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# The Divisor Function and Divisor Problem

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**Abstract:** The purpose of this text is twofold. First we discuss some problems involving Paul Erdős (1913-1996), whose centenary of birth is this year. In the second part some recent results on divisor problems are discussed, and their connection with the powers moments of $|\zeta(\frac{1}{2}+it)|$ is pointed out.

**Keywords:** Dirichlet divisor problem, Riemann zeta-function, integral of the error term, mean square estimates, short intervals

## 1 Introduction

The classical number of divisors function of a positive integer $n$ is

$$d(n) := \sum_{\delta|n} 1.$$

We have $d(mn) = d(m)d(n)$ whenever $(m,n) = 1$, so that $d(n)$ is a multiplicative arithmetic function. Further $d(p^\alpha) = \alpha+1$ for $\alpha \in \mathbb{N}$, where $p, p_j$ denote generic primes and $\mathbb{N}$ is the set of natural numbers. Therefore, if $n = \prod_{j=1}^r p_j^{\alpha_j}$ is the canonical decomposition of $n$ into prime powers, then

$$d(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1).$$

In general

$$\zeta^k(s) = \sum_{n=1}^\infty d_k(n)n^{-s} \qquad (k \in \mathbb{N},\ \Re e\, s > 1),$$

where the (general) divisor function $d_k(n)$ represents the number of ways $n$ can be written as a product of $k$ factors, so that in particular $d_1(n) \equiv 1$ and $d(n) \equiv d_2(n)$. The Riemann zeta-function is

$$\zeta(s) := \sum_{n=1}^\infty n^{-s} = \prod_p (1-p^{-s})^{-1} \qquad (\Re e\, s > 1),$$

otherwise it is defined by analytic continuation. It is the simplest and most important example of the so-called class of $L$-functions $\sum_{n=1}^\infty f(n)n^{-s}$ satisfying certain natural properties. See e.g., J. Kaczorowski [37] for a survey of the *Selberg class* of $L$-functions generalizing $\zeta(s)$.

The function $d_k(n)$ is a also multiplicative function of $n$, meaning that $d_k(mn) = d_k(m)d_k(n)$ if $m$ and $n$ ($\in \mathbb{N}$) are coprime, and

$$d_k(p^\alpha) = (-1)^\alpha \binom{-k}{\alpha} = \frac{k(k+1)\cdots(k+\alpha-1)}{\alpha!}$$

for primes $p$ and $\alpha \in \mathbb{N}$.

## 2 Iterations of $d(n)$

From the wealth of problems involving the divisor function $d(m)$ we shall concentrate on some problems connected with the work of Paul Erdős (1913-1996), one of the greatest mathematicians of the XXth century. We begin with the iterations of $d(n)$. Thus let, for $k \in \mathbb{N}$ fixed,

$$d^{(1)}(n) := d(n),\ d^{(k)}(n) := d\left(d^{(k-1)}(n)\right) \quad (k > 1)$$

be the $k$-th iteration of $d(n)$. Already $d^{(2)}(n)$ **is not** multiplicative! This fact makes the problems involving $d^{(k)}(n)$ and iterates of other multiplicative functions quite difficult.

The great Indian mathematician S. Ramanujan (1887-1920) [40] proved in 1915 that (for the connection between Erdős and Ramanujan see [9])

$$d^{(2)}(n) > 4^{\sqrt{2\log n}/\log\log n}$$

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for infinitely many $n$. This lower bound follows if one considers ($p_j$ is the $j$-th prime)

$$N = 2^1 \cdot 3^2 \cdot 5^4 \cdot \ldots \cdot p_k^{p_k - 1} \tag{2.1}$$

and lets $k \to \infty$. Namely

$$d(N) = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_k, \quad d^{(2)}(N) = 2^k,$$

and one easily bounds $k$ from (2.1) by the prime number theorem. Ramanujan's paper [40] contains many other results on the divisor function $d(n)$

