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On Non-Commutative Weighted Ergodic Theorems for Multi-Parameter Semigroups

Abdusalom Karimov
Mathematics Department, Institute of Textile and Light Industry, Tashkent, Uzbekistan
E-mail address: karimov57@rambler.ru

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In the present paper we consider a von Neumann algebra M with a faithful normal semi-finite trace τ, and $\{\alpha_{t_1}\}_{t_1\geq 0}, \ldots, \{\alpha_{t_N}\}_{t_N\geq 0}$, N strongly continuous semigroups of absolute contractions on $L_p(M,\tau)$ $(p > 1)$. We prove that for every $x \in L_p(M,\tau)$ and Besicovitch function $b(t_1, \ldots t_N)$ the averages

$$\frac{1}{T_1T_2\cdots T_N}\int_0^{T_N}\cdots\int_0^{T_1} b(t_1,\ldots,t_N)(\alpha_{t_N}\cdots\alpha_{t_1})(x)dt_1dt_2\cdots dt_N$$

converge b.a.u. in $L_p(M)$ as $\max\{T_1,\ldots,T_N\} \to 0$ and $\min\{T_1,\ldots,T_N\} \to \infty$, respectively.

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1 Introduction

Individual ergodic theorem with respect to almost everywhere convergence in von Neumann algebras was studied by many authors [3, 4, 7, 14]. In [7] various maximal ergodic theorems in non-commutative L_p-spaces were proved and as applications of such results the corresponding individual and local ergodic theorems were obtained. Almost everywhere convergence of the Besicovitch weighted ergodic averages in von Neumann algebras was firstly proved in [6]. Further, in [1] by means of the Banach principle the Besicovitch weighted ergodic theorem was proved. In [10] bilateral almost uniform convergence of weighted multi-parameter averages was proved with respect to bounded Besicovitch families for positive contractions in non-commutative L_p-spaces. In [11], recently, a Besicovitch function weighted local ergodic theorem has been proved in the L_p-spaces.

In this paper we prove weighted local and individual ergodic theorems for multi-parameter strongly continuous semigroups of absolute contractions, with respect to bounded Besicovitch families, in the non-commutative L_p-spaces. As a particular case, we will obtain a result of [11]. To prove the main result we use the maximal ergodic inequality given in [4] and the Banach principle for semigroups proved in [14].

2 Preliminaries and Notations

Let M be a semifinite von Neumann algebra acting on a Hilbert space H, let τ be a faithful normal semifinite trace on M, and let $P(M)$ be the complete lattice of all projections in M. A densely-defined closed operator x in H is said to be *affiliated* with M if $y'x \subset xy'$ for every $y' \in M'$, where M' is the commutant of the algebra M. An operator x, affiliated with M, is said to be τ*-measurable* if for each $\varepsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \leq \varepsilon$ such that $eH \subset D_x$, where $e^{\perp} = \mathbb{1} - e$, $\mathbb{1}$ is the unit of M, D_x is the domain of x. Let $L_0(M)$ be the set of all $\tau-$measurable operators affiliated with M. Let $\|\cdot\|$ stand for the uniform norm in M.

For a positive self-adjoint operator $x = \int_0^\infty \lambda de_\lambda$ affiliated with M, one can define

$$\tau(x) = \sup_n \tau\bigg(\int_0^n \lambda de_\lambda\bigg) = \int_0^\infty d\tau(e_\lambda).$$

If $0 < p \leq \infty$, then

$$L_p = L_p(M) = \begin{cases} \{x \in L_0(M) : \|x\|_p = \tau(|x|^p)^{1/p} < \infty\} & \text{for } p \neq \infty, \\ (M, \|\cdot\|) & \text{for } p = \infty. \end{cases}$$

Here, $|x|$ is the *absolute value* of x, i.e. the square root of x^*x. By L_+ (resp. L_{sa}) we denote the set of positive (resp. self-adjoint) elements of L. We refer the reader to [13] for more information about non-commutative integration and to [15, 16] for general terminology of von Neumann algebras.

There are several different types of convergences in $L_0(M)$, each of them, in the commutative case with finite measure, reduces to the almost everywhere convergence (see for example [12]). In this paper we deal with the so called *bilateral almost uniform* (b.a.u.) convergence in $L_0(M)$ for which $x_\alpha \to x$ means that for every $\varepsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \leq \varepsilon$ such that $\|e(x_\alpha - x)e\| \to 0$. It is clear that b.a.u. implies convergence in measure.

