma(-\alpha)}\lim_{N_x\to\infty}\frac{1}{h^{\alpha}}\sum_{k=0}^{N_x}\frac{\Gamma(k-\alpha)}{\Gamma(k+1)}u[x-(k-1)h,y,t], \tag{2}$$

where N_x is a positive integer, $h=(x-x_L)/N_x$ and $\Gamma(.)$ is the gamma function. The normalized Grünwald weights are defined as follows:

$$g_{\alpha,k}=\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}=(-1)^k\binom{\alpha}{k}. \tag{3}$$

For more details on these normalized weights see [4]. For the numerical approximation scheme, let us define $\Delta x=(x_H-x_L)/N_x=h_x>0$, as the grid size in $x-$direction, with $x_i=x_L+i\Delta x$, for $i=0,1,...,N_x$; $\Delta y=(y_H-y_L)/N_y=h_y>0$, as the grid size in $y-$direction, with $y_j=y_L+j\Delta y$, for $j=0,1,...,N_y$. $t_n=n\Delta t$, to be the integration time, $0\leqslant t_n\leqslant T$, Define $u^n_{i,j}$ as the numerical approximation to $u(x_i,y_j,t_n)$. The initial conditions are set by $u^0_{i,j}=f_{i,j}=f(x_i,y_j)$. The Dirichlet boundary condition on the boundary of this rectangular region are at $x=x_L$, $u^n_{0,j}=B^n_{0,j}=B(x_L,y_j,t_n)=0$; at $x=x_H$, $u^n_{N_x,j}=B^n_{N_x,j}=B(x_H,y_j,t_n)$; at $y=y_L$, $u^n_{i,0}=B^n_{i,0}=B(x_i,y_L,t_n)=0$; and at $y=y_H$, $u^n_{i,N_y}=B^n_{i,N_y}=B(x_i,y_H,t_n)$. Define $d_{i,j}=d(x_i,y_j)$, $e_{i,j}=e(x_i,y_j)$, and $q^n_{i,j}=q(x_i,y_j,t_n)$. Applying Micken's scheme by replacing the step size h by a function $\psi(h)$ and using the shifted Grünwald estimates, then substituting into (1) we can claim the following:

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\phi(\Delta t)}=d_{i,j}\delta_{\alpha,x}u^n_{i,j}+e_{i,j}\delta_{\beta,y}u^n_{i,j}+q^n_{i,j}+T(x,y,t), \tag{4}$$

where $\phi(\Delta t):=1-e^{-\Delta t}$,

$$\delta_{\alpha,x}u^n_{i,j}:=\frac{1}{\psi_1(\Delta x)^{\alpha}}\sum_{k=0}^{i+1}g_{\alpha,k}u^n_{i-k+1,j},$$

$$\delta_{\beta,y}u^n_{i,j}:=\frac{1}{\psi_2(\Delta y)^{\beta}}\sum_{k=0}^{j+1}g_{\beta,k}u^n_{i,j-k+1},$$

$\psi_1(\Delta x):=sinh(\Delta x)$, $\psi_2(\Delta y):=sinh(\Delta y)$ and $T(x,y,t)$ is the truncation term [8]. Using (4) we can claim:

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{\phi(\Delta t)}=\frac{d_{i,j}}{\psi_1(\Delta x)^{\alpha}}\sum_{k=0}^{i+1}g_{\alpha,k}u^n_{i-k+1,j}+\frac{e_{i,j}}{\psi_2(\Delta y)^{\beta}}\sum_{k=0}^{j+1}g_{\beta,k}u^n_{i,j-k+1}+q^n_{i,j}. \tag{5}$$

Or

$$u^{n+1}_{i,j}=u^n_{i,j}+\bar{s_1}\sum_{k=0}^{i+1}g_{\alpha,k}u^n_{i-k+1,j}+\bar{s_2}\sum_{k=0}^{j+1}g_{\beta,k}u^n_{i,j-k+1}+\phi(\Delta t)q^n_{i,j}, \tag{6}$$

where $\bar{s_1}=s_1d_{i,j}$, $s_1=\frac{\phi(\Delta t)}{\psi_1(\Delta x)^{\alpha}}$; $\bar{s_2}=s_2e_{i,j}$, $s_2=\frac{\phi(\Delta t)}{\psi_2(\Delta y)^{\beta}}$.

