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An Analogue Result of q-Beta Integral

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Abstract: We give an analogue result of q-beta integral by applying the q-Chu-Vandermonde formula and get several identities which include q-series ${}_3\phi_2$ and q-integral.

Keywords: q-beta integral, q-Chu-Vandermonde formula, transformations formula.
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1 Introduction and main result

Throughout this paper we suppose $|q| < 1, \mathbb{N} = \{1, 2, \ldots\}$. The q-shifted factorial are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \tag{1.1}$$

$$(a;q)_\infty = \lim_{n\to\infty}\prod_{k=0}^{n-1}(1-aq^k) = \prod_{k=0}^{\infty}(1-aq^k), \qquad n \geq 1. \tag{1.2}$$

Clearly,

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1.3}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n$$
$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty$$

The basic hypergeometric series ${}_{r+1}\phi_r$, or q-series are defined by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, b_2, \ldots, b_r; q)_n} z^n. \tag{1.4}$$

The q-Chu-Vandermonde convolution formula is

$${}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q\right) = \frac{a^n (c/a;q)_n}{(c;q)_n} \tag{1.5}$$

F. H. Jackson defined the q-integral as follows (see [4])

$$\int_0^d f(t)d_qt = d(1-q)\sum_{n=0}^{\infty} f(dq^n)q^n, \tag{1.6}$$

and

$$\int_c^d f(t)d_qt = \int_0^d f(t)d_qt - \int_0^c f(t)d_qt. \tag{1.7}$$

He also defined q-integral on $(0, \infty)$ by

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{1.8}$$

on the interval $(-\infty, \infty)$ the bilateral q-integral is defined by

$$\int_{-\infty}^{\infty} f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty}[f(q^n) + f(-q^n)]q^n. \tag{1.9}$$

Askey obtained an elegant formula of the q-beta integral (see [2]):

$$\int_{-\infty}^{\infty}\frac{(a\omega, b\omega;q)_\infty}{(-d\omega, e\omega;q)_\infty}d_q\omega = \frac{2(1-q)(q^2;q^2)_\infty^2(de, q/de, a/e, -a/d, b/e, -b/d;q)_\infty}{(q;q)_\infty(d^2, e^2, q^2/d^2, q^2/e^2;q^2)_\infty(-ab/deq;q)_\infty} \tag{1.10}$$

provided that $|q| < 1, |ab/deq| < 1$ and there are no zero factors in the denominator of the integrals.

The q-beta integral is an important formula in basic hypergeometric series. For instance, Wang gave some extensions of q-beta integral (see [7, 8, 9, 10]).

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In this paper, we give an interesting analogue result of q-beta integral using the q-Chu-Vandermonde formula. Making use of the similar method of Wang, we derive the following main result of this paper.

Theorem 1.1 *If $|q| < 1, |e/d| < 1$ and there are no zero factors in the denominator of the integrals, then for any nonnegative integers n, we have*

$$\int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty (a\omega;q)_n}{(-d\omega;q)_\infty (e\omega;q)_n}\, d_q\omega = \frac{2dqa^n(1-q)(q^2;q^2)_\infty^2 (de,q/de,-e/d,b/e;q)_\infty}{e^n(b+dq)(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty} \times \left[{}_3\phi_2\left(\begin{matrix} q^{-n},e/a,-b/dq \\ q,-e/d \end{matrix} ;q,q\right) - {}_3\phi_2\left(\begin{matrix} q^{-n},e/a,-b/dq \\ q,-e/d \end{matrix} ;q,q^2\right)\right] \quad (1.11)$$

2 The proof of Theorem

Recalling the q-Chu-Vandermonde convolution formula

$${}_2\phi_1\left(\begin{matrix} q^{-n},c \\ a \end{matrix} ;q,q\right) = \frac{c^n(a/c;q)_n}{(a;q)_n}. \quad (2.1)$$