Now take any set $A \subseteq R_+^N$, where $N \geq 1$. Recall that the space $L_p(M; \ell_\infty(A))$ is defined as the set of all families $x = \{x_t\}_{t \in A}$ in $L_p(M)$ which admit a factorization of the following form: there are $a, b \in L_{2p}(M)$ and $y = \{y_t\} \subset M$ such that $x_t = a y_t b \ \ \forall t \in A$. Then we define

$$\|x\|_{L_p(M;\ell_\infty(A))} = \inf\{\|a\|_{2p} \sup_{t \in A} \|y_t\|_\infty \|b\|_{2p}\},$$

where the infimum runs over all factorizations as above. Then $(L_p(M;\ell_\infty(A)), \|x\|_{L_p(M;\ell_\infty(A))})$ is a Banach space [7]. There it was shown that a family of positive elements $x = \{x_t\}_{t\in A}$ belongs to $L_p(M;\ell_\infty(A))$ iff there is $a \in L_p^+(M)$ such that $x_t \leq a$ for all $t \in A$, moreover, $\|x\|_{L_p(M;\ell_\infty(A))} = \inf\{\|a\|_p : a \in L_p^+(M),\, x_t \leq a,\, \forall t \in A\}$. The norm of x in $L_p(M;\ell_\infty(A))$ will be often denoted by $\|\sup_t x_t\|_p$. In the sequel we will be interested with the spaces $L_p(M;\ell_\infty(R_+^N))$ and $L_p(M;\ell_\infty((0,1]^N))$.

For $\mathbf{t} = (t_1,\ldots,t_N) \in R_+^N$, we denote $m(\mathbf{t}) = \min\{t_1,\ldots,t_N\}$, $M(\mathbf{t}) = \max\{t_1,\ldots,t_N\}$, $\Lambda_{[m,n]} = \{\mathbf{t} = (t_1,\ldots,t_N) \in R_+^N : m \leq m(\mathbf{t}), M(\mathbf{t}) \leq n\}$.

In order to prove ergodic theorem by the corresponding maximal ergodic theorems, it is convenient to use a subspace $L_p(M;c_0(R_+^N))$ of $L_p(M;\ell_\infty(R_+^N))$ which is defined as the space of all families $x = \{x_\mathbf{t}\}_{\mathbf{t}\in R_+^N} \subset L_p(M)$ such that there are $a, b \in L_{2p}(M)$ and $\{y_\mathbf{t}\} \subset M$ satisfying $x_\mathbf{t} = ay_\mathbf{t}b$ and $\lim_{m(\mathbf{t})\to\infty} \|y_\mathbf{t}\|_\infty = 0$, and the subspace $L_p(M;c_0((0,1]^N))$ of $L_p(M;\ell_\infty((0,1]^N))$ which is defined as the space of all families $\{x_\mathbf{t}\}_{\mathbf{t}\in(0,1]^N} \subset L_p(M)$ such that there are $a, b \in L_{2p}(M)$ and $\{y_\mathbf{t}\} \subset M$ satisfying $x_\mathbf{t} = ay_\mathbf{t}b$ and $\lim_{M(\mathbf{t})\to 0} \|y_\mathbf{t}\|_\infty = 0$. It is easy to check that $L_p(M;c_0(R_+^N))$ and $L_p(M;c_0((0,1]^N))$ are closed subspaces of $L_p(M;\ell_\infty(R_+^N))$ and $L_p(M;\ell_\infty((0,1]^N))$, respectively.

For the sake of completeness, we provide the proof of the next lemma, which is an analog of Lemma 6.2 in [7].