Important results on the order of $d^{(k)}(n)$ were obtained in 1967 by P. Erdős and I. Kátai [13]. Let $\ell_k$ denote the $k$-th Fibonacci number:

$$\ell_{-1} = 0,\ \ell_0 = 1,\ \ell_k = \ell_{k-1} + \ell_{k-2} \quad (k \geqslant 1).$$

Then the result of P. Erdős and I. Kátai says that

$$d^{(k)}(n) \ < \ \exp\Big((\log n)^{1/\ell_k + \varepsilon}\Big) \tag{2.2}$$

for fixed $k$ and $n \geqslant n_0(\varepsilon, k)$, and that for every $\varepsilon > 0$

$$d^{(k)}(n) \ > \ \exp\Big((\log n)^{1/\ell_k - \varepsilon}\Big) \tag{2.3}$$

for infinitely many $n$. Here and later $\varepsilon$ denotes arbitrarily small positive constants, not necessarily the same ones at each occurrence. The lower bound in (2.3) follows for $n = N_j$, where inductively $N_1 = 2 \cdot 3 \cdot \ldots \cdot p_r$, and if

$$N_j = \prod_{i=1}^{S_j} p_i^{r_i},$$

say, then

$$N_{j+1} = (p_1 \cdot \ldots \cdot p_{r_1})^{p_i - 1} (p_{r_i+1} \cdot \ldots \cdot p_{r_1+r_2})^{p_2 - 1}$$

$$\cdots (p_{r_1+\cdots r_{S_j-1}+1} \cdot \ldots \cdot p_{r_1+\cdots r_{S_j}})^{p_{S_j} - 1}.$$

Then one has $d^{(k)}(N_k) = 2^r$, and the proof reduces to finding the lower bound for $r$. The proof of the upper bound in (2.2) is more involved.

Improvements of (2.2) and (2.3) in the general case have been obtained by A. Smati [43]. Extensive work has been done in the case when $k = 2$. Thus in [11] P. Erdős and A. Ivić proved, for $n \geqslant n_0$ and suitable $C > 0$,

$$d^{(2)}(n) < \exp\left(C\left(\frac{\log n \log\log n}{\log\log\log n}\right)^{1/2}\right). \tag{2.4}$$

This follows from

$$\log d(n) = \sum_{i=1}^{r} \log(\alpha_i + 1) \ll r \log\log r = \omega(n) \log\log \omega(n),$$

and the bound ($\omega(n) = \sum_{p|n} 1$ is the number of distinct prime factors of $n > 1$, $\omega(1) = 0$)

$$\omega(d(n)) \ll \left(\frac{\log n \log\log n}{\log\log\log n}\right)^{1/2}.$$

For some other work of P. Erdős and A. Ivić on $d(n)$ see [12] and [5], the last paper bring a joint work with S.W. Graham and C. Pomerance.

A. Smati [42], [43] improved the upper bound in (2.4) to

$$d^{(2)}(n) < \exp\big(C\sqrt{\log n}\big) \qquad (C > 0, n \geqslant n_0), \tag{2.5}$$

which turned out to be only by a factor of $\log\log n$ (in the exponent) smaller than the true upper bound. Namely, in 2011 Y. Buttkewitz, C. Elsholtz, K. Ford and J.-C. Schlage-Puchta [3] practically settled the problem of the maximal order of $d^{(2)}(n)$ by proving that

$$\max_{n \leqslant x} \log d^{(2)}(n) \ = \ \frac{\sqrt{\log x}}{\log\log x}\left(D + O\left(\frac{\log\log\log x}{\log\log x}\right)\right),$$

where $D = 2.7958\ldots$ is an explicit constant. Note: We use throughout the paper the notation ($C$ denotes generic postive constants)

$$f(x) \ll g(x) \Longleftrightarrow f(x) = O(g(x)) \Longleftrightarrow |f(x)| \leqslant Cg(x) \quad (x \geqslant x_0).$$

R. Bellman and H.N. Shapiro [2] conjectured that, for fixed $k \geq 1$,

$$\sum_{n \leq x} d^{(k)}(n) = (1 + o(1)) c_k x \log_k x \qquad (x \to \infty), \tag{2.6}$$

where $\log_k$ is the $k$ times iterated natural logarithm. For $k = 1$ this is trivial, but for $k > 1$ it is a difficult problem. P. Erdős [8] and I. Kátai [34] obtained (2.6) for $k = 2$, while I. Kátai [35] proved it for $k = 3$. Finally Erdős and Kátai [14] proved it for $k = 4$, where the matter seems to stand at present.