By the following relation

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad (2.2)$$

the formula (2.1) can be written as

$$\sum_{k=0}^{n} \frac{(q^{-n},c;q)_k q^k}{(q;q)_k} \cdot \frac{(aq^k;q)_\infty}{(a;q)_\infty} = c^n \cdot \frac{(a/c;q)_n}{(a;q)_n}. \quad (2.3)$$

Letting $a \longmapsto a\omega$ in (2.3) and multiplying the both sides of equation (2.3) by

$$\frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty},$$

we obtain that

$$\sum_{k=0}^{n} \frac{(q^{-n},c;q)_k q^k}{(q;q)_k} \cdot \frac{(aq^k\omega,b\omega;q)_\infty}{(-d\omega,a\omega;q)_\infty} = c^n \cdot \frac{(a\omega/c;q)_n}{(a\omega;q)_n} \cdot \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty}. \quad (2.4)$$

To Calculate the q-integral on both sides of (2.4) with respect to variable ω, we have

$$\sum_{k=0}^{n} \frac{(q^{-n},c;q)_k q^k}{(q;q)_k} \cdot \int_{-\infty}^{\infty} \frac{(aq^k\omega,b\omega;q)_\infty}{(-d\omega,a\omega;q)_\infty} d_q\omega = c^n \cdot \int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty} \cdot \frac{(a\omega/c;q)_n}{(a\omega;q)_n} d_q\omega. \quad (2.5)$$

Applying the Askey's result (1.10) to the integral on the left-hand side of (2.5), we find that

$$\sum_{k=0}^{n} \frac{(q^{-n},c;q)_k q^k}{(q;q)_k} \cdot \frac{2(1-q)(q^2;q^2)_\infty^2 (da,q/da,q^k,-aq^k/d,b/a,-b/d;q)_\infty}{(q;q)_\infty (d^2,a^2,q^2/d^2,q^2/a^2;q^2)_\infty (-bq^k/dq;q)_\infty} = c^n \cdot \int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty} \cdot \frac{(a\omega/c;q)_n}{(a\omega;q)_n} d_q\omega, \quad (2.6)$$

which can be rewritten as

$$\sum_{k=0}^{n} \frac{(q^{-n},c;q)_k (-b/dq;q)_k q^k}{(q;q)_k (-a/d;q)_k (q;q)_{k-1}} \frac{2dq(1-q)(q^2;q^2)_\infty^2 (da,q/da,-a/d,b/a;q)_\infty}{c^n(b+dq)(d^2,a^2,q^2/d^2,q^2/a^2;q^2)_\infty} = \int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty} \cdot \frac{(a\omega/c;q)_n}{(a\omega;q)_n} d_q\omega. \quad (2.7)$$

We therefore obtain

$$\int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty} \cdot \frac{(a\omega/c;q)_n}{(a\omega;q)_n} d_q\omega = \frac{2dq(1-q)(q^2;q^2)_\infty^2 (da,q/da,-a/d,b/a;q)_\infty}{c^n(b+dq)(d^2,a^2,q^2/d^2,q^2/a^2;q^2)_\infty} \times \left[{}_3\phi_2\left(\begin{matrix} q^{-n},c,-b/dq \\ q,-a/d \end{matrix} ;q,q\right) - {}_3\phi_2\left(\begin{matrix} q^{-n},c,-b/dq \\ q,-a/d \end{matrix} ;q,q^2\right)\right]. \quad (2.8)$$

Replacing c by a/e and interchanging a and e in (2.8), we obtain the formula (1.11) at once. The proof is complete.

3 Some applications

In this section, we give three identities using the formula (1.11) which include a new identity for ${}_3\phi_2$ and two identities of q-integral.