Lemma 2.1. *Let $\{x_\mathbf{t}\} \in L_p(M;c_0((0,1]^N))$ with $1 \leq p < \infty$, then $\{x_\mathbf{t}\}$ converges b.a.u. to 0 as $M(\mathbf{t}) \to 0$.*

Proof. Let $\{x_\mathbf{t}\} \in L_p(M;c_0((0,1]^N))$. Then there are $a, b \in L_{2p}(M)$ and $\{y_\mathbf{t}\} \in M$ such that $x_\mathbf{t} = ay_\mathbf{t}b$ and $\|a\|_{2p} < 1$, $\|b\|_{2p} < 1$, $\lim_{M(\mathbf{t})\to 0} \|y_\mathbf{t}\|_\infty = 0$. We can assume $a, b \geq 0$. Let e_a be a spectral projection of a such that $\tau(e_a^\perp) < \varepsilon/2$ and $\|e_a a\|_\infty \leq (2/\varepsilon)^{1/2p}$. Similarly, we find a spectral projection e_b of b. Set $e = e_a \wedge e_b$. Then $\tau(e^\perp) \leq \tau(e_a^\perp) + \tau(e_b^\perp) < \varepsilon$ and $\|ex_\mathbf{t}e\|_\infty \leq \|ea\|_\infty \|y_\mathbf{t}\|_\infty \|ba\|_\infty \leq \|y_\mathbf{t}\|_\infty \|e_a a\|_\infty \|e_b b\|_\infty < (2/\varepsilon)^{1/p} \|y_\mathbf{t}\|_\infty$. Thus $\lim_{M(\mathbf{t})\to 0} \|ex_\mathbf{t}e\|_\infty = 0$ and so $x_\mathbf{t} \to 0$ b.a.u. as $M(\mathbf{t}) \to 0$. □

Let $(Z, \Re, \mu)$ be a measurable space with a probability measure μ. Let $\widetilde{M}$ be the von Neumann algebra of all essentially bounded ultraweakly measurable functions $h : (Z, \mu) \to M$ equipped with the trace $\widetilde{\tau}(h) = \int_Z \tau(h(z))d\mu(z)$, and let $\widetilde{L}_p = L_p(\widetilde{M}, \widetilde{\tau})$.

Lemma 2.2 ([10]). *a) Let $\{x_\mathbf{t}\}_{\mathbf{t}\in R_+^N} \in L_p(\widetilde{M};c_0(R_+^N))$. Then $\{x_\mathbf{t}(z)\}_{\mathbf{t}\in R_+^N} \in L_p(M;c_0(R_+^N))$ for almost all $z \in Z$.*

b) If for every $\mathbf{p} \in R_+^N$ $\{x_{\mathbf{t}+\mathbf{p}} - x_\mathbf{t}\}_{\mathbf{t}\in R_+^N} \in L_p(M;c_0(R_+^N))$ with $1 \leq p < \infty$, then $\{x_t\}$ convergence b.a.u. as $m(\mathbf{t}) \to \infty$ to some x from $L_p(M)$.

Notice that Lemma 2.2. is true for the spaces $L_p(\widetilde{M};c_0((0,1]^N))$ and $L_p(M;c_0((0,1]^N))$. Recall that a positive linear map $\alpha : L_1(M,\tau) \to L_1(M,\tau)$ is called an *absolute contraction* if $\alpha(x) \leq \mathbb{1}$ and $\tau(\alpha(x)) \leq \tau(x)$ for every $x \in M \cap L_1$ with $0 \leq x \leq \mathbb{1}$. If α is a positive contraction in L_1, then, as is shown in [17], $\|\alpha(x)\|_p \leq \|x\|_p$ holds for $x = x^* \in M \cap L_p$ and all $1 \leq p \leq \infty$. Besides, there exist unique continuous extensions $\alpha : L_p \to L_p$ for all $1 \leq p < \infty$ and a unique ultra-weakly continuous extension $\alpha : M \to M$ (see [7,17]). This implies that, for every $x \in L_p$ and any positive integer k, one has

$$\|\alpha^k(x)\|_p \leq 2\|x\|_p.$$

Let $\{\alpha_t\}_{t\geq 0}$ be semigroup of absolute contraction on L_1. This means that each α_t is an absolute contraction on L_1, $\alpha_0 = Id$ and $\alpha_t\alpha_s = \alpha_{t+s}$ for all $t, s \geq 0$. By the same symbol α_t we will denote its extension to L_p $(1 \leq p < \infty)$. In the sequel we assume that the semigroup $\{a_t\}$ is strongly continuous in L_p, for fixed p, i.e. $\lim_{t\to s} \|\alpha_t f - \alpha_s f\|_p = 0$ for all $s \geq 0$ and $f \in L_p$.