Finally we mention a problem related to the iteration of $d(n)$. In 1992 the author [22] conjectured that

$$\sum_{n \leqslant x} d\big(n + d(n)\big) = Bx \log x + O(x) \quad (B > 0). \tag{2.7}$$

I. Kátai [36] obtained this formula with the error term $O(x \log x / \log\log x)$. He indicated that a formula analogous to (2.7) holds also for the summatory function of $d(n + f(n))$, where for example,

$$f(n) = \omega(n), \Omega(n) := \sum_{p^\alpha || n} \alpha, d_k(n).$$

## 3 P. Erdős's work on $d(n)$ in short intervals

From the rich legacy of P. Erdős concerning results and problems involving $d(n)$ we single out his classical paper

[7] (for some of his other papers involving $d(n)$, see [6] (with J.-L. Nicolas and A. Sárközy) and [10] and [15], the last two written jointly with R.R. Hall and L. Mirsky, respectively). He begins in [7] (we keep his German original): *$d(n)$ sei der Anzahl der Teiler von $n$. Folgende asymptotische Formel ist wohl-bekannt:*

$$\sum_{n=1}^{x} d(n) = x\log x + (2C-1)x + O(x^{\alpha}), \quad \alpha = 15/46 \tag{3.1}$$

*(C ist die Eulersche Konstante).*

Note that the function in the $O$-term is commonly denoted by $\Delta(x)$, thus

$$\Delta(x) := \sum_{n\leqslant x} d(n) - x(\log x + 2C - 1). \tag{3.2}$$

The constant 15/46 = 0.32608..., due to H.-E. Richert [41] (1952), can be replaced by M.N. Huxley's (2003) [19] value 131/416 = 0.31493... .

Erdős's theorem is as follows: *Es sei $h(x)$ eine beliebige wachsende Funktion, die mit $x$ gegen $\infty$ strebt. Es sei*

$$f(x) > (\log x)^{2\log 2-1}\exp\Big(h(x)\sqrt{\log\log x}\Big).$$

*Dann gilt für fast alle $x$*

$$\sum_{n\leqslant f(x)} d(x+n) = (1+o(1))\,f(x)\log x \qquad (x\to\infty). \tag{3.3}$$

*Diese Formel lässt sich nicht weiter verschärfen. Ist nämlich*

$$f(x) = (\log x)^{2\log 2-1}\exp\Big(c\sqrt{\log\log x}\Big) \qquad (c>0),$$

*so gilt (3.3) nicht mehr für fast alle $x$.*

It is commonly conjectured that the error term in (3.1) is $O_\varepsilon(x^{1/4+\varepsilon})$, while it is known long ago that it is $\Omega(x^{1/4})$ ($f(x) = \Omega(g(x))$ as $x\to\infty$ means that $f(x) = o(g(x))$ does not hold). The conjecture on the upper bound is one of the most difficult problems in analytic number theory, as it does not appear to follow from the Lindelöf hypothesis (LH, $\zeta(\frac12+it) \ll_\varepsilon |t|^\varepsilon$) or from the Riemann hypothesis (RH, all complex zeros of $\zeta(s)$ satisfy $\mathbb{R}e\, s = \frac12$). Both the LH and RH are unsettled to this day, and it is known that the RH implies the LH (see e.g., Chapter 1 of [21].