3 Stability Analysis of NSFDM

In this section we will use a kind of von Neumann method to study the stability analysis of the non-standard finite difference scheme (6).

Theorem 3.1. The non-standard finite-difference scheme (6) for SFDE is conditionally stable and the stability condition is:

$$\phi(\Delta t)\leq S,$$

where

$$S=\frac{-2A}{A^2+B^2},$$

$$A=\frac{d_{i,j}}{(\psi_1(\Delta x))^{\alpha}}\sum_{k=0}^{i+1}g_{\alpha,k}[\cos(q_1(k-1)\psi_1(\Delta x))]+\frac{e_{i,j}}{(\psi_2(\Delta y))^{\beta}}\sum_{k=0}^{j+1}g_{\beta,k}[\cos(q_2(k-1)\psi_2(\Delta y))].$$

$$B=\frac{d_{i,j}}{(\psi_1(\Delta x))^{\alpha}}\sum_{k=0}^{i+1}g_{\alpha,k}[\sin(q_1(k-1)\psi_1(\Delta x))]+\frac{e_{i,j}}{(\psi_2(\Delta y))^{\beta}}\sum_{k=0}^{j+1}g_{\beta,k}[\sin(q_2(k-1)\psi_2(\Delta y))].$$

Proof. Let us analyze the stability of (6) by substituting in a separated solution $u_{i,j}^{n}=\zeta_n e^{mq_1 i\psi_1(\Delta x)}e^{mq_2 j\psi_2(\Delta y)}$ $=\zeta_n e^{mq_1 i\psi_1(\Delta x)+mq_2 j\psi_2(\Delta y)}$ where $m=\sqrt{-1}$, q_1,q_2 are real spatial wave-number. Inserting this expression we get

$$\zeta_{n+1}e^{mq_1 i\psi_1(\Delta x)+mq_2 j\psi_2(\Delta y)}=\zeta_n e^{mq_1 i\psi_1(\Delta x)+mq_2 j\psi_2(\Delta y)}$$

$$+\bar{s_1}\sum_{k=0}^{i+1}g_{\alpha,k}\zeta_n e^{mq_1(i-k+1)\psi_1(\Delta x)+mq_2 j\psi_2(\Delta y)}+\bar{s_2}\sum_{k=0}^{j+1}g_{\beta,k}\zeta_n e^{mq_1 i\psi_1(\Delta x)+mq_2(j-k+1)\psi_2(\Delta y)}, \tag{7}$$

divided (7) by $e^{mq_1 i\psi_1(\Delta x)+mq_2 j\psi_2(\Delta y)}$ then we get:

$$\zeta_{n+1}=\zeta_n+\bar{s_1}\sum_{k=0}^{i+1}g_{\alpha,k}\zeta_n e^{-mq_1(k-1)\psi_1(\Delta x)}+\bar{s_2}\sum_{k=0}^{j+1}g_{\beta,k}\zeta_n e^{-mq_2(k-1)\psi_2(\Delta y)}. \tag{8}$$

Using the known Euler's formula $e^{m\theta}=\cos\theta+m\sin\theta$, $m=\sqrt{-1}$, we have:

$$\begin{aligned}\zeta_{n+1}=\zeta_n&+\bar{s_1}\sum_{k=0}^{i+1}g_{\alpha,k}\zeta_n[\cos(q_1(k-1)\psi_1(\Delta x))-m\sin(q_1(k-1)\psi_1(\Delta x))]\\&+\bar{s_2}\sum_{k=0}^{j+1}g_{\beta,k}\zeta_n[\cos(q_2(k-1)\psi_2(\Delta y))-m\sin(q_2(k-1)\psi_2(\Delta y))].\end{aligned} \tag{9}$$