Let us consider $\{\alpha_{t_1}\}_{t_1\geq 0}, \ldots, \{\alpha_{t_N}\}_{t_N\geq 0}$ semigroups of absolute contraction of $L_p(M)$. We form their ergodic averages

$$\beta_{T_1T_2\cdots T_N}(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_N}) = \frac{1}{T_1T_2\cdots T_N} \int\limits_0^{T_1,T_2,\cdots,T_N} \alpha_{t_N}\alpha_{t_{N-1}}\cdots\alpha_{t_1}dt_1dt_2\cdots dt_N.$$

The last one is always denoted by

$$\beta_{\mathbf{T}}(\alpha_{\mathbf{t}}) = \frac{1}{\Pi(\mathbf{T})}\int_0^{\mathbf{T}} \alpha_{\mathbf{t}}d\mathbf{t},$$

where $\mathbf{T} = (T_1, T_2, \ldots, T_N)$, $\Pi(\mathbf{T}) = T_1T_2\cdots T_N$, $\alpha_{\mathbf{t}} = \alpha_{t_N}\alpha_{t_{N-1}}\cdots\alpha_{t_1}$, and $d\mathbf{t} = dt_1dt_2\cdots dt_N$.

In [7] the following maximal inequality has been proved.

Theorem 2.1. *Let* $\{\alpha_{t_1}\}_{t_1\geq 0}, \ldots, \{\alpha_{t_N}\}_{t_N\geq 0}$ *be semigroups as above. Then for any* $1 < p < \infty$ *one has*

$$\left\|\sup_{m(\mathbf{T})>0}\frac{1}{\Pi(\mathbf{T})}\int_0^{\mathbf{T}} \alpha_{\mathbf{t}}(x)d\mathbf{t}\right\|_p \leq C_p^N\|x\|_p \ \ \forall x \in L_p(M).$$

Definition 2.1. Let $(B, \|\cdot\|, \geq)$ be an ordered real Banach space with the closed convex cone $B_+, B = B_+ - B_+$. A subset $B_0 \subset B_+$ is said to be minorantly dense in B_+ if for every $x \in B_+$ there is a sequence $\{x_n\}$ in B_0 such that $x_n \leq x$ for each n, and $\|x - x_n\| \to 0$ as $n \to \infty$.

For example, $M_+ \cap L_1(M) \cap L_2(M)$ is a minorantly dense subset of $L_1(M)_{sa}$.

In [14] Banach principle has been proved for N semigroups.

Theorem 2.2. *Let X be an ordered real Banach space with the closed convex cone X_+ $X = X_+ - X_+$, and for each $\mathbf{t} = (t_1, \dots, t_N) \in R_+^N$, $N \in \aleph$, $\aleph$ -natural numbers. Let $\alpha_{\mathbf{t}} : X \to L_0(M)$ be a continuous positive linear map. Assume that the following conditions are satisfied:*

(i) *for each $b \in X_+$ and $\delta > 0$ there exists $y \in M^+$ $0 \neq y \leq 1$ and $n \in \aleph$ such that*

$$\sup_{m(\mathbf{t})>n} \|y\alpha_{\mathbf{t}}(b)y\|_\infty < \infty$$

and $\tau(1-y) \leq \delta$,

(i) *there exists X_0, a minorantly dense subset of X_+ such that for each $b \in X_0$ the family operators $\alpha_{\mathbf{t}}(b) - \alpha_{\mathbf{s}}(b)(\mathbf{t}, \mathbf{s} \in R_+^N)$ b.a.u. converge to 0 as $m(\mathbf{t}) \to \infty, m(\mathbf{s}) \to \infty$.*

Then for each $b \in X$, $\alpha_{\mathbf{t}}(b)$ is b.a.u. convergent to some element of $L_0(M)$ as $m(\mathbf{t}) \to \infty$.

Notice that Theorem 2.2. is true when we replace R_+^N by $(0,1]^N$.

3 Multiparameter Weighted Ergodic Theorem for Non-Commutative L_p-Spaces

Recall the following ergodic theorem for N semigroups, which has been proved in [7, Theorem 6.8].