Let $D_k^+(n) = \max_{0\le h<k} d(n+h)$, where $d(n)$ is the number of divisors of $n$. P. Erdős and R.R. Hall [10] showed that, for fixed $k$,

$$\text{(3.4)} \qquad \sum_{n\le x} D_k^+(n) = k(1+o(1))x\log x \qquad (x\to\infty).$$

R.R. Hall conjectured that (3.4) is true so long as $k \le (\log x)^\alpha$ with $\alpha < \log 4 - 1$. He showed that (3.4) fails if $k \ge (\log x)^\alpha$ with $\alpha > \log 4 - 1$. R.R. Hall [16] showed later that in fact (3.4) is true for $k \le (\log x)^{\log 4-1}\exp\{-\xi(x)\sqrt{\log\log x}\}$ where $\xi(x)\to\infty$. Furthermore $\sum_{n\le x} D_k^+(n) = o(kx\log x)$ if

$$k > (\log x)^{\log 4-1}\exp\Big\{\xi(x)\sqrt{\log\log x}\Big\}.$$

Further results on this and on related topics were obtained by R.R. Hall and G. Tenenbaum [17].

## 4 The additive divisor problem

We turn now to modern developments involving the divisor function in short intervals. The importance of these results is that they have applications to power moments of $|\zeta(\frac12+it)|$, which is one of central topics in the theory of the Riemann zeta-function. The author [24] proved in 1997 the folllowing.

THEOREM 1. *For a fixed integer $k\ge 3$ and any fixed $\varepsilon>0$, we have*

$$\int_0^T |\zeta(\tfrac12+it)|^{2k}dt \ll_{k,\varepsilon} T^{1+\varepsilon}\Big(1+\sup_{T^{1+\varepsilon}<M\ll T^{k/2}} \frac{G_k(M;T)}{M}\Big), \tag{4.1}$$

*if, for $M < M' \le 2M, T^{1+\varepsilon} \le M \ll T^{k/2}$,*

$$G_k(M;T) := \sup_{\substack{M\le x\le M'\\ 1\le t\le M^{1+\varepsilon}/T}} \Big|\sum_{h\le t}\Delta_k(x,h)\Big|.$$

The bound (4.1) provides a direct link between upper bounds for the $2k$-th moment of $|\zeta(\frac12+it)|$ and sums of $\Delta_k(x,h)$ over the shift parameter $h$, showing also the limitations of the method, where $\Delta_k(x,h)$ denotes the error term in the asymptotic formula for the sum $\sum_{n\leqslant x} d_k(n)d_k(n+h)$. Of course the problem greatly increases in complexity as $k$ increases, and this is one of the reasons why in [23] only the case $k=3$ was considered. The case $k=2$ was not treated at all, since for the fourth moment of $|\zeta(\frac12+it)|$ we have an asymptotic formula with precise results for the corresponding error term (see e.g., Chapter 5 of [21] and the paper of Y. Motohashi and the author [28]).

As for the function $\Delta_k(x,h)$, one writes

$$\sum_{n\leqslant x} d_k(n)d_k(n+h) = xP_{2k-2}(\log x;h) + \Delta_k(x,h),$$

where it is assumed that $k\ge 2$ is a fixed integer, and $P_{2k-2}(\log x;h)$ is a suitable polynomial of degree $2k-2$ in $\log x$, whose coefficients depend on $k$ and $h$, while $\Delta_k(x,h)$ is supposed to be the error term. This means that we should have

$$\Delta_k(x,h) = o(x) \quad as \quad x\to\infty,$$

but unfortunately this is not yet known to hold for any $k\ge 3$, even for fixed $h$, while for $k=2$ there are many results.

This is the so-called *binary additive divisor problem* in the case when $k=2$, and the *general additive divisor problem* when $k>2$. The binary additive divisor problem consists of the evaluation of the sum

$$D(N;f) := \sum_{n\le N} d(n)d(n+f), \tag{4.2}$$

where $f$ is a natural number, not necessarily fixed. One can write

$$D(N;f) = M(N;f)+E(N;f), \tag{4.3}$$

where $M(N;f)$ and $E(N;f)$ are to be considered the "main term" and the "error term", respectively, in the asymptotic formula for the sum $D(N;f)$ in (4.2) as $N\to\infty$. Already A.E. Ingham [20] showed that the main term in (4.3) has the form

$$M(N;f) = \Big\{c_1(f)\log^2 N + c_2(f)\log N + c_3(f)\Big\}N,$$

where the coefficients $c_j(f)$ (which depend on $f$) can be written down explicitly. For some modern results on $E(N;f)$ we refer the reader to [39], [29] and [38].

