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On a Ramanujan Quantity

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Abstract: In this paper, We study the several modular equations of Ramanujan Quantities $R(1,2,4;q)$ (established by Nikos Bagis) and $R(1,2,4;q^n)$ for $n = 4,6,8,9,10,11,13,14,15,17,19,23$ *and* 25.

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1 Introduction

In Chapter 16 of his second notebook [1], Ramanujan develops the theory of theta-function and is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, |\, ab \,| < 1, \qquad (1)$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

where $(a;q)_0 = 1$ and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

Following Ramanujan, we defined

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \qquad (2)$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \qquad (3)$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \qquad (4)$$

and

$$\chi(q) := (-q;q^2)_\infty. \qquad (5)$$

In [3],[4] Nikos Bagis define Ramanujan Quantities $R(a,b,p;q)$ as

$$R(a,b,p;q) = q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{\prod_{n=0}^{\infty}(1-q^a q^{np})(1-q^{p-a}q^{np})}{\prod_{n=0}^{\infty}(1-q^b q^{np})(1-q^{p-b}q^{np})}, \qquad (6)$$

where a, b,and p are positive rationales such that $a+b<p$. General Theorem such

$$\frac{q^{B-A}}{1-a_1b_1} + \frac{(a_1-b_1q_1)(b_1-a_1q_1)}{(1-a_1b_1)(q_1^2+1)} + \frac{(a_1-b_1q_1^3)(b_1-a_1q_1^3)}{(1-a_1b_1)(q_1^4+1)} + \cdots$$

$$= \frac{\prod_{n=0}^{\infty}(1-q^a q^{np})(1-q^{p-a}q^{np})}{\prod_{n=0}^{\infty}(1-q^b q^{np})(1-q^{p-b}q^{np})} \qquad (7)$$

where $a_1 = q^A$, $b_1 = q^B$, $q_1 = q^{A+B}$, $a = 2A+3p/4$, $2B+p/4$, and $p = 4(A+B)$, $|q|<1$, are proved.

Now we define a modular equation in brief. The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $|\, x \,| < 1$.
Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right) \qquad (8)$$

and

$$q := q(x) := \exp\left(-\pi\,\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right), \qquad (9)$$

where $0<x<1$.
Let r denote a fixed natural number and assume that the

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following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}. \qquad (10)$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (10). We often say that β is of degree r over α and $m:=\dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

In this paper, we obtain several new modular relation between $R(1,2,4;q)$ and $R(1,2,4;q^n)$ for $n = 4,6,8,9,10,11,13,14,15,17,19,23$ *and* 25.

2 Preliminary results

Definition 1[4]

$$[a,p;q]=(q^{p-a};q^p)_\infty(q^a;q^p)_\infty \qquad (11)$$

where $q=e^{-\pi\sqrt{r}}$ *and* $a,p,r>0$.

Definition 2[4]

$$R(a,b,p;q):=q^{-(a-b)/2+(a^2-b^2)/(2p)}\frac{[a,p;q]}{[b,p;q]} \qquad (12)$$

Conjecture 3[4] *If* $x=R(1,2,4;q)$ *and* $y=R(1,2,4;q^2)$, *then*

$$x^4-y^2+4x^4y^4=0 \qquad (13)$$

Conjecture 4[4] *If* $x=R(1,2,4;q)$ *and* $y=R(1,2,4;q^3)$, *then*

$$x^4-xy+4x^3y^3-y^4=0 \qquad (14)$$

Conjecture 5[4] *If* $x=R(1,2,4;q)$ *and* $y=R(1,2,4;q^5)$, *then*

$$x^6-xy+5x^4y^2-5x^2y^4+16x^5y^5-y^6=0 \qquad (15)$$

Conjecture 6[4] *If* $x=R(1,2,4;q)$ *and* $y=R(1,2,4;q^7)$, *then*

$$\begin{aligned}&x^8-xy+7x^2y^2-28x^3y^3+70x^4y^4-112x^5y^5\\&+112x^6y^6-64x^6y^6-64x^7y^7+y^8=0\end{aligned} \qquad (16)$$