Theorem 3.1. *Let $\{\alpha_{t_1}\}_{t_1 \geq 0}, \dots, \{\alpha_{t_N}\}_{t_N \geq 0}$ be N semigroups. Let $1 < p < \infty$, $x \in L_p(M)$, and*

$$\beta_{\mathbf{T}}(\alpha_{\mathbf{t}}(x)) = \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} \alpha_{\mathbf{t}}(x) d\mathbf{t}.$$

Then

1) $\{\beta_{\mathbf{T}+\mathbf{P}}(\alpha_{\mathbf{t}}(x)) - \beta_{\mathbf{T}}(\alpha_{\mathbf{t}}(x))\}_{\mathbf{T} \in R_+^N} \in L_p(M; c_0(R_+^N))$, *for every $\mathbf{P} \in R_+^N$.*
2) $\{\beta_{\mathbf{T}+\mathbf{P}}(\alpha_{\mathbf{t}}(x)) - \beta_{\mathbf{T}}(\alpha_{\mathbf{t}}(x))\}_{\mathbf{T} \in (0,1]^N} \in L_p(M; c_0(0,1]^N)$, *for every $\mathbf{P} \in (0,1]^N$.*

Recall that a function $P : (0,1]^N \to C$ ($P : R_+^N \to C$) is called trigonometric polynomial in N variables if it is of the form

$$P(\mathbf{t}) = \sum_{j=1}^{n} k_j e^{2\pi i (\mathbf{p}_j, \mathbf{t})},$$

where $(\mathbf{p}_j, \mathbf{t}) = \sum_{i=1}^N \psi_i^{(j)} t_i$, $\mathbf{t} \in (0,1]^N$ ($\mathbf{t} \in R_+^N$), $\mathbf{p}_j = (\psi_i^{(j)}) \in R^N$ for some $(k_j) \subset C$. By $S((0,1]^N)$ we denote the set of all trigonometric polynomials defined on $(0,1]^N$.

Definition 3.1. We say that measurable function $b : (0,1]^N \to C$ is a 0-Besicovitch function if (i) $b \in L^\infty((0,1]^N)$,
(ii) given any $\varepsilon > 0$ there is $P \in S((0,1]^N)$ such that

$$\limsup_{M(\mathbf{T})\to 0} \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} |b(\mathbf{t}) - P(\mathbf{t})| d\mathbf{t} < \varepsilon.$$

Similarly in the last definition if we require $m(\mathbf{T}) \to \infty$ instead of $M(\mathbf{T}) \to 0$, then the function $b : R_+^N \to C$ is called Besicovitch function.

Lemma 3.1. *Let $\{\alpha_{t_1}\}_{t_1 \geq 0}, \dots, \{\alpha_{t_N}\}_{t_N \geq 0}$ be N semigroups on $L_p(M)$. Then for every trigonometric polynomial $P(\mathbf{t})$ on $(0,1]^N$ (respectively on R_+^N) and every $x \in L_p(M)$ the averages*

$$\widetilde{\beta}_{\mathbf{T}}(x) = \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} P(\mathbf{t}) \alpha_{\mathbf{t}}(x) d\mathbf{t}$$

converge b.a.u. as $M(\mathbf{T}) \to 0$ (respect. as $m(\mathbf{T}) \to \infty$).

Proof. Let $B = \{z \in C : |z| = 1\}$ be the unit circle in C with normalized Lebesque measure σ. Denote

$$\mathbf{B} = \underbrace{B \times \cdots \times B}_{N}, \qquad \mu = \underbrace{\sigma \otimes \cdots \otimes \sigma}_{N}.$$

Now consider $\widetilde{L}_p = L_p(\widetilde{M})$ with $\widetilde{M} = M \otimes L_\infty(\mathbf{B}, \mu)$ and $\widetilde{\tau} = \tau \otimes \mu$. Let us fix $\mathbf{s} \in \mathbf{B}$ and define $\widetilde{\alpha}_{\mathbf{t}}^{(\mathbf{s})} : \widetilde{L}_p \to \widetilde{L}_p$ by

$$(\widetilde{\alpha}_{\mathbf{t}}^{(\mathbf{s})}(f))(\mathbf{z}) = \alpha_{\mathbf{t}}(f(\mathbf{s}^{\mathbf{t}} \circ \mathbf{z})), \;\; f \in \widetilde{L}_p \,, \;\; \mathbf{z} \in \mathbf{B}. \tag{3.1}$$

Here $\mathbf{s} \circ \mathbf{z} = (s_1 z_1, s_2 z_2, \dots, s_N z_N)$ with $\mathbf{s} = (s_1, s_2, \dots, s_N)$, $\mathbf{z} = (z_1, z_2, \dots, z_N)$ and $\mathbf{s}^{\mathbf{t}} = (s_1^{t_1}, s_2^{t_2}, \dots, s_N^{t_N})$, where $\mathbf{t} = (t_1, t_2, \dots, t_N)$.