It seems reasonable to expect that for the quantity $G_k$ in (4.1) we shall have a bound of the form

$$G_k \ll_{k,\varepsilon} T^{a_k+\varepsilon}M^{b_k+\varepsilon} \qquad (a_k\ge 0, b_k\ge 1) \tag{4.4}$$

with suitable constants $a_k, b_k$. Hence assuming (4.4) it follows that

$$G_k M^{-1} \ll T^{a_k+\varepsilon}M^{(b_k-1+\varepsilon)} \ll T^{a_k+\frac{k}{2}(b_k-1)+\varepsilon}$$

for $M \ll T^{k/2}$. Therefore from Theorem 1 we obtain.

**Corollary 1**. *If (4.4) holds, then for a fixed integer $k\ge 3$ we have*

$$\int_0^T |\zeta(\tfrac12+it)|^{2k}\,dt \ll_{k,\varepsilon} T^{1+\varepsilon}\left(1+T^{a_k+\frac{k}{2}(b_k-1)}\right). \tag{4.5}$$

The case $k=3$ was investigated by the author in [24]. Therein it was conjectured that

$$\sum_{h\le H}\Delta_3(x,h) \ll_\varepsilon Hx^{\frac23+\varepsilon} \qquad (1\le H\le x^{\frac13+\delta}) \tag{4.6}$$

for some $\delta>0$. If $k=3$ we have $T^{1+\varepsilon}\le M \ll T^{3/2}$,and

$$G_3 = \sup_{M\le x\le M', 1\le t\le M^{1+\varepsilon}/T}\Big|\sum_{h\le t}\Delta_3(x,h)\Big|.$$

Moreover $t\le x^{\frac13+\delta}$ is satisfied, since

$$t \le \frac{M^{1+\varepsilon}}{T} \le M^{\frac13+\delta}$$

because $M^{\frac23+\varepsilon-\delta}\le T$ for $\varepsilon<\delta$ and sufficiently large $T$. Thus if (4.6) holds we have

$$G_3M^{-1} \ll \sup_{M\le x\le 2M, 1\le t\le M^{1+\varepsilon}/T} tx^{\frac23+\varepsilon}M^{-1}$$

$$\ll M^{1+\varepsilon}T^{-1}M^{\frac23+\varepsilon}M^{-1} \ll T^{\varepsilon},$$

namely for $k=3$ we have $a_3=0, b_3=1$ in (4.4). Hence we obtain from (4.5)

$$\int_0^T |\zeta(\tfrac12+it)|^6\,dt \ll_{k,\varepsilon} T^{1+\varepsilon},$$

which was already shown to hold in [24] if (5.10) is assumed. We also havefrom (4.5)

**Corollary 2**. *For a fixed integer $k\ge 3$ we have*

$$\int_0^T |\zeta(\tfrac12+it)|^{2k}\,dt \ll_{k,\varepsilon} T^{1+\varepsilon}. \tag{4.7}$$

*provided that (4.4) holds with $a_k=0, b_k=1$.*

The bounds (4.5) and (4.7) provide the means to bound the $2k$-th moment of $|\zeta(\frac12+it)|$. It remains to be seen, of course, whether there is any hope of proving the $2k$–th moment for $k\ge 4$ by (4.7), namely whether $a_k=0, b_k=1$ can hold at all for sufficiently large $k$. It would be highly interesting if one could even get any non-trivial results concerning $a_k, b_k$ and improve unconditionally the existing bounds (see Chapters 7 and 8 of [21] for the moments of $|\zeta(\frac12+it)|$. For more connections between divisor problems and power moments of $|\zeta(\frac12+it)|$, see e.g. the author's papers [25] and [26].