Lemma 1.[1, Ch. 17, Entry 10-11, p.122-123]

$$\varphi(-q^2)=\sqrt{z}(1-\alpha)^{1/8} \qquad (17)$$

$$\psi(-q)=\sqrt{\frac{1}{2}z}\{\alpha(1-\alpha)q^{-1}\}^{1/8} \qquad (18)$$

where $q=e^{-y}$

Lemma 2.[2, Entry 17.3.2, p.385] *If* β *has degree 4 over* α, *then*

$$(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta})=2\sqrt[4]{\beta(1-\alpha)}. \qquad (19)$$

Lemma 3.[1] *If* β *has degree 8 over* α, *then*

$$(1-(1-\alpha)^{1/4})(1-\beta^{1/4})=2\sqrt{2}(\beta(1-\alpha))^{1/8}. \qquad (20)$$

Lemma 4.[1, Entry 3(x),(xi), p.352] *If* β *has degree 9 over* α, *then*

$$\left(\frac{\beta}{\alpha}\right)^{1/8}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/8}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}=\sqrt{m}. \qquad (21)$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}=\frac{3}{\sqrt{m}}. \qquad (22)$$

Lemma 5.[1, Entry 7, p.363] *If* β *has degree 11 over* α, *then*

$$(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}+2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}=1. \qquad (23)$$

Lemma 6.[1, Entry 63, p.387] *If* β *has degree 13 over* α, *then*

$$\sqrt{U}(U^3+8W)=\sqrt{W}(11U^2+V). \qquad (24)$$

where

$U=1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}$,

$V=64\left(\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}-\sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right)$,

and $W=32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}$.

Lemma 7.[1, Entry 21, p.435] *Let* α *and* β *has degrees 3, 5 or 1, 15 respectively, then*

$$\begin{aligned}&(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}\pm\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\\&=\left\{\frac{1}{2}\left(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right)\right\}^{1/2},\end{aligned} \qquad (25)$$

where the minus sign is chosen in the first case and the plus sign is selected in the second case.

Lemma 8.[1, Entry 17.3.26, p.392] *If* β *has degree 17 over* α, *then*

$$\begin{aligned}m=&\left(\frac{\beta}{\alpha}\right)^{1/4}+\left(\frac{(1-\beta)}{(1-\alpha)}\right)^{1/4}+\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}\\&-2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left\{1+\left(\frac{\beta}{\alpha}\right)^{1/8}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/8}\right\}.\end{aligned} \qquad (26)$$

$$\begin{aligned}\frac{17}{m}=&\left(\frac{\alpha}{\beta}\right)^{1/4}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/4}+\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}\\&-2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}\left\{1+\left(\frac{\alpha}{\beta}\right)^{1/8}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}\right\}.\end{aligned} \qquad (27)$$

Lemma 9. *[1, Entry 58, p.386] If β has degree 19 over α, then*

$$X^5 - 7X^2 Z - YZ = 0. \tag{28}$$

where

$$X = 1 - \sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)},$$
$$Y = 16\left(\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} - \sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}\right),$$
$$and Z = 16\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}.$$

Lemma 10. *[1, Entry 15, p.411] If β has degree 23 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1. \tag{29}$$

Lemma 11. *[1, Entry 17.3.27, p.392] If β has degree 25 over α, then*

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} = \sqrt{m}. \tag{30}$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12} = \frac{5}{\sqrt{m}}. \tag{31}$$

3 Modular relations Ramanujan Quantities of $R(1,2,4;q)$

In this section, we obtain certain modular relations between $R(1,2,4;q)$ and $R(1,2,4;q^n)$.

Theorem 7. *If $u := R(1,2,4;q)$ and $v := R(1,2,4;q^4)$ then*

$$u^8 - v^2 + 8v^2u^8 + 24v^4u^8 - 4v^6 + 32v^6u^8 + 16v^8u^8 = 0. \tag{32}$$

Proof. Employing Definition (1) and (2) with $a = 1, b = 2$ *and* $p = 4$, we get

$$R(1,2,4;q) = q^{1/8}\frac{(q;q^4)_\infty(q^3;q^4)_\infty}{(q^2;q^4)_\infty(q^2;q^4)_\infty} \tag{33}$$

Using the equations (1), (2) and (3), then the above equation can be written as

$$R(1,2,4;q) = q^{1/8}\frac{f(-q,-q^3)}{f(-q^2,-q^2)} = q^{1/8}\frac{\psi(-q)}{\varphi(-q^2)} \tag{34}$$

employing the equations (17) and (18), we obtain

$$q^{1/8}\frac{\psi(-q)}{\varphi(-q^2)} = \frac{\alpha^{1/8}}{\sqrt{2}} \tag{35}$$

using the above two equation (34) and (35), we get If $0 < \alpha < 1$, then

$$\alpha^{1/8} = \sqrt{2}R(1,2,4;q) \tag{36}$$

employing the above equation (36) with the lemma (2), we obtain (32).