One can see that the mapping $\widetilde{\alpha}_{\mathbf{t}}^{(\mathbf{s})}$ is a multiple semigroup of absolute contractions. Now according to Theorem 3.1 we have

$$\left\{ \frac{1}{\Pi(\mathbf{T}+\mathbf{P})} \int_0^{\mathbf{T}+\mathbf{P}} \widetilde{\alpha}_{\mathbf{t}}^{(\mathbf{s})}(f) d\mathbf{t} - \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} \widetilde{\alpha}_{\mathbf{t}}^{(\mathbf{s})}(f) d\mathbf{t} \right\}_{\mathbf{T} \in (0,1]^N} \in L_p(\widetilde{M}; c_0((0,1]^N))$$

for every $\mathbf{P} \in (0,1]^N$ and $f \in \widetilde{L}_p$. Hence, Lemma 2.2 implies that

$$\left\{ \frac{1}{\Pi(\mathbf{T}+\mathbf{P})} \int_0^{\mathbf{T}+\mathbf{P}} (\widetilde{\alpha}_{\mathbf{t}}^{(\mathbf{s})}(f))(\mathbf{z}) d\mathbf{t} - \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} (\widetilde{\alpha}_{\mathbf{t}}^{(\mathbf{s})}(f))(\mathbf{z}) d\mathbf{t} \right\}_{\mathbf{T} \in (0,1]^N} \in L_p(M; c_0((0,1]^N))$$

a.e. $\mathbf{z} \in \mathbf{B}$.

Now instead of f we take $f_x(\mathbf{z}) = \Pi(\mathbf{z})x$ with fixed $x \in L_p(M)$. Then according the equality $\widetilde{\alpha}_{\mathbf{t}}^{(\mathbf{s})}(f_x(\mathbf{z})) = \Pi(\mathbf{s}^{\mathbf{t}})\Pi(\mathbf{z})\alpha_{\mathbf{t}}(x)$, we get

$$\Pi(\mathbf{z}) \left\{ \frac{1}{\mathbf{\Pi}(\mathbf{T}+\mathbf{P})} \int_0^{\mathbf{T}+\mathbf{P}} \Pi(\mathbf{s}^{\mathbf{t}}) \alpha_{\mathbf{t}}(x) d\mathbf{t} - \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} \Pi(\mathbf{s}^{\mathbf{t}}) \alpha_{\mathbf{t}}(x) d\mathbf{t} \right\}_{\mathbf{T} \in (0,1]^N} \in L_p(M; c_0((0,1]^N))$$

for almost all $\mathbf{z} \in \mathbf{B}$. Consequently, by $\Pi(\mathbf{z}) \neq 0$, one concludes that

$$\left\{ \frac{1}{\mathbf{\Pi}(\mathbf{T}+\mathbf{P})} \int\limits_0^{\mathbf{T}+\mathbf{P}} \Pi(\mathbf{s^t})\alpha_{\mathbf{t}}(x)d\mathbf{t} - \frac{1}{\Pi(\mathbf{T})} \int\limits_0^{\mathbf{T}} \Pi(\mathbf{s^t})\alpha_{\mathbf{t}}(x)d\mathbf{t} \right\}_{\mathbf{T}\in(0,1]^N} \in L_p(M; c_0((0,1]^N)).$$

Since every polynomial $P(\mathbf{t})$ is linear combination of $\Pi(\mathbf{s^t})$, we get the assertion for every trigonometric polynomial in N variables. $\square$

Theorem 3.2. *Let $\{\alpha_{t_1}\}_{t_1 \geq 0}, \ldots, \{\alpha_{t_N}\}_{t_N \geq 0}$ be N strongly continuous semigroups of absolute contractions on $L_p(M)$. Then, for every $x \in L_p(M)$*

(i) *The averages*

$$\beta_{\mathbf{T}}(x) = \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} b(\mathbf{t})\alpha_{\mathbf{t}}(x)d\mathbf{t}$$

converge b.a.u. in $L_p(M)$ as $M(\mathbf{T}) \to 0$, where $b(\mathbf{t})$- is bounded 0-Besicovitch function on $(0,1]^N$;

(ii) *The averages*

$$\beta_{\mathbf{T}}(x) = \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} b(\mathbf{t})\alpha_{\mathbf{t}}(x)d\mathbf{t}$$

converge b.a.u. in $L_p(M)$ as $m(\mathbf{T}) \to \infty$, where $b(\mathbf{t})$- is bounded Besicovitch function on R_+^N.