## 5 New bounds for the sums $\Delta_k(x,h)$ over the shift parameter $h$

S. Baier, T.D. Browning, G. Marasingha and L. Zhao [1] recently proved that

$$\sum_{h\le H}\Delta_3(N;h) \ll_\varepsilon N^\varepsilon\left(H^2+H^{1/2}N^{13/12}\right) \qquad (1\le H\le N), \tag{5.1}$$

$$\Delta_3(N;h) = \sum_{N<n\leqslant 2N} d_3(n)d_3(n+h) - NP_4(\log N;h),$$

and if $N^{1/3+\varepsilon}\le H\le N^{1-\varepsilon}$, then there exists $\delta=\delta(\varepsilon)>0$ for which

$$\sum_{h\le H}|\Delta_3(N;h)|^2 \ll_\varepsilon HN^{2-\delta(\varepsilon)}.$$

Note that (5.1), in the interval $N^{1/6+\varepsilon}\le H\le N^{1-\varepsilon}$, gives an asymptotic formula for the averaged sum $\sum_{h\le H} D_3(N,h)$.

Jie Wu and the author [30] proved the following:

THEOREM 2. *For a fixed integer $k\ge 3$ we have*

(5.2)

$$\sum_{h\le H}\Delta_k(N;h) \ll_\varepsilon N^\varepsilon\left(H^2+N^{1+\beta_k}\right) \qquad (1\le H\le N),$$

*where $\beta_k$ is defined by*

$$\beta_k := \inf\left\{ b_k \;:\; \int_1^X |\Delta_k(x)|^2 \dot{x} \ll x^{1+2b_k}\right\}$$

*and $\Delta_k(x)$ is the remainder term in the asymptotic formula for $\sum_{n\le x} d_k(n)$.*

We have (see e.g., Chapter 13 of [21])

$$\sum_{n\le x} d_k(n) = xp_{k-1}(\log x) + \Delta_k(x),$$

where

$$p_{k-1}(\log x) = \operatorname*{Res}_{s=1}\left(\zeta^k(s)\frac{x^{s-1}}{s}\right),$$

so that $p_{k-1}(z)$ is a polynomial of degree $k-1$ in $z$, all of whose coefficients depend on $k$. In particular,

$$p_1(z) \quad = \quad z+2\gamma-1 \qquad (\gamma = -\Gamma'(1)).$$

It is known that $\beta_k = (k-1)/(2k)$ for $k = 2,3,4$, $\beta_5 \le 9/20, \beta_6 \le 1/2$, etc. and $\beta_k \ge (k-1)/(2k)$ for every $k \in \mathbb{N}$. It is conjectured that $\beta_k = (k-1)/(2k)$ for every $k \in \mathbb{N}$, and this is equivalent to the Lindelöf Hypothesis (that $\zeta(\frac{1}{2}+it) \ll_\varepsilon (|t|+1)^\varepsilon$).From the Theorem 1 we obtain, for $1 \le H \le N$,

$$\sum_{h\le H} \Delta_3(N;h) \ll_\varepsilon N^\varepsilon\left(H^2+N^{4/3}\right),$$

$$\sum_{h\le H} \Delta_4(N;h) \ll_\varepsilon N^\varepsilon\left(H^2+N^{11/8}\right),$$

$$\sum_{h\le H} \Delta_5(N;h) \ll_\varepsilon N^\varepsilon\left(H^2+N^{29/20}\right),$$

$$\sum_{h\le H} \Delta_6(N;h) \ll_\varepsilon N^\varepsilon\left(H^2+N^{3/2}\right).$$

Since it is known that $\beta_k < 1$ for any $k$, this means that the bound in (5.2) improves on the trivial bound $HN^{1+\varepsilon}$ in the range $N^{\beta_k+\varepsilon} \le H \le N^{1-\varepsilon}$. Our result thus supports the assertion that $\Delta_k(N;h)$ is really the error term in the asymptotic formula for $D_k(N,h)$, as given above. In the case when $k=3$, we have an improvement on the result of Baier et al. when $H \ge N^{1/2}$.

