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Conditional Expectation of Generalized Order Statistics and Characterization of Probability Distributions

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Abstract: In this paper, two general form of distributions $1-F(x)=e^{-ah(x)}$ and $1-F(x)=[ah(x)+b]^c$, $x\in(\alpha,\beta)$ are characterized through the conditional expectation of power of difference of two generalized order statistics. Further, some of its important deductions and particular cases are also discussed.

Keywords: Generalized order statistics, order statistics, record values, conditional expectation and characterization.

1 Introduction

Kamps [7] introduced the concept of generalized order statistics (*gos*) to unify several models of ordered random variables, e.g. order statistics, record values, progressively type II censored order statistics and sequential order statistics. This common approach make it possible to deduce several distributional and moment properties at once. These models can be effectively applied in reliability theory and survival analysis.

The random variables (*rvs*), $X(1,n,m,k), X(2,n,m,k),\ldots X(n,n,m,k)$, $k>0, m\in\mathbb{R}$ are n *gos* from an absolutely continuous distribution function (*df*) $F(x)$ and probability density function (*pdf*) $f(x)$, if their joint density function is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^m f(x_i)\right)[1-F(x_n)]^{k-1}f(x_n) \tag{1}$$

on the cone $F^{-1}(0)<x_1\le\ldots\le x_n<F^{-1}(1)$,

where $\gamma_j=k+(m+1)(n-j)$ for all $j, 1\le j\le n$, k is a positive integer and $m\ge -1$.

If $m=0$ and $k=1$, then $X(r,n,m,k)$, the $r-th$ *gos* reduces to the $r-th$ order statistics and (1) will be the joint *pdf* of order statistics $X_{1:n}\le X_{2:n}\le\ldots\le X_{n:n}$ from *df* $F(x)$. If $m=-1$ and $k=1$, then (1) will be the joint *pdf* of the first n upper record values. In view of (1), the *pdf* of $X(r,n,m,k)$, the $r-th$ *gos* is given by

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)), \qquad \alpha\le x\le\beta \tag{2}$$

where, $\bar{F}(x)=1-F(x)$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1\le r<s\le n$, is

$$\begin{aligned} f_{X(r,n,m,k),X(s,n,m,k)}(x,y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x)) \\ &\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y), \qquad \alpha\le x<y\le\beta \end{aligned} \tag{3}$$

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where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i \ , \ \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & , \ m \neq -1 \\ -\log(1-x) & , \ m = -1 \end{cases}$$

and $\quad g_m(x) = h_m(x) - h_m(0), \ x \in (0,1).$

The problem of characterization of distributions has always been the topic of interest among researchers. Various approaches are available in literature. Conditional expectation of ordered random variables are extensively used in characterizing the probability distributions. Khan and Abu-Salih [10] have characterized some general form of distributions through conditional expectation of function of order statistics fixing adjacent order statistics. Khan and Abouammoh [9] extended the result of Khan and Abu-Salih [10] and characterized the distributions when the conditioning is not adjacent. Further, Samuel [16] characterized the distributions considered by Khan and Abu-Salih [10] for generalized order statistics (*gos*).

Keseling [8] has generalized the result of Franco and Ruiz [4] in terms of generalized order statistics and characterized some general form of distributions. Khan *et al.* [13] established characterizing relationship for the distributions through generalized order statistics and characterized several distributions through conditional expectation of function of generalized order statistics.

For more detailed survey on characterization one may refer to Franco and Ruiz [4,5], López-Blázquez and Moreno-Rebollo [14], Dembińska and Wesolowski [2,3], Wu and Ouyang [18], Khan and Athar [12], Wesolowski and Ahsanullah [17], Athar *et al.* [1] and references there in.

In this paper, an attempt is made to characterize two general forms of distributions $F(x) = 1 - e^{-ah(x)}$ and $1 - F(x) = [ah(x)+b]^c, x \in (\alpha, \beta)$ through conditional expectation of $p-th$ power of difference of functions of two generalized order statistics.