Theorem 8. *If $u := R(1,2,4;q)$ and $v := R(1,2,4;q^6)$ then*

$$\begin{aligned}&u^{16} + 96v^{12}u^8 + 48v^8u^8 - v^2u^4 + 256u^{16}v^{16} + 96v^8u^{16}\\&+ v^8 + 256v^{12}u^{16} - 448v^{10}u^{12} - 112v^6u^{12} + 6v^4u^8\\&+ 12v^2u^{12} + 16v^4u^{16} + 768u^{12}v^{14} - 64u^4v^{14} = 0.\end{aligned} \tag{37}$$

Proof. Using the Conjecture (3), we obtain

$$y = \sqrt{\frac{1-16x^8}{8x^4}} \tag{38}$$

Replace q to q^2 in the equation (4) and using the above equation (38), we arrive at the equation (37).

Theorem 9. *If $u := R(1,2,4;q)$ and $v := R(1,2,4;q^8)$ then*

$$\begin{aligned}&u^{16} - 28v^6 - v^2 - 64v^8 - 8v^4 - 112v^{10} - 128v^{12} - 64v^{14} + 448v^6u^8\\&+ 16v^2u^8 + 1024v^8u^8 + 128v^4u^8 + 1792v^{10}u^8 + 2048v^{12}u^8\\&+ 1024v^{14}u^8 - 16v^2u^{16} + 1120v^8u^{16} - 448v^6u^{16} + 112v^4u^{16}\\&- 1792v^{10}u^{16} + 256v^{16}u^{16} - 1024v^{14}u^{16} + 1792v^{12}u^{16} = 0.\end{aligned} \tag{39}$$

Proof. Using the lemma (3) with the equation (36), we obtain (39)

Theorem 10. *If $u := R(1,2,4;q)$ and $v := R(1,2,4;q^9)$ then*

$$\begin{aligned}&u^{12} - vu^3 + 8vu^{11} + v^2u^2 + 10v^2u^{10} - v^3u - 24v^3u^9 + 15v^4u^8\\&+ 48u^7v^5 - 84u^6v^6 + 48u^5v^7 + 15u^4v^8 - 24v^9u^3 - 256v^9u^{11}\\&+ 10v^{10}u^2 + 256v^{10}u^{10} + 8v^{11}u - 256v^{11}u^9 + v^{12} = 0.\end{aligned} \tag{40}$$

Proof. Using the lemma (4) with the equation (36), we get

$$\begin{aligned}&(u^{12} - vu^3 + 8vu^{11} + v^2u^2 + 10v^2u^{10} - v^3u - 24v^3u^9\\&+ 15u^8v^4 + 48u^7v^5 - 84u^6v^6 + 48u^5v^7 + 15u^4v^8 + v^{12}\\&- 24v^9u^3 - 256v^9u^{11} + 10v^{10}u^2 + 256v^{10}u^{10} + 8v^{11}u\\&- 256v^{11}u^9)(u^2+v^2)^2(u^4+v^4)^2 = 0\end{aligned} \tag{41}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem first factor vanishes identically. This completes the proof.