The basic idea of proof is to start from

$$\sum_{h\le H} \Delta_k(N,h)$$

$$= \sum_{N<n\le 2N} d_k(n)\sum_{h\le H} d_k(n+h) - \sum_{h\le H}\int_N^{2N}\mathfrak{S}_k(x,h)\,dx$$

$$= M_k(N,H) + R_k(N,H) - \sum_{h\le H}\int_N^{2N}\mathfrak{S}_k(x,h)\,dx,$$

say. Here

$$\mathfrak{S}_k(x,h) := \sum_{q=1}^{\infty}\frac{c_q(h)}{q^2}Q_k(x,q)^2,$$

where $\mu(n)$ is the Möbius function, $c_q(h) := \sum_{d|(h,q)} d\mu(q/d)$ is the Ramanujan sum and $Q_k(x,q)$ is a polynomial of degree $2k-2$ whose coefficients depend on $q$, and may be explicitly evaluated (see e.g., [1]). Further we set

$$M_k(N,H) := \sum_{N<n\le 2N} d_k(n)\operatorname*{Res}_{s=1}\left(\zeta(s)^k\frac{(n+H)^s-n^s}{s}\right),$$

$$R_k(N,H) := \sum_{N<n\le 2N} d_k(n)\big(\Delta_k(n+H)-\Delta_k(n)\big),$$

and use complex integration to estimate $M_k(N,h)$ and then connect $R_k(N,H)$ to mean square estimates for $\Delta_k(x)$. We have

$$M_k(N,H) = H\int_N^{2N}\left(\operatorname*{Res}_{s=1}\zeta(s)^k x^{s-1}\right)^2 dx$$

$$+O_\varepsilon\left(H^2N^\varepsilon + NH^{\alpha_k+\varepsilon} + N^{1+\beta_k+\varepsilon}\right)$$

and

$$\sum_{h\le H}\int_N^{2N}\mathfrak{S}_k(x,h)\,dx$$

The constants $\alpha_k, \beta_k$ are defined as

$$\alpha_k = \inf\Big\{\ a_k\ :\ \Delta_k(x) \ll a_k\ \Big\}$$

and

$$\beta_k = \inf\Big\{\ b_k\ :\ \int_1^X \Delta_k^2(x)\,dx \ll X^{1+2b_k}\ \Big\},$$

and $(k-1)/(2k) \leqslant \beta_k \leqslant \alpha_k < 1$ for $k = 2,\ldots$. By completing the estimations one obtains the assertion of the theorem.

## 6 New results involving $\Delta(x+U)-\Delta(x)$

The final topic will be a discussion (see (3.2)) of

$$\Delta(x+U)-\Delta(x) = \sum_{x<n\le x+U} d(n) + O(Ux^\varepsilon) \quad (1 \ll U \le x), \tag{6.1}$$

so that we are considering the divisor function $d(n)$ in "short intervals" $[x,x+U]$ if $U = o(x)$ as $x\to\infty$. The interest in this topic comes from the work of M. Jutila [32], who proved that

$$\int_T^{T+H}\big(\Delta(x+U)-\Delta(x)\big)^2 dx =$$

$$\frac{1}{4\pi^2}\sum_{n\le\frac{T}{2U}}\frac{d^2(n)}{n^{3/2}}\int_T^{T+H} x^{1/2}\left|\exp\left(2\pi i U\sqrt{\frac{n}{x}}\right)-1\right|^2 dx$$

$$+O_\varepsilon(T^{1+\varepsilon}+HU^{1/2}T^\varepsilon), \tag{6.2}$$

for $1\le U\ll T^{1/2}\ll H\le T$. From (6.2) one deduces ($a\asymp b$ means $a\ll b\ll a$)

$$\int_T^{T+H}\Big(\Delta(x+U)-\Delta(x)\Big)^2dx\asymp HU\log^3\left(\frac{\sqrt{T}}{U}\right) \quad (6.3)$$

for $HU\gg T^{1+\varepsilon}$ and $T^\varepsilon\ll U\le\frac12\sqrt{T}$. In [33] Jutila proved that the integral in (6.3) is

$$\ll_\varepsilon T^\varepsilon(HU+T^{2/3}U^{4/3}) \qquad (1\ll H,U\ll X).$$

In the case when $H=T$ the author [27] improved (6.2) and proved.