One of the fundamental transformations in the theory of basic hypergeometric series is the following Sears' ${}_3\phi_2$ transformation (see [6]), which will be used to prove 3.1 below.

$${}_3\phi_2\left(\begin{matrix} q^{-n},b,c \\ d,e \end{matrix} ;q,deq^n/bc\right) = \frac{(e/c;q)_n}{(e;q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n},c,d/b \\ d,cq^{1-n}/e \end{matrix} ;q,q\right). \quad (3.1)$$

Theorem 3.1 *If $n \in \mathbb{N}$, then we have*

$${}_3\phi_2\left(\begin{matrix} q^{-n},a,bq^{n-1} \\ q,c \end{matrix} ;q,q\right) = a^n \frac{(a^{-1},b;q)_n}{(c;q)_n (q;q)_{n-1}} + \frac{(aq^{1-n};q)_n}{(c;q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n},bq^{n-1},q/a \\ q,a^{-1} \end{matrix} ;q,q\right). \quad (3.2)$$

Proof. Letting $b = aq^n$ in (1.11), we obtain

$$\int_{-\infty}^{\infty} \frac{(a\omega;q)_\infty}{(-d\omega;q)_\infty (e\omega;q)_n} d_q\omega = \frac{2da^n(1-q)(q^2;q^2)_\infty^2 (de,q/de,-e/d,a/e;q)_\infty}{e^n(d+aq^{n-1})(a/e;q)_n (d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty} \times \left[{}_3\phi_2\left(\begin{matrix} q^{-n},e/a,-a/dq^{1-n} \\ q,-e/d \end{matrix} ;q,q\right) - {}_3\phi_2\left(\begin{matrix} q^{-n},e/a,-a/dq^{1-n} \\ q,-e/d \end{matrix} ;q,q^2\right)\right]. \quad (3.3)$$

On the other hand, setting $b = eq^n$ in (1.10) and noting that (1.10), we obtain

$$\int_{-\infty}^{\infty} \frac{(a\omega;q)_\infty}{(-d\omega;q)_\infty (e\omega;q)_n} d_q\omega = \int_{-\infty}^{\infty} \frac{(a\omega,eq^n\omega;q)_\infty}{(-d\omega,e\omega;q)_\infty} d_q\omega = \frac{2dq(1-q)(q^2;q^2)_\infty^2 (de,q/de,a/e,-e/d;q)_\infty}{(a+dq)(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty} \frac{(-a/dq;q)_n}{(-e/d;q)_n (q;q)_{n-1}}. \quad (3.4)$$

Comparing the equations (3.3) and (3.4), we find that

$${}_3\phi_2\left(\begin{matrix} q^{-n},e/a,-a/dq^{1-n} \\ q,-e/d \end{matrix} ;q,q\right) = \frac{e^n(a/e,-a/d;q)_n}{a^n(-e/d;q)_n (q;q)_{n-1}} + {}_3\phi_2\left(\begin{matrix} q^{-n},e/a,-a/dq^{1-n} \\ q,-e/d \end{matrix} ;q,q^2\right). \quad (3.5)$$

Applying (3.1) in (3.5), we get

$${}_3\phi_2\left(\begin{matrix} q^{-n},e/a,-a/dq^{1-n} \\ q,-e/d \end{matrix} ;q,q\right) = \frac{e^n(a/e,-a/d;q)_n}{a^n(-e/d;q)_n (q;q)_{n-1}} + \frac{(eq^{1-n}/a;q)_n}{(-e/d;q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n},-a/dq^{1-n},aq/e \\ q,a/e \end{matrix} ;q,q\right). \quad (3.6)$$

Replacing $(e/a,-a/d,-e/d)$ by (a,b,c) in (3.6), we obtain (3.2) immediately.

Theorem 3.2 *we have*

$$\int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty} \cdot \frac{(a\omega;q)_{2n}}{(a^2\omega^2;q^2)_n} d_q\omega =$$
$$\frac{2dq^{n^2+1}(1-q)(q^2;q^2)_\infty^2(-da,-q/da,a/d,-b/a;q)_\infty}{(-1)^n(b+dq)(d^2,a^2,q^2/d^2,q^2/a^2;q^2)_\infty}$$
$$\times \left[{}_3\phi_2\left(\begin{matrix} q^{-n},-q^{-n},-b/dq \\ q,a/d \end{matrix} ;q,q \right) - {}_3\phi_2\left(\begin{matrix} q^{-n},-q^{-n},-b/dq \\ q,a/d \end{matrix} ;q,q^2 \right) \right] \tag{3.7}$$

provided that no zero factors in the denominator of the integrals.