Theorem 11. *If* $u := R(1,2,4;q)$ *and* $v := R(1,2,4;q^{10})$ *then*

$$\begin{aligned}&1400v^{12}u^8 - v^2u^4 + 10v^{10}u^4 + 240v^8u^{16} + 655v^4u^{16} + 15v^8u^8\\&- 40v^4u^8 - 19040v^{12}u^{16} + 3840v^{16}u^{16} + u^{24} - 15296v^{10}u^{20}\\&- 260v^6u^{12} - 70v^2u^{20} + 400v^{10}u^{12} + 1280v^{12}u^{24} + 4096u^{24}v^{24}\\&+ 240v^8u^{24} + 24v^4u^{24} + 3720v^6u^{20} + 3840v^{16}u^{24} + 20v^2u^{12}\\&- 61184v^{14}u^{20} + 6144v^{20}u^{24} + 167680v^{20}u^{16} - 10240v^{20}u^8\\&+ 1600v^{14}u^{12} + 240u^8v^{16} - 16640u^{12}v^{18} - 1024u^4v^{22} + v^{12}\\&+ 20480u^{12}v^{22} - 71680u^{20}v^{22} + 238080v^{18}u^{20} + 40u^4v^{14} = 0.\end{aligned} \tag{42}$$

Proof. Replace q to q^2 in the equation (15) and using the above equation (38), we arrive at the equation (42).

Theorem 12. *If* $u := R(1,2,4;q)$ *and* $v := R(1,2,4;q^{11})$ *then*

$$\begin{aligned}&165v^4u^8 - 44u^{11}v^3 - 11u^3v^3 - 165u^4v^8 - 66u^5v^5 - 11v^6u^2\\&- uv + 704u^9v^9 + 11u^9v + 11uv^9 + 176v^6u^{10} - 44u^3v^{11} - v^{12}\\&+ 264u^7v^7 + 11u^6v^2 + u^{12} - 176v^{10}u^6 + 1024u^{11}v^{11} = 0.\end{aligned} \tag{43}$$

Proof. Using the lemma (5) with the equation (36), we get

$$\begin{aligned}&(-90112u^{14}v^{22} + 1488u^6v^{22} + u^2v^2 + 1488u^{22}v^6 + 241359u^{16}v^8\\&- 238u^4v^{20} - 238u^{20}v^4 + 947u^{12}v^4 - 3853824u^{16}v^{16} + v^{24} + u^{24}\\&+ 947u^4v^{12} + 242432u^{20}v^{12} - 29380u^{12}v^{12} - 3801088u^{20}v^{20}\\&+ 413152u^{14}v^{14} + 6533120u^{18}v^{18} + 1048576u^{22}v^{22} - 58u^4v^4\\&+ 1595u^6v^6 + 25822u^{10}v^{10} - 25668u^6v^{14} - 25668u^{14}v^6\\&- 15054u^8v^8 + 241359u^8v^{16} + 242432u^{12}v^{20} - 90112u^{22}v^{14}\\&- 22u^2v^{10} + 93u^2v^{18} - 22u^{10}v^2 - 410688u^{10}v^{18} + 93u^{18}v^2\\&- 410688u^{18}v^{10})(44u^{11}v^3 + 165v^4u^8 + 11u^3v^3 - 165v^8u^4\\&+ 66u^5v^5 + uv - 704u^9v^9 - 11u^9v - 11uv^9 + 176v^6u^{10}\\&- 264u^7v^7 - 11v^6u^2 + 11u^6v^2 + u^{12} - 176v^{10}u^6 - v^{12}\\&- 1024u^{11}v^{11} + 44u^3v^{11})(-44u^{11}v^3 + 165v^4u^8 - 11u^3v^3\\&- 66u^5v^5 - uv + 704u^9v^9 + 11u^9v + 11uv^9 + 176v^6u^{10}\\&- 44u^3v^{11} + 264u^7v^7 - 11v^6u^2 + 11u^6v^2 + u^{12} - 176v^{10}u^6\\&- v^{12} + 1024u^{11}v^{11} - 165v^8u^4) = 0\end{aligned} \tag{44}$$

By the Identity Theorem third factor vanishes identically. This completes the proof.

Theorem 13. *If* $u := R(1,2,4;q)$ *and* $v := R(1,2,4;q^{13})$ *then*

$$\begin{aligned}&13uv^9 - 429v^8u^6 - 52u^3v^7 + 832u^{11}v^7 - uv + 429u^{10}v^4 - 208u^5v^{13}\\&+ 4096u^{13}v^{13} - 130u^5v^5 - 52u^7v^3 + 429v^6u^8 - v^{14} + 832u^7v^{11} + u^{14}\\&+ 13u^9v + 65v^2u^{12} + 2080u^9v^9 - 429v^{10}u^4 - 65v^{12}u^2 - 208u^{13}v^5 = 0.\end{aligned} \tag{45}$$

Proof. Using the lemma (6) with the equation (36), we obtain (45).