Where $\zeta(x)$ means the Riemann zeta function. The stability will be determined by the behaviour of ζ_n. If we write $\zeta_{n+1}=\eta\zeta_n$ and assume that $\eta\equiv\eta(q)$ is independent of time, then we can obtain

$$\begin{aligned}\eta\zeta_n=\zeta_n&+\bar{s_1}\sum_{k=0}^{i+1}g_{\alpha,k}\zeta_n[\cos(q_1(k-1)\psi_1(\Delta x))-m\sin(q_1(k-1)\psi_1(\Delta x))]\\&+\bar{s_2}\sum_{k=0}^{j+1}g_{\beta,k}\zeta_n[\cos(q_2(k-1)\psi_2(\Delta y))-m\sin(q_2(k-1)\psi_2(\Delta y))],\end{aligned} \tag{10}$$

divided by ζ_n to obtain the following formula of η:

$$\begin{aligned}\eta=1&+\bar{s_1}\sum_{k=0}^{i+1}g_{\alpha,k}[\cos(q_1(k-1)\psi_1(\Delta x))-m\sin(q_1(k-1)\psi_1(\Delta x))]\\&+\bar{s_2}\sum_{k=0}^{j+1}g_{\beta,k}[\cos(q_2(k-1)\psi_2(\Delta y))-m\sin(q_2(k-1)\psi_2(\Delta y))].\end{aligned} \tag{11}$$

$$\begin{aligned}\eta=1&+\frac{\phi(\Delta t)d_{i,j}}{(\psi_1(\Delta x))^{\alpha}}\sum_{k=0}^{i+1}g_{\alpha,k}[\cos(q_1(k-1)\psi_1(\Delta x))]\\&-m\frac{\phi(\Delta t)d_{i,j}}{(\psi_1(\Delta x))^{\alpha}}\sum_{k=0}^{i+1}g_{\alpha,k}[\sin(q_1(k-1)\psi_1(\Delta x))]\\&+\frac{\phi(\Delta t)e_{i,j}}{(\psi_2(\Delta y))^{\beta}}\sum_{k=0}^{j+1}g_{\beta,k}[\cos(q_2(k-1)\psi_2(\Delta y))]\\&-m\frac{\phi(\Delta t)e_{i,j}}{(\psi_2(\Delta y))^{\beta}}\sum_{k=0}^{j+1}g_{\beta,k}[\sin(q_2(k-1)\psi_2(\Delta y))].\end{aligned} \tag{12}$$

$$\begin{aligned}\eta = 1+\phi(\Delta t)(\frac{d_{i,j}}{(\psi_1(\Delta x))^{\alpha}}\sum_{k=0}^{i+1} g_{\alpha,k}[\cos(q_1(k-1)\psi_1(\Delta x))] \\ +\frac{e_{i,j}}{(\psi_2(\Delta y))^{\beta}}\sum_{k=0}^{j+1} g_{\beta,k}[\cos(q_2(k-1)\psi_2(\Delta y))]) \\ -m\phi(\Delta t)(\frac{d_{i,j}}{(\psi_1(\Delta x))^{\alpha}}\sum_{k=0}^{i+1} g_{\alpha,k}[\sin(q_1(k-1)\psi_1(\Delta x))] \\ +\frac{e_{i,j}}{(\psi_2(\Delta y))^{\beta}}\sum_{k=0}^{j+1} g_{\beta,k}[\sin(q_2(k-1)\psi_2(\Delta y))]).\end{aligned} \tag{13}$$

Let

$$A = \frac{d_{i,j}}{(\psi_1(\Delta x))^{\alpha}}\sum_{k=0}^{i+1} g_{\alpha,k}[\cos(q_1(k-1)\psi_1(\Delta x))]+\frac{e_{i,j}}{(\psi_2(\Delta y))^{\beta}}\sum_{k=0}^{j+1} g_{\beta,k}[\cos(q_2(k-1)\psi_2(\Delta y))].$$

$$B = \frac{d_{i,j}}{(\psi_1(\Delta x))^{\alpha}}\sum_{k=0}^{i+1} g_{\alpha,k}[\sin(q_1(k-1)\psi_1(\Delta x))]+\frac{e_{i,j}}{(\psi_2(\Delta y))^{\beta}}\sum_{k=0}^{j+1} g_{\beta,k}[\sin(q_2(k-1)\psi_2(\Delta y))].$$

$$\eta = 1+\phi(\Delta t)A - m\phi(\Delta t)B. \tag{14}$$

The mode will be stable as long as $|\eta| \leq 1$, i.e.,

$$|1+\phi(\Delta t)A - m\phi(\Delta t)B| \leq 1, \tag{15}$$

this means that:

$$(1+\phi(\Delta t)A)^2 + (\phi(\Delta t)B)^2 \leq 1. \tag{16}$$

$$1+2\phi(\Delta t)A + (\phi(\Delta t))^2(A)^2 + (\phi(\Delta t))^2(B)^2 \leq 1. \tag{17}$$