2 Characterization

Let $X(r,n,m,k)$, $r = 1,2,\ldots,n$ be *gos*, then the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$ in view of (2) and (3) is

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!} \frac{[\bar{F}(y)]^{\gamma_s - 1}}{(m+1)^{s-r-1}[\bar{F}(x)]^{\gamma_{r+1}}} \left[[\bar{F}(x)]^{m+1} - [\bar{F}(y)]^{m+1} \right]^{s-r-1} f(y).$$

Theorem 2.1. Let X be a random variable with an absolutely continuous df $F(x)$ and pdf $f(x)$ in the interval (α, β), where α and β may be finite or infinite, then for $1 \le r < s \le n$,

$$E[\{h(X(s,n,m,k)) - h(X(r,n,m,k))\}^2 | X(r,n,m,k) = x] = g_{r,s} = 2! \frac{1}{a^2} \sum_{i_1=r}^{s-1} \sum_{i_2=i_1}^{s-1} \frac{1}{\gamma_{i_1+1}} \frac{1}{\gamma_{i_2+1}}, \tag{4}$$

if and only if

$$\bar{F}(x) = e^{-ah(x)}, \qquad a > 0, \tag{5}$$

where $h(x)$ be a continuous, differentiable and monotonic function of x.

Proof. To prove the necessary part, we have

$$E[\{h(X(s,n,m,k)) - h(X(r,n,m,k))\}^2 | X(r,n,m,k) = x]$$
$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \int_x^\beta (h(y)-h(x))^2 \left[1 - \left\{ \frac{\bar{F}(y)}{\bar{F}(x)} \right\}^{m+1} \right]^{s-r-1} \left[\frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_s - 1} \frac{f(y)}{\bar{F}(x)} dy. \tag{6}$$

Let $\frac{\bar{F}(y)}{\bar{F}(x)} = u$, then $(h(y) - h(x))^2 = \frac{1}{a^2}(\ln u)^2$.

Therefore, *RHS* of (6) becomes

$$= \frac{1}{a^2} \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \int_0^1 (\ln u)^2 \left(1-u^{m+1}\right)^{s-r-1} u^{\gamma_s - 1} du. \tag{7}$$

Now set $u^{m+1} = t$, then (7) reduces to

$$= \frac{1}{a^2} \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r+2}} \int_0^1 (\ln t)^2 \left(1-t\right)^{s-r-1} t^{\frac{\gamma_s}{m+1}-1} dt. \tag{8}$$

Since

$$\int_0^1 (\ln u)^2 u^{\mu-1}(1-u)^{\nu-1} du = 2\,\beta(\mu,\nu) \sum_{k_1=0}^{\nu-1} \frac{1}{\mu+k_1} \sum_{k_2=k_1}^{\nu-1} \frac{1}{\mu+k_2}$$

(*c.f.* Gradshteyn and Ryzhik, [6]; pp 543),

where $\beta(\mu,\nu)$ is complete beta function.

Therefore, (8) becomes

$$= \frac{1}{a^2} \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r+2}} 2\beta\left(\frac{\gamma_s}{m+1}, s-r\right) \sum_{k_1=0}^{s-r-1} \sum_{k_2=k_1}^{s-r-1} \frac{1}{\frac{\gamma_s}{m+1}+k_1} \frac{1}{\frac{\gamma_s}{m+1}+k_2}$$

$$= 2! \frac{1}{a^2} \sum_{i_1=r}^{s-1} \sum_{i_2=i_1}^{s-1} \frac{1}{\gamma_{i_1+1}} \frac{1}{\gamma_{i_2+1}}.$$

To prove the sufficiency part, let

$$E[\{h(X(s,n,m,k)) - h(X(r,n,m,k))\}^2 | X(r,n,m,k) = x] = g_{r,s}$$

or

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \int_x^\beta (h(y)-h(x))^2 \left[[\bar{F}(x)]^{m+1} - [\bar{F}(y)]^{m+1}\right]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y) dy$$
$$= g_{r,s} [\bar{F}(x)]^{\gamma_{r+1}}.$$

Differentiating both sides *w.r.t* x, we get

$$\frac{C_{s-1}}{C_{r-1}(s-r-2)!(m+1)^{s-r-2}} \int_x^\beta (h(y)-h(x))^2 \left[[\bar{F}(x)]^{m+1} - [\bar{F}(y)]^{m+1}\right]^{s-r-2} [\bar{F}(y)]^{\gamma_s - 1} [\bar{F}(x)]^m f(x) f(y) dy$$
$$+ \frac{2C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} h'(x) \int_x^\beta (h(y)-h(x)) \left[[\bar{F}(x)]^{m+1} - [\bar{F}(y)]^{m+1}\right]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y) dy$$
$$= \gamma_{r+1} [\bar{F}(x)]^{\gamma_{r+1}-1} f(x) g_{r,s}$$