Proof. (i) Let $b(\mathbf{T})$ be a bounded 0-Besicovitch function on $(0,1]^N$. For $\varepsilon > 0$ there is a trigonometric polynomial $P_\varepsilon(\mathbf{t})$ on $(0,1]^N$ such that

$$\lim \sup_{M(\mathbf{T})\to 0} \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} |b(\mathbf{t}) - P_\varepsilon(\mathbf{t})|d\mathbf{t} < \varepsilon.$$

Then, for $x \in M \cap L_1(M)$ one has

$$\begin{aligned} \left\| \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} b(\mathbf{t})\alpha_{\mathbf{t}}(x)d\mathbf{t} - \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} P_\varepsilon(\mathbf{t})\alpha_{\mathbf{t}}(x)d\mathbf{t} \right\|_\infty &\leq \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} |b(\mathbf{t}) - P_\varepsilon(\mathbf{t})|d\mathbf{t}\|x\|_\infty \\ &< \varepsilon\|x\|_\infty. \end{aligned} \tag{3.2}$$

We may suppose that $|b(\mathbf{t})| \leq 1$ for almost all $\mathbf{t} \in (0,1]^N$. Let $x \in L_p(M)_+$ then Theorem 4.5 in [7] implies

$$\left\| \sup_{\mathbf{T}} \beta_{\mathbf{T}}(x) \right\|_{L_p(M;\ell_\infty((0,1]^N))} \leq C_p^N \|x\|_p. \tag{3.3}$$

Now consider

$$\widetilde{\beta}_{\mathbf{T}}^{(r)}(x) = \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} \Re(b(\mathbf{t}))\alpha_{\mathbf{t}}(x)d\mathbf{t} \quad \widetilde{\beta}_{\mathbf{T}}^{(i)}(x) = \frac{1}{\Pi(\mathbf{T})} \int_0^{\mathbf{T}} \Im(b(\mathbf{t}))\alpha_{\mathbf{t}}(x)d\mathbf{t}$$

and

$$\widetilde{\beta}^R_{\mathbf{T}}(x) = \widetilde{\beta}^r_{\mathbf{T}}(x) + \beta_{\mathbf{T}}(x) \ \ \widetilde{\beta}^I_{\mathbf{T}}(x) = \widetilde{\beta}^i_{\mathbf{T}}(x) + \beta_{\mathbf{T}}(x).$$

Then we have $\widetilde{\beta}^R_{\mathbf{T}}(x) \le 2\beta_{\mathbf{T}}(x) \ \ \widetilde{\beta}^I_{\mathbf{T}}(x) \le 2\beta_{\mathbf{T}}(x)$ and, from (3.3) one gets

$$\left\| \sup_{\mathbf{T}} \widetilde{\beta}^{(R)}_{\mathbf{T}}(x) \right\|_{L_p(M;\ell_\infty((0,1]^N))} \le 2C_p^N \|x\|_p,$$

$$\left\| \sup_{\mathbf{T}} \widetilde{\beta}^{(I)}_{\mathbf{T}}(x) \right\|_{L_p(M;\ell_\infty((0,1]^N))} \le 2C_p^N \|x\|_p.$$

Consequently

$$\left\| \sup_{\mathbf{T}} \beta_{\mathbf{T}}(x) \right\|_{L_p(M;\ell_\infty((0,1]^N))} \le 4C_p^N \|x\|_p \text{ for all } x \in L_p(M)_+. \tag{3.4}$$

Any $x \in L_p(M)$ can be represented as $x = \sum_{k=0}^{3} i^k x_k$, where $x_k \in L_p(M)_+ \ \ k = 0, 1, 2, 3$. Therefore, the inequality (3.4) implies that

$$\left\| \sup_{\mathbf{T}} \beta_{\mathbf{T}}(x) \right\|_{L_p(M;\ell_\infty((0,1]^N))} \le 16C_p^N \|x\|_p \quad \text{for any } \ x \in L_p(M). \tag{3.5}$$