So,

$$2A+\phi(\Delta t)(A)^2+\phi(\Delta t)(B)^2 \leq 0. \tag{18}$$

$$\phi(\Delta t) \leq S, \tag{19}$$

where

$$S = \frac{-2A}{A^2+B^2}, \quad A^2+B^2 > 0. \qquad \square \tag{20}$$

Theorem 3.2. The truncation error of SFDE is:

$$T(x,y,t) = O(\phi(\Delta t)) + O(\psi_1(\Delta x)) + O(\psi_2(\Delta y)).$$

Proof. Evaluating (1) at the point (x_i, y_j, t_n), gives

$$[\frac{\partial u}{\partial t} - d\frac{\partial^{\alpha} u}{\partial x^{\alpha}} - e\frac{\partial^{\beta} u}{\partial y^{\beta}} - q]_{(x_i,y_j,t_n)} = 0, \tag{21}$$

by the difference equation

$$\phi(\Delta_t)u^n_{i,j} - d\psi_1(\Delta_x)u^n_{i+1,j} - e\psi_2(\Delta_y)u^n_{i,j+1} = T(x_i,y_j,t_n). \tag{22}$$

Neglecting the truncation error term $T(x_i,y_j,t_n)$, we get the explicit difference scheme (6). From (1)-(6) and (22), we get

$$[\frac{\partial u}{\partial t}|_{(x_i,y_j,t_n)} - \phi(\Delta_t)u^n_{i,j}] - d[\frac{\partial^{\alpha} u}{\partial x^{\alpha}}|_{(x_i,y_j,t_n)} - \psi_1(\Delta_x)u^n_{i+1,j}] - e[\frac{\partial^{\beta} u}{\partial y^{\beta}}|_{(x_i,y_j,t_n)} - \psi_2(\Delta_y)u^n_{i,j+1}] = T(x_i,y_j,t_n), \tag{23}$$

$$\frac{\partial}{\partial t}u(x_i,y_j,t_n) = \phi(\Delta_t)u(x_i,y_j,t_n) + O(\phi(\Delta t)), \tag{24}$$

$$\psi_1(\Delta_x)u^n_{i+1,j} = \frac{\partial^\alpha u}{\partial x^\alpha}|_{(x_i,y_j,t_n)} + O(\psi_1(\Delta x))^2, \tag{25}$$

$$\frac{\partial^\alpha u}{\partial x^\alpha}|_{(x_{i+1},y_j,t_n)} = \frac{\partial^\alpha u}{\partial x^\alpha}|_{(x_i,y_j,t_n)} + \psi_1(\Delta_x)\frac{d\partial^\alpha u}{dx\partial x^\alpha}|_{(x_i,y_j,t_n)} + O(\psi_1(\Delta x))^2, \tag{26}$$

so that

$$\psi_1(\Delta_x)u^n_{i+1,j} = \psi_1(\Delta_x)u^n_{i,j} + O(\psi_1(\Delta x)) + O(\psi_1(\Delta x))^2. \tag{27}$$

And

$$\psi_2(\Delta_y)u^n_{i,j+1} = \frac{\partial^\beta u}{\partial y^\beta}|_{(x_i,y_j,t_n)} + O(\psi_2(\Delta y))^2, \tag{28}$$

$$\frac{\partial^\beta u}{\partial y^\beta}|_{(x_i,y_{j+1},t_n)} = \frac{\partial^\beta u}{\partial y^\beta}|_{(x_i,y_j,t_n)} + \psi_2(\Delta_y)\frac{d\partial^\beta u}{dy\partial y^\beta}|_{(x_i,y_j,t_n)} + O(\psi_2(\Delta y))^2, \tag{29}$$

so that

$$\psi_2(\Delta_y)u^n_{i,j+1} = \psi_2(\Delta_y)u^n_{i,j} + O(\psi_2(\Delta y)) + O(\psi_2(\Delta y))^2. \tag{30}$$