Proof. Using the formula

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k, \qquad (a^2;q^2)_n = (a;q)_n(-a;q)_n$$

we easily get

$$\frac{(a\omega;q)_{2n}}{(a^2\omega^2;q^2)_n} = \frac{(a\omega;q)_n(aq^n\omega;q)_n}{(a\omega;q)_n(-a\omega;q)_n} = \frac{(aq^n\omega;q)_n}{(-a\omega;q)_n}. \tag{3.8}$$

Replacing a by aq^n and setting $e = -a$ in (1.11), noting that (3.8), we obtain that

$$\int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty} \cdot \frac{(a\omega;q)_{2n}}{(a^2\omega^2;q^2)_n} d_q\omega$$
$$= \int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty} \cdot \frac{(aq^n\omega;q)_n}{(-a\omega;q)_n}$$
$$= \frac{2dq^{n^2+1}(1-q)(q^2;q^2)_\infty^2(-da,-q/da,a/d,-b/a;q)_\infty}{(-1)^n(b+dq)(d^2,a^2,q^2/d^2,q^2/a^2;q^2)_\infty}$$
$$\times \left[{}_3\phi_2\left(\begin{matrix} q^{-n},-q^{-n},-b/dq \\ q,a/d \end{matrix} ;q,q \right) - {}_3\phi_2\left(\begin{matrix} q^{-n},-q^{-n},-b/dq \\ q,a/d \end{matrix} ;q,q^2 \right) \right].$$

This proof is complete.

Theorem 3.3 *we have*

$$\int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty} d_q\omega = 0 \tag{3.9}$$

provided that no zero factors in the denominator of the integrals.

Proof. We recall the Ramanujan's bilateral summation formula (see [5])

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(q;q)_\infty(b/a;q)_\infty(ax;q)_\infty(q/ax;q)_\infty}{(b;q)_\infty(q/a;q)_\infty(x;q)_\infty(b/ax;q)_\infty}. \tag{3.10}$$

Rewritting the above formula as

$$\sum_{n=-\infty}^{\infty} \frac{(bq^n;q)_\infty}{(aq^n;q)_\infty} x^n = \frac{(q;q)_\infty(b/a;q)_\infty(ax;q)_\infty(q/ax;q)_\infty}{(a;q)_\infty(q/a;q)_\infty(x;q)_\infty(b/ax;q)_\infty}. \tag{3.11}$$

Applying the formula (3.11) and noting that definition (1.9), we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{(b\omega;q)_\infty}{(-d\omega;q)_\infty} d_q\omega &= (1-q) \sum_{n=-\infty}^{\infty} \left[\frac{(bq^n;q)_\infty}{(-dq^n;q)_\infty} + \frac{(-bq^n;q)_\infty}{(dq^n;q)_\infty} \right] q^n \\
&= (1-q) \left[\sum_{n=-\infty}^{\infty} \frac{(bq^n;q)_\infty}{(-dq^n;q)_\infty} q^n + \sum_{n=-\infty}^{\infty} \frac{(-bq^n;q)_\infty}{(dq^n;q)_\infty} q^n \right] \\
&= (1-q) \left[\frac{(-dq;q)_\infty(-1/d;q)_\infty(-b/d;q)_\infty}{(-d;q)_\infty(-q/d;q)_\infty(-b/dq;q)_\infty} + \frac{(dq;q)_\infty(1/d;q)_\infty(-b/d;q)_\infty}{(d;q)_\infty(q/d;q)_\infty(-b/dq;q)_\infty} \right] \\
&= (1-q) \left[\frac{1+1/d}{(1+d)(1+b/dq)} + \frac{1-1/d}{(1-d)(1+b/dq)} \right] \\
&= (1-q) \left[\frac{q}{dq+b} - \frac{q}{dq+b} \right] \\
&= 0.
\end{aligned}$$

This proof is complete.

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