From (3.2) and [7, Proposition 2.5] we have

$$\left\| \sup_{\mathbf{T}\in\Lambda[2^{-n},2^{-k}]} \Big(\beta_{\mathbf{T}}(x) - \widetilde{\beta}_{\mathbf{T}}(x)\Big) \right\|_{L_p(M;\ell_\infty((0,1]^N))} \le \left\| \sup_{\mathbf{T}\in\Lambda[2^{-n},2^{-k}]} \Big(\beta_{\mathbf{T}}(x) - \widetilde{\beta}_{\mathbf{T}}(x)\Big) \right\|_\infty^{1-q/p}$$

$$\left\| \sup_{\mathbf{T}\in\Lambda[2^{-n},2^{-k}]} \Big(\beta_{\mathbf{T}}(x) - \widetilde{\beta}_{\mathbf{T}}(x)\Big) \right\|_q^{\frac{q}{p}} \le \varepsilon^{1-q/p} \|x\|^{1-q/p} \left\| \sup_{\mathbf{T}\in\Lambda[2^{-n},2^{-k}]} \Big(\beta_{\mathbf{T}}(x) - \widetilde{\beta}_{\mathbf{T}}(x)\Big) \right\|_q^{q/p}. \tag{3.6}$$

We define a sequence $b^{(k)} = (b^{(k)}_{\mathbf{T}})_{\mathbf{T}\in(0,1]^N} \in L_p(M; c_0((0,1]^N))$ as follows:

$$b^{(k)}_{\mathbf{T}} = \begin{cases} \beta_{\mathbf{T}}(x) - \widetilde{\beta}_{\mathbf{T}}(x), & \text{if } \mathbf{T} \in \Lambda[2^{-k}, 1], \\ 0, & \text{if } \mathbf{T} \notin \Lambda[2^{-k}, 1]. \end{cases}$$

From (3.6) one finds that $b^{(k)} \to \{\beta_{\mathbf{T}}(x) - \widetilde{\beta}_{\mathbf{T}}(x)\}$ in $L_p(M; \ell_\infty((0,1]^N))$ as $k \to \infty$. Since $L_p(M; c_0((0,1]^N))$ is closed subspace in $L_p(M; \ell_\infty((0,1]^N))$ we have

$$\{\beta_{\mathbf{T}}(x) - \widetilde{\beta}_{\mathbf{T}}(x)\} \in L_p(M; c_0((0,1]^N)).$$

From Lemma 3.1, we obtain $\{\widetilde{\beta}_{\mathbf{T}+\mathbf{P}}(x) - \widetilde{\beta}_{\mathbf{T}}(x)\}_{\mathbf{T}\in(0,1]^N} \in L_p(M; c_0((0,1]^N))$ and the equality

$$\beta_{\mathbf{T}+\mathbf{P}}(x) - \beta_{\mathbf{T}}(x) = \beta_{\mathbf{T}+\mathbf{P}}(x) - \widetilde{\beta}_{\mathbf{T}+\mathbf{P}}(x) + \widetilde{\beta}_{\mathbf{T}}(x) - \beta_{\mathbf{T}}(x) + \widetilde{\beta}_{\mathbf{T}+\mathbf{P}}(x) - \widetilde{\beta}_{\mathbf{T}}(x)$$

implies that $\{\beta_{\mathbf{T}+\mathbf{P}}(x) - \beta_{\mathbf{T}}(x)\}_{\mathbf{T}\in(0,1]^N} \in L_p(M; c_0((0,1]^N))$. Now from minorant density of $L_1(M) \cap M$ in $L_p(M)$, by using Theorem 2.2 and (3.5), we get $\{\beta_{\mathbf{T}+\mathbf{P}}(x) - \beta_{\mathbf{T}}(x)\}_{\mathbf{T}\in(0,1]^N} \in L_p(M; c_0((0,1]^N))$ for any $x \in L_p(M)$.

(ii) This statement can be similarly proved by using the same arguments as used in (i) with application of the spaces $L_p(M; \ell_\infty(R_+^N)),\ L_p(M; c_0(R_+^N))$. □

Remark. Note that the proved theorem extends the results of the papers [1] to semigroups setting. Moreover, it generalizes a result of [11].

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