From this results and from (24), we claim that

$$T(x,y,t) = O(\phi(\Delta t)) + O(\psi_1(\Delta x)) + O(\psi_2(\Delta y)). \qquad \Box \tag{31}$$

4 Numerical Results

Example 4.1. Consider the following space fractional diffusion equation:

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\frac{\partial^{1.7}u(x,y,t)}{\partial x^{1.7}} + e(x,y)\frac{\partial^{1.7}u(x,y,t)}{\partial y^{1.7}} + q(x,y,t), \tag{32}$$

on a finite rectangular domain $0 < x < 1$, $0 < y < 1$, for $0 \leqslant t \leqslant T_{end}$. The diffusion coefficients are

$$d(x,y) = \Gamma(2.2)x^{2.7}y/6,$$

and

$$e(x,y) = 2xy^{2.7}/\Gamma(4.7),$$

the forcing function:

$$q(x,y,t) = -(1+2xy)e^{-t}x^3y^{3.7},$$

with the initial condition:

$$u(x,y,0) = x^3y^{3.7},$$

and Dirichlet boundary conditions:

$$u(0,y,t) = u(x,0,t) = 0,\ u(1,y,t) = e^{-t}y^{3.7},\ u(x,1,t) = e^{-t}x^3,\ \forall t \geqslant 0.$$

The exact solution to this two-dimensional fractional diffusion equation is given by [33]:

$$u(x,y,t) = e^{-t}x^3y^{3.7}. \tag{33}$$

The numerical studies are given as follows: Table 1 shows the maximum absolute numerical errors between the exact solutions and the numerical solutions using SFDM and NSFDM at time $T_{end} = 5$.

Table 1: The maximum errors obtained by SFDM and NSFDM when $T_{end} = 5$, with different time step size.

Δt	$\Delta x = \Delta y$	maximum error (SFDM)	maximum error (NSFDM)
$\frac{5}{15}$	$\frac{1}{5}$	$7.4733462E-3(convergent)$	$2.2021735E-3(convergent)$
$\frac{5}{12}$	$\frac{1}{5}$	$1.2386004E-2(convergent)$	$2.6472456E-3(convergent)$
$\frac{5}{10}$	$\frac{1}{5}$	$1.8325816E-2(convergent)$	$3.0567317E-3(convergent)$
$\frac{5}{8}$	$\frac{1}{5}$	$1.9665325E-1(convergent)$	$3.6103985E-3(convergent)$
$\frac{5}{5}$	$\frac{1}{5}$	$2.9566822E0(divergent)$	$1.1095885E-2(convergent)$

In order to test the numerical schemes, Figure 1 shows the approximate solution when $\alpha = \beta = 1.7$, at $T_{end} = 5$, $\Delta x = \Delta y = 0.2$ and $\Delta t = 1$. For this unstable solution behaviour, the stability condition is not satisfied and $S = 0.796$, where the value of Δt is larger than the stability bound S, while Figure 2 shows the exact solution at $t = 2$.

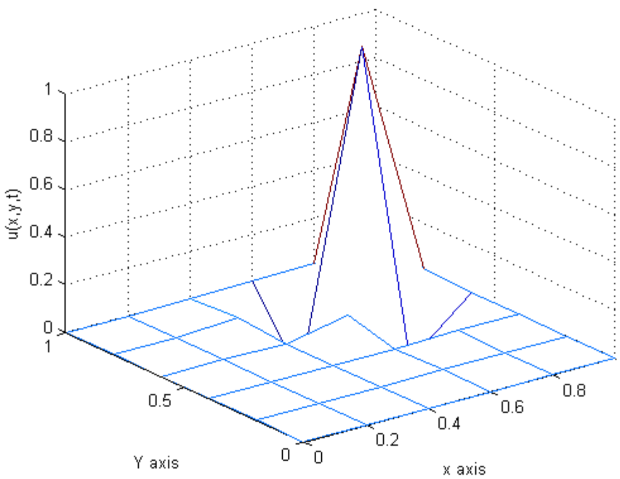

Fig. 1: SFDM solution when $\Delta x = \Delta y = 0.2$, $\Delta t = 1$ and $S = 0.796$.