THEOREM 3. *For* $1\ll U=U(T)\le\frac12\sqrt{T}$ *we have* ($c_3=8\pi^{-2}$)

$$\int_T^{T+H}\Big(\Delta(x+U)-\Delta(x)\Big)^2dx=$$

$$TU\sum_{j=0}^3 c_j\log^j\left(\frac{\sqrt{T}}{U}\right)+O_\varepsilon(T^{1/2+\varepsilon}U^2)+O_\varepsilon(T^{1+\varepsilon}U^{1/2}). \quad (6.4)$$

In (6.4) all the constants $c_j$ may be made explicit. Note that, for $T^\varepsilon\le U=U(T)\le T^{1/2-\varepsilon}$ (6.4) is a true asymptotic formula. From (6.4) one can deduce that, for $1\ll U\le\frac12\sqrt{T}$, we have ($c_3=8\pi^{-2}$)

$$\sum_{T\le n\le 2T}\Big(\Delta(n+U)-\Delta(n)\Big)^2=$$

$$TU\sum_{j=0}^3 c_j\log^j\left(\frac{\sqrt{T}}{U}\right)+O_\varepsilon(T^{1/2+\varepsilon}U^2)+O_\varepsilon(T^{1+\varepsilon}U^{1/2}). \quad (6.5)$$

The asymptotic formula (6.5) is a considerable improvement over a result of Coppola–Salerno [4], who had shown that ($T^\varepsilon\le U\le\frac12\sqrt{T}, L=\log T$) $\sum_{T\le n\le 2T}\Big(\Delta(n+U)-\Delta(n)\Big)^2=\frac{8}{\pi^2}TU\log^3\left(\frac{\sqrt{T}}{U}\right)+O(TUL^{5/2})$.

The starting point for the above results is the explicit expression (see e.g., Chapter 3 of [21])

$$\Delta(x)=\frac{1}{\pi\sqrt2}x^{\frac14}\sum_{n\le N}d(n)n^{-\frac34}\cos(4\pi\sqrt{nx}-\frac14\pi)$$

$$+O_\varepsilon(x^{\frac12+\varepsilon}N^{-\frac12}) \quad (2\le N\ll x),$$

which is flexible, since the parameter $N$ may be arbitrarily chosen. It is sometimes called the *truncated Voronoï formula*, in honour of G.F. Voronoï [44], who more than a century ago obtained an explicit formula for $\Delta(x)$ containing the familiar Bessel functions.

The most recent results on $\Delta(x+U)-\Delta(x)$ have been obtained by the author and W. Zhai [31]. We state just two of their theorems.

THEOREM 4. *Suppose* $\log^2T\ll U\leqslant T^{1/2}/2, T^{1/2}\ll H\leqslant T$, *then we have*

$$\int_T^{T+H}\max_{0\leqslant u\leqslant U}\Big|\Delta(x+u)-\Delta(x)\Big|^2dx\ll$$

$$HUL^5+TL^4\log L+H^{1/3}T^{2/3}U^{2/3}L^{10/3}(\log L)^{2/3},$$

where $L:=\log T$.

This generalizes and sharpens a result of D.R. Heath-Brown & K.-M. Tsang (1994) [18]. From Theorem 3 we obtain then.

THEOREM 5. *Suppose* $T,U,H$ *are large parameters and* $C>1$ *is a large constant such that*

$$T^{131/416+\varepsilon}\ll U\leqslant C^{-1}T^{1/2}L^{-5},\quad CT^{1/4}UL^5\log L\leqslant H\le T.$$

*Then in the interval* $[T,T+H]$ *there are* $\gg HU^{-1}$ *subintervals of length* $\gg U$ *such that on each subinterval one has* $\pm\Delta(x)\ge c_\pm T^{1/4}$ *for some* $c_\pm>0$.

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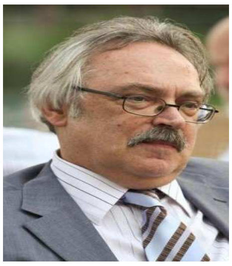

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