or

$$\gamma_{r+1} \frac{f(x)}{\bar{F}(x)} \frac{C_{s-1}}{C_r(s-r-2)!(m+1)^{s-r-2}} \int_x^\beta (h(y)-h(x))^2 \frac{\left[[\bar{F}(x)]^{m+1} - [\bar{F}(y)]^{m+1}\right]^{s-r-2} [\bar{F}(y)]^{\gamma_s - 1}}{[\bar{F}(x)]^{\gamma_{r+2}}} f(y) dy$$
$$+ 2\,h'(x) \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \int_x^\beta (h(y)-h(x)) \frac{\left[[\bar{F}(x)]^{m+1} - [\bar{F}(y)]^{m+1}\right]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1}}{[\bar{F}(x)]^{\gamma_{r+1}}} f(y) dy$$
$$= \gamma_{r+1} \frac{f(x)}{\bar{F}(x)} g_{r,s}$$

or $$\gamma_{r+1}\frac{f(x)}{F(x)}g_{r+1,s}+2\,h'(x)\frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{\gamma_{j+1}}=\gamma_{r+1}\frac{f(x)}{F(x)}g_{r,s}.$$

After rearranging the above expression, we get

$$\frac{f(x)}{F(x)}=\frac{1}{\gamma_{r+1}}\frac{2\,h'(x)\frac{1}{a}\sum_{i=r}^{s-1}\frac{1}{\gamma_{i+1}}}{[g_{r,s}-g_{r+1,s}]}$$

or

$$\frac{f(x)}{F(x)}=\frac{1}{\gamma_{r+1}}\frac{2\,h'(x)\frac{1}{a}\sum_{i=r}^{s-1}\frac{1}{\gamma_{i+1}}}{\left[\frac{2}{a^2}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\frac{1}{\gamma_{i_1+1}}\frac{1}{\gamma_{i_2+1}}-\frac{2}{a^2}\sum_{i_1=r+1}^{s-1}\sum_{i_2=i_1}^{s-1}\frac{1}{\gamma_{i_1+1}}\frac{1}{\gamma_{i_2+1}}\right]}$$

$$=\frac{1}{\gamma_{r+1}}\frac{a\,h'(x)\sum_{i=r}^{s-1}\frac{1}{\gamma_{i+1}}}{\frac{1}{\gamma_{r+1}}\sum_{i_2=r}^{s-1}\frac{1}{\gamma_{i_2+1}}},$$

which implies,

$$\frac{f(x)}{F(x)}=ah'(x).$$

Hence the theorem.

Remark 2.1. Setting $m=0$ and $k=1$ in (4), Theorem 2.1 reduces for order statistics

$$E[\{h(X_{s:n})-h(X_{r:n})\}^2|X_{r:n}=x]=2!\frac{1}{a^2}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\frac{1}{(n-i_1)}\frac{1}{(n-i_2)}$$

if and only if $\bar{F}(x)=e^{-ah(x)}, \qquad x\in(\alpha,\beta).$

Remark 2.2. At $m=-1$ and $k=1$, (4) reduces for upper record values

$$E[\{h(X_{U(s)})-h(X_{U(r)})\}^2|X_{U(r)}=x]=\frac{1}{a^2}(s-r)(s-r-1)$$

if and only if $\bar{F}(x)=e^{-ah(x)}, \qquad x\in(\alpha,\beta).$

Lemma 2.1. For any positive integers μ and ν with $n\in\mathbb{N}$

$$\int_0^1(\ln u)^n(1-u)^{\nu-1}u^{\mu-1}du=(-1)^n\,n!\;\beta(\mu,\nu)\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\cdots\sum_{i_n=i_{n-1}}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2}\cdots\frac{1}{\mu+i_n}, \tag{9}$$

where $\beta(\mu,\nu)$ is complete beta function.

Proof. Lemma can be established using the results of Gradshteyn and Ryzhik ([6]; pp 540, 543).

Theorem 2.2. Under the condition as stated in Theorem 2.1

$$E[\{h(X(s,n,m,k))-h(X(r,n,m,k))\}^p|X(r,n,m,k)=x]=g_{r,s,p}$$

$$=p!\,\frac{1}{a^p}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_p=i_{p-1}}^{s-1}\frac{1}{\gamma_{i_1+1}}\frac{1}{\gamma_{i_2+1}}\cdots\frac{1}{\gamma_{i_p+1}} \tag{10}$$

if and only if

$$\bar{F}(x)=e^{-ah(x)}, \qquad a>0, \tag{11}$$

where $h(x)$ be a continuous, differentiable and monotonic function of x.