Theorem 14. *If* $u := R(1,2,4;q)$ *and* $v := R(1,2,4;q^{14})$ *then*

$$\begin{aligned}&13468v^4u^{24} - 816256v^{14}u^{20} + 853888v^{10}u^{28} + 5894v^8u^{16} + 32v^4u^{32}\\&+ 39088v^6u^{20} + 140v^{12}u^8 - 208320v^{12}u^{24} + 4587520u^{28}v^{30} + u^{32}\\&+ 54648832v^{22}u^{28} - 896v^8u^8 - 426496u^{12}v^{22} - 229376u^8v^{24}\\&+ 28v^2u^{12} - 728v^4u^{16} + 12768v^{12}u^{16} + 136864v^8u^{24} - 210v^2u^{20}\\&- 2268v^6u^{12} + 47488v^{10}u^{20} + 458752u^{12}v^{30} + 35037184v^{24}u^{24}\\&- 2731520v^{14}u^{28} - v^2u^4 + 448v^8u^{32} - 45920v^6u^{28} + 280v^2u^{28}\\&+ 60256v^{14}u^{12} + 3584v^{12}u^{32} + 55164928v^{28}u^{24} - 2981888v^{28}u^{16}\\&- 47022080v^{26}u^{28} + 114688v^{24}u^{32} - 10926080v^{18}u^{28} + 896u^4v^{22}\\&+ 3039232v^{22}u^{20} - 3333120v^{20}u^{24} - 28672v^{28}u^8 - 2322432v^{26}u^{12}\\&+ 1508864v^{24}u^{16} + 57344v^{20}u^{32} + 131072v^{28}u^{32} - 179648v^{16}u^{16}\\&+ 241024v^{18}u^{12} + 2240v^{20}u^8 + 40026112v^{26}u^{20} - 4322304v^{16}u^{24}\\&+ 204288v^{20}u^{16} - 3265024v^{18}u^{20} - 16384u^4v^{30} + 28896u^8v^{16}\\&- 3440640u^{20}v^{30} - 7v^4u^8 + 14v^{10}u^4 + 17920v^{16}u^{32} + v^{16}\\&+ 65536u^{32}v^{32} - 6664v^{10}u^{12} = 0.\end{aligned} \tag{46}$$

Proof. Replace q to q^2 in the equation (16) and using the above equation (38), we arrive at the equation (46).

Theorem 15. *If* $u := R(1,2,4;q)$ *and* $v := R(1,2,4;q^{15})$ *then*

$$\begin{aligned}&u^{24} - 15v^{10}u^2 + 1260v^{14}u^6 - 245v^{12}u^4 + 6240u^{13}v^{13} - 21280u^{14}v^{14}\\&+ 8399u^8v^{16} + 8399u^{16}v^8 - 130560u^{16}v^{16} + 4080u^7v^{15} + 4080u^{15}v^7\\&- 55680u^{15}v^{15} - 15u^{10}v^2 + 1260u^{14}v^6 - 245u^{12}v^4 + 9660u^{12}v^{12}\\&+ 1560u^{11}v^{11} - 870u^9v^9 - 1330u^{10}v^{10} - 280u^7v^7 + v^{24} - 15u^5v^5\\&+ 245v^{18}u^2 + 245u^{18}v^2 + 5776u^{21}v^5 + 960u^{23}v^7 + 1444u^{19}v^3 - uv^9\\&+ 20160u^{18}v^{10} + 1444v^{19}u^3 + 5776v^{21}u^5 + 960v^{23}u^7 + 16320u^{17}v^9\\&+ 4050u^{20}v^4 + 3920u^{22}v^6 - 61440u^{22}v^{14} - 92160u^{21}v^{13} - 510u^8v^8\\&- 62720u^{20}v^{12} + 4050v^{20}u^4 - 16384u^{23}v^{15} - 245760u^{19}v^{19} - u^9v\\&- 61440v^{22}u^{14} - 92160v^{21}u^{13} - 6400v^{19}u^{11} - 62720v^{20}u^{12} - 91u^6v^6\\&- 65536v^{20}u^{20} - 372736v^{18}u^{18} - 16384v^{23}u^{15} + 15uv^{17} + 15u^{17}v\\&+ 3920v^{22}u^6 - u^4v^4 - 90u^{11}v^3 - 90u^3v^{11} - 100u^{13}v^5 - 100u^5v^{13}\\&- 286720u^{17}v^{17} + 20160v^{18}u^{10} + 16320v^{17}u^9 - 6400u^{19}v^{11} = 0.\end{aligned} \tag{47}$$

Proof. Using the lemma (7) with the equation (36), we obtain (47).