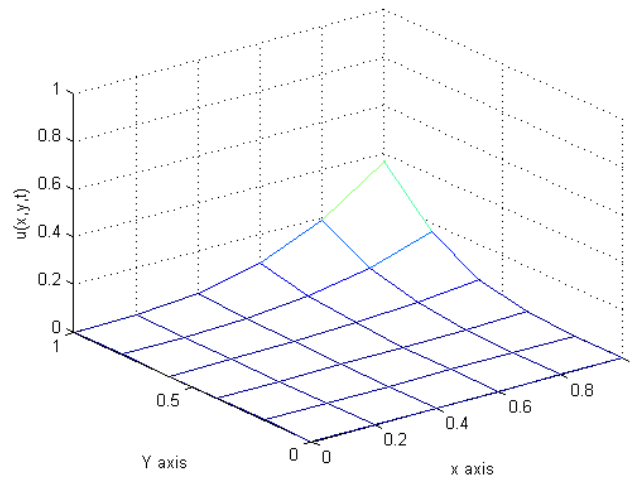

Fig. 2: Exact solution at $t = 2$.

Using the same process for NSFDM with $\psi_1(\Delta x) = \psi_2(\Delta y) = 0.201$. Figure 3 shows the approximate solution when $\Delta x = \Delta y = 0.2$, $\Delta t = 1$ and $\phi(\Delta t) = 0.632$. For this stable solution behaviour, the stability condition is not satisfied and $S = 0.805$, where the value of $\phi(\Delta t)$ is less than the stability bound S, while Figure 4 shows the exact solution at $t = 2$, for more details on the stability conditions see Theorem 1.

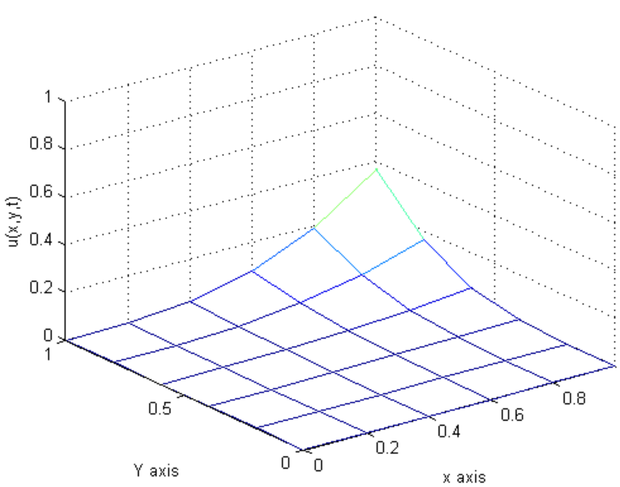

Fig. 3: NSFDM solution when $\psi_1(\Delta x) = \psi_2(\Delta y) = 0.201$, $\phi(\Delta t) = 0.632$ and $S = 0.805$.

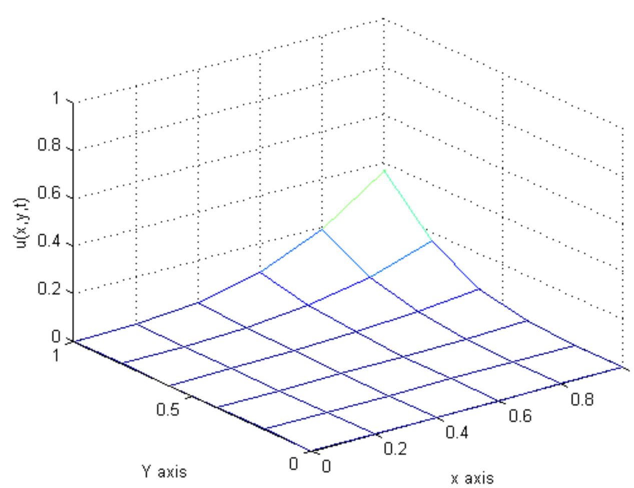

Fig. 4: Exact solution at $t = 2$.