Theorem 16. *If* $u := R(1,2,4;q)$ *and* $v := R(1,2,4;q^{17})$ *then*

$$\begin{aligned}
&78336v^{13}u^{13} - 4352v^{10}u^{16} + 22644v^{12}u^{6} - 65280v^{12}u^{14} - vu \\
&- 13464v^{13}u^{5} + 4352v^{9}u^{17} + 44030v^{10}u^{8} + 17v^{9}u - 33456v^{11}u^{7} \\
&+ 26112v^{11}u^{15} + 102v^{7}u^{3} - 33456v^{7}u^{11} - 17v^{8}u^{2} + 44030v^{8}u^{10} \\
&- 49164v^{9}u^{9} + 306v^{5}u^{5} - 13464v^{5}u^{13} - 255v^{6}u^{4} + 22644v^{6}u^{12} \\
&- 17v^{2}u^{8} + 425v^{2}u^{16} + 102v^{3}u^{7} - 2448v^{3}u^{15} - 255v^{4}u^{6} + u^{18} \\
&+ 7140v^{4}u^{14} + 17vu^{9} - 34vu^{17} + v^{18} - 34v^{17}u + 4352v^{17}u^{9} \\
&- 65536v^{17}u^{17} + 425v^{16}u^{2} - 4352v^{16}u^{10} + 26112v^{15}u^{11} \\
&- 2448v^{15}u^{3} - 65280v^{14}u^{12} + 7140v^{14}u^{4} = 0.
\end{aligned} \tag{48}$$

Proof. Using the lemma (8) with the equation (36), we obtain (48).

Theorem 17. *If* $u := R(1,2,4;q)$ *and* $v := R(1,2,4;q^{19})$ *then*

$$\begin{aligned}
&38v^{6}u^{2} + 646v^{2}u^{14} + 41496u^{11}v^{11} - 418u^{3}v^{11} - 21242v^{12}u^{8} \\
&+ 6859v^{4}u^{16} - 247v^{4}u^{8} - 63232v^{12}u^{16} - v^{20} + u^{20} + 247u^{4}v^{8} \\
&- 418u^{11}v^{3} + 19u^{3}v^{3} - 171u^{5}v^{5} - 10374u^{9}v^{9} + 19u^{9}v - uv \\
&+ 2622v^{6}u^{10} - 3192u^{7}v^{7} - 38u^{6}v^{2} - 2622v^{10}u^{6} - 1140u^{5}v^{13} \\
&+ 204288u^{13}v^{13} - 1140u^{13}v^{5} + 26752v^{9}u^{17} - 57vu^{17} - 57v^{17}u \\
&+ 26752v^{17}u^{9} - 311296v^{17}u^{17} - 19456u^{19}v^{11} + 41952u^{14}v^{10} \\
&+ 21242u^{12}v^{8} - 10336u^{6}v^{18} + 155648u^{14}v^{18} + 63232v^{16}u^{12} \\
&+ 19uv^{9} - 6859v^{16}u^{4} + 4560u^{7}v^{15} + 10336v^{6}u^{18} + 4560u^{15}v^{7} \\
&+ 175104u^{15}v^{15} - 646v^{14}u^{2} - 155648u^{18}v^{14} - 41952u^{10}v^{14} \\
&+ 228u^{19}v^{3} + 228u^{3}v^{19} + 262144u^{19}v^{19} - 19456u^{11}v^{19} = 0.
\end{aligned} \tag{49}$$

Proof. Using the lemma (9) with the equation (36), we obtain (49).

Theorem 18. *If* $u := R(1,2,4;q)$ *and* $v := R(1,2,4;q^{23})$ *then*

$$\begin{aligned}
&596890u^{21}v^{9} - u^{3}v^{3} - 9223680u^{25}v^{13} + 43909120u^{20}v^{26} + 25u^{13}v \\
&+ 2430u^{20}v^{2} - 16777216u^{25}v^{29} + 180935u^{20}v^{10} + 156701u^{26}v^{4} \\
&+ v^{2}u^{4} - v^{5}u + v^{4}u^{2} - vu^{5} + 180935u^{10}v^{20} + 46202880u^{25}v^{21} \\
&+ 379790u^{25}v^{5} + 4320u^{7}v^{15} - 320u^{7}v^{7} + 607340u^{19}v^{11} + 74uv^{29} \\
&+ 847635u^{14}v^{16} - 255u^{8}v^{6} + 596890u^{9}v^{21} - 18920u^{8}v^{14} + u^{30} \\
&- 250u^{3}v^{11} + 379790u^{5}v^{25} + 705u^{5}v^{9} + 25uv^{13} - 16384000u^{19}v^{27} \\
&- 16384000u^{27}v^{19} - 16777216u^{27}v^{27} - 18920u^{14}v^{8} - 150uv^{21} \\
&+ 1307790u^{17}v^{13} - 150u^{21}v + 16777216u^{26}v^{28} - 5214720u^{26}v^{12} \\
&+ 43909120u^{26}v^{20} - 9223680u^{13}v^{25} + 1307790u^{13}v^{17} + 670u^{4}v^{10} \\
&- 2181120u^{14}v^{24} - 16711680u^{22}v^{24} - 46380u^{13}v^{9} - 36030u^{17}v^{5} \\
&+ 26324u^{27}v^{3} - 11873280u^{17}v^{21} - 11873280u^{21}v^{17} + 156701u^{4}v^{26} \\
&- 255u^{6}v^{8} - 16711680u^{24}v^{22} + 248335u^{24}v^{6} - 20370u^{4}v^{18} + 74u^{29}v \\
&+ 2520u^{19}v^{3} + 1638400u^{17}v^{29} + 847635u^{16}v^{14} - 8520u^{16}v^{6} + v^{30} \\
&- 4843520u^{16}v^{22} - 9681920u^{18}v^{20} - 11468800u^{18}v^{28} + 670u^{10}v^{4} \\
&+ 4320u^{15}v^{7} + 663385u^{18}v^{12} - 20370u^{18}v^{4} - 16777216u^{29}v^{25} \\
&- 37820u^{12}v^{10} - 732560u^{15}v^{15} + 622080u^{10}v^{28} - 9681920u^{20}v^{18} \\
&- 36030u^{5}v^{17} + 1105920u^{15}v^{23} + 2430u^{2}v^{20} + 663385u^{12}v^{18} \\
&- 175u^{2}v^{12} + 16777216u^{28}v^{26} - 11468800u^{28}v^{18} + 622080u^{28}v^{10} \\
&- 37820u^{10}v^{12} + 645120u^{11}v^{27} - 4843520u^{22}v^{16} + 1105920u^{23}v^{15} \\
&- 8520u^{6}v^{16} - 38400u^{9}v^{29} - 13680u^{11}v^{11} - 175u^{12}v^{2} + 705u^{9}v^{5} \\
&+ 1638400u^{29}v^{17} - 46380u^{9}v^{13} + 432285u^{22}v^{8} - 61560u^{7}v^{23} \\
&- 38400u^{29}v^{9} + 46202880u^{21}v^{25} + 2520u^{3}v^{19} - 61560u^{23}v^{7} \\
&- 2181120u^{24}v^{14} - 20971520u^{23}v^{23} + 645120u^{27}v^{11} - 250u^{11}v^{3} \\
&+ 607340u^{11}v^{19} + 26324u^{3}v^{27} + 248335u^{6}v^{24} - 3502080u^{19}v^{19} \\
&- 5214720u^{12}v^{26} + 432285u^{8}v^{22} + 2051u^{28}v^{2} + 2051u^{2}v^{28} = 0.
\end{aligned} \tag{50}$$

Proof. Using the lemma (10) with the equation (36), we obtain (50).