Example 4.2. Consider the following space fractional diffusion equation :

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\frac{\partial^{1.9}u(x,y,t)}{\partial x^{1.9}} + e(x,y)\frac{\partial^{1.6}u(x,y,t)}{\partial y^{1.6}} + q(x,y,t) \tag{34}$$

on a finite rectangular domain $0 < x < 1$, $0 < y < 1$, for $0 \leqslant t \leqslant T_{end}$. The diffusion coefficients are

$$d(x,y) = x^3y^{1.4}/\Gamma(3.9), \quad e(x,y) = x^{1.1}y^3/\Gamma(3.6),$$

the forcing function: $q(x,y,t) = -(1+2x^{1.1}y^{1.4})e^{-t}x^{2.9}y^{2.6}$,
with the initial conditions: $u(x,y,0) = x^{2.9}y^{2.6}$,
and Dirichlet boundary conditions: $u(0,y,t) = u(x,0,t) = 0$, $u(1,y,t) = e^{-t}y^{2.6}$, $u(x,1,t) = e^{-t}x^{2.9}$, $\forall t \geqslant 0$.
The exact solution to this two-dimensional fractional diffusion equation is given by [34]:

$$u(x,y,t) = e^{-t}x^{2.9}y^{2.6}. \tag{35}$$

The numerical studies are given as follows: Table 2 shows the maximum absolute numerical errors between the exact solutions and the numerical solutions using SFDM and NSFDM at time $T_{end} = 5$.

Table 2: The maximum errors obtained by SFDM and NSFDM when $T_{end} = 5$, with different time step size.

Δt	$\Delta x = \Delta y$	maximum error (SFDM)	maximum error (NSFDM)
$\frac{5}{15}$	$\frac{1}{5}$	$1.1118891E-2(convergent)$	$1.8383829E-3(convergent)$
$\frac{5}{12}$	$\frac{1}{5}$	$1.5760271E-1(convergent)$	$2.2099309E-3(convergent)$
$\frac{5}{10}$	$\frac{1}{5}$	$2.1694026E0(divergent)$	$4.3608783E-3(convergent)$
$\frac{5}{8}$	$\frac{1}{5}$	$1.1524792E1(divergent)$	$3.2262976E-2(convergent)$
$\frac{5}{5}$	$\frac{1}{5}$	$2.3375558E1(divergent)$	$8.5214990E-2(convergent)$

In order to test the numerical schemes, Figure 5 shows the approximate solution when $\alpha = 1.9$, $\beta = 1.6$ at $T_{end} = 5$, $\Delta x = \Delta y = 0.2$ and $\Delta t = 1$. The unstable solution behaviour, the stability condition in this case is not satisfied and $S = 0.451$, where the value of Δt is larger than the stability bound S, while Figure 6 shows the exact solution at $t = 2$.

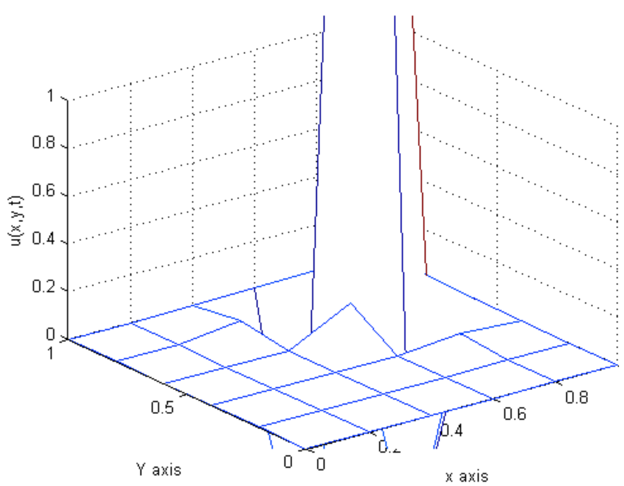

Fig. 5: SFDM solution when $\Delta x = \Delta y = 0.2$, $\Delta t = 1$ and $S = 0.451$.