Theorem 19. *If* $u := R(1,2,4;q)$ *and* $v := R(1,2,4;q^{25})$ *then*

$$\begin{aligned}
&847635u^{14}v^{16} - 150uv^{21} + 596890u^{21}v^{9} - 255u^{8}v^{6} - 11468800u^{18}v^{28} \\
&- 16777216u^{25}v^{29} + 607340u^{11}v^{19} + 156701u^{26}v^{4} - 20971520u^{23}v^{23} \\
&+ 43909120u^{20}v^{26} + 180935u^{20}v^{10} - 9681920u^{20}v^{18} - 9223680u^{25}v^{13} \\
&+ 596890u^{9}v^{21} - 18920u^{8}v^{14} + 26324u^{3}v^{27} - 250u^{3}v^{11} + 379790u^{5}v^{25} \\
&+ 705u^{5}v^{9} + 25uv^{13} - 16384000u^{19}v^{27} - 16384000u^{27}v^{19} - 320u^{7}v^{7} \\
&- 16777216u^{27}v^{27} - 18920u^{14}v^{8} + 74uv^{29} + 25u^{13}v + 1307790u^{17}v^{13} \\
&- 150u^{21}v + 16777216u^{26}v^{28} - 5214720u^{26}v^{12} + 43909120u^{26}v^{20} \\
&- 9223680u^{13}v^{25} + 1307790u^{13}v^{17} + 248335u^{6}v^{24} - 2181120u^{14}v^{24} \\
&- 175u^{2}v^{12} + 16777216u^{28}v^{26} - 11468800u^{28}v^{18} + 622080u^{28}v^{10} \\
&- 37820u^{10}v^{12} + 645120u^{11}v^{27} - 4843520u^{22}v^{16} + 1105920u^{23}v^{15} \\
&- 250u^{11}v^{3} + 74u^{29}v + 1638400u^{29}v^{17} - 46380u^{9}v^{13} + 432285u^{22}v^{8} \\
&- 16711680u^{22}v^{24} - 36030u^{17}v^{5} + 26324u^{27}v^{3} - 11873280u^{21}v^{17} \\
&- 11873280u^{17}v^{21} + 432285u^{8}v^{22} - 46380u^{13}v^{9} + 156701u^{4}v^{26} \\
&- 16711680u^{24}v^{22} + 248335u^{24}v^{6} - 20370u^{4}v^{18} + 4320u^{15}v^{7} \\
&+ 2520u^{19}v^{3} + 1638400u^{17}v^{29} + 847635u^{16}v^{14} - 37820u^{12}v^{10} \\
&- 3502080u^{19}v^{19} - 4843520u^{16}v^{22} - 9681920u^{18}v^{20} + 670u^{4}v^{10} \\
&- 5214720u^{12}v^{26} + 663385u^{18}v^{12} - 20370u^{18}v^{4} + 645120u^{27}v^{11} \\
&+ 663385u^{12}v^{18} - 8520u^{16}v^{6} + 705u^{9}v^{5} + 46202880u^{25}v^{21} + u^{30} \\
&- 732560u^{15}v^{15} + 622080u^{10}v^{28} - 175u^{12}v^{2} - 36030u^{5}v^{17} + v^{30} \\
&+ 1105920u^{15}v^{23} + 2051u^{28}v^{2} + 2051u^{2}v^{28} + 2430u^{2}v^{20} - u^{3}v^{3} \\
&- 8520u^{6}v^{16} - 38400u^{9}v^{29} - 16777216u^{29}v^{25} - 13680u^{11}v^{11} \\
&- 61560u^{7}v^{23} - 38400u^{29}v^{9} + 670u^{10}v^{4} + 46202880u^{21}v^{25} \\
&+ 2520u^{3}v^{19} - 61560u^{23}v^{7} - 2181120u^{24}v^{14} + 180935u^{10}v^{20} \\
&+ 379790u^{25}v^{5} + 4320u^{7}v^{15} + 607340u^{19}v^{11} + v^{2}u^{4} - 255u^{6}v^{8} \\
&- v^{5}u + v^{4}u^{2} - vu^{5} + 2430u^{20}v^{2} = 0.
\end{aligned} \tag{51}$$

Proof. Using the lemma (11) with the equation (36), we obtain (51).

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