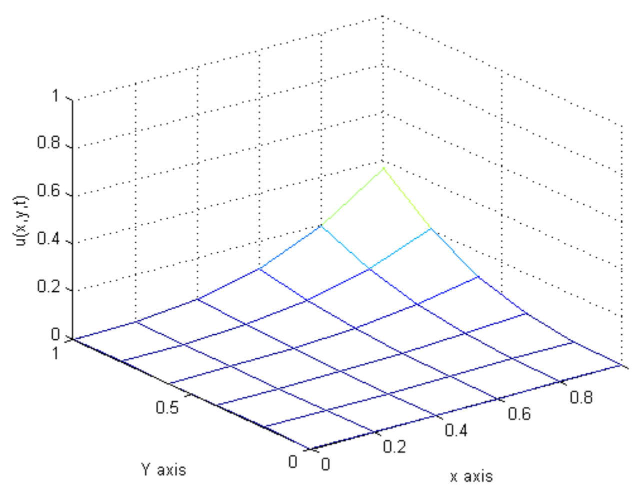

Fig. 6: Exact solution at $t = 2$.

Using the same process for NSFDM with $\psi_1(\Delta x) = \psi_2(\Delta y) = 0.201$. Figure 7 shows the approximate solution when $\Delta x = \Delta y = 0.2$, $\Delta t = 1$ and $\phi(\Delta t) = 0.632$. The stable solution behaviour, the stability condition in this case is satisfied and $S = 0.729$, where the value of $\phi(\Delta t)$ is less than the stability bound S, while Figure 8 shows the exact solution at $t = 2$, for more details on the stability conditions see Theorem 1.

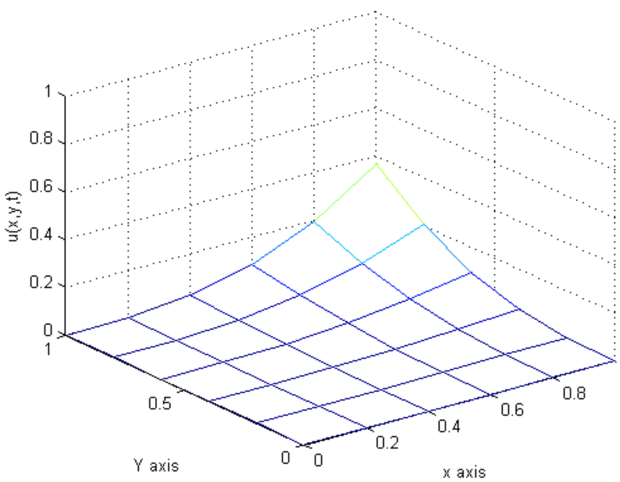

Fig. 7: NSFDM solution when $\psi_1(\Delta x) = \psi_2(\Delta y) = 0.201$, $\phi(\Delta t) = 0.632$ and $S = 0.729$.

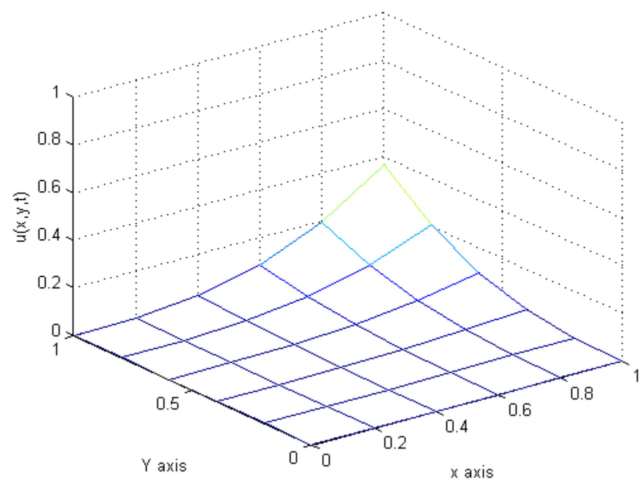

Fig. 8: Exact solution at $t = 2$.

Conclusions

In this paper, a two-dimensional space fractional order diffusion equation is numerically studied using SFDM and NSFDM, where the fractional derivative is defined in the right-shifted Grünwald sense. Stability analysis of the proposed method for SFDE is discussed by means of a fractional version of the von Neumann stability analysis. Two numerical test examples are presented. We can conclude from the comparison between SFDM and NSFDM that for some kind of non-linear fractional differential equations, NSFDM leads to faster convergence, more accurate results and large stability regions than SFDM.

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