Proof. To prove the necessary part, we have

$$E[\{h(X(s,n,m,k))-h(X(r,n,m,k))\}^p|X(r,n,m,k)=x]$$
$$=\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_x^\beta (h(y)-h(x))^p\left[1-\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}^{m+1}\right]^{s-r-1}\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_s-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{12}$$

Let $\frac{\bar{F}(y)}{\bar{F}(x)}=u$, then $(h(y)-h(x))^p=\frac{(-1)^p}{a^p}(\ln u)^p$.

Therefore, *RHS* of (12) becomes

$$=\frac{(-1)^p}{a^p}\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_0^1(\ln u)^p\,(1-u^{m+1})^{s-r-1}u^{\gamma_s-1}du. \tag{13}$$

Setting $u^{m+1}=t$, then (13) reduces to

$$=\frac{(-1)^p}{a^p}\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r+p}}\int_0^1(\ln t)^p\,(1-t)^{s-r-1}\,t^{\frac{\gamma_s}{m+1}-1}dt. \tag{14}$$

Now using Lemma 2.1, we get

$$=\frac{(-1)^p}{a^p}\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r+p}}(-1)^p\;p!\;\beta(\frac{\gamma_s}{m+1},s-r)$$
$$\times\sum_{i_1=0}^{s-r-1}\sum_{i_2=i_1}^{s-r-1}\cdots\sum_{i_p=i_{p-1}}^{s-r-1}\frac{1}{\frac{\gamma_s}{m+1}+i_1}\frac{1}{\frac{\gamma_s}{m+1}+i_2}\cdots\frac{1}{\frac{\gamma_s}{m+1}+i_p}$$
$$=\frac{p!}{a^p}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_p=i_{p-1}}^{s-1}\frac{1}{\gamma_{i_1+1}}\frac{1}{\gamma_{i_2+1}}\cdots\frac{1}{\gamma_{i_p+1}}.$$

Hence the (10).

To prove the sufficiency part, let

$$E[\{h(X(s,n,m,k))-h(X(r,n,m,k))\}^p|X(r,n,m,k)=x]=g_{r,s,p}$$

or

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_x^\beta (h(y)-h(x))^p\left[[\bar{F}(x)]^{m+1}-[\bar{F}(y)]^{m+1}\right]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dy$$
$$=g_{r,s,p}\,[\bar{F}(x)]^{\gamma_{r+1}}.$$

Differentiating both sides *w.r.t x*, we get

$$\frac{p\, h'(x)\, C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_x^\beta (h(y)-h(x))^{p-1}\frac{\left[[\bar{F}(x)]^{m+1}-[\bar{F}(y)]^{m+1}\right]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}}{[\bar{F}(x)]^{\gamma_{r+1}}}f(y)dy$$
$$+\gamma_{r+1}\frac{f(x)}{[F(x)]}\frac{C_{s-1}}{C_r(s-r-2)!(m+1)^{s-r-2}}\int_x^\beta (h(y)-h(x))^{p}\frac{\left[[\bar{F}(x)]^{m+1}-[\bar{F}(y)]^{m+1}\right]^{s-r-2}[\bar{F}(y)]^{\gamma_s-1}}{[\bar{F}(x)]^{\gamma_{r+2}}}f(y)dy$$
$$=\gamma_{r+1}\frac{f(x)}{F(x)}g_{r,s,p}$$

or

$$h'(x)\ p\ \frac{(p-1)!}{a^{p-1}}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_{p-1}=i_{p-2}}^{s-1}\frac{1}{\gamma_{i_1}+1}\frac{1}{\gamma_{i_2}+1}\cdots\frac{1}{\gamma_{i_{p-1}}+1}+\gamma_{r+1}\frac{f(x)}{F(x)}g_{r+1,s,p}$$
$$=\gamma_{r+1}\frac{f(x)}{F(x)}\,g_{r,s,p} \tag{15}$$

which implies,

$$\frac{f(x)}{F(x)}=\frac{1}{\gamma_{r+1}}\frac{h'(x)\ \frac{p!}{a^{p-1}}\ \sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_{p-1}=i_{p-2}}^{s-1}\frac{1}{\gamma_{i_1}+1}\frac{1}{\gamma_{i_2}+1}\cdots\frac{1}{\gamma_{i_{p-1}}+1}}{[g_{r,s,p}\ -\ g_{r+1,s,p}]}. \tag{16}$$

Consider,

$$g_{r,s,p}\ -\ g_{r+1,s,p}$$
$$=\frac{p!}{a^p}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_p=i_{p-1}}^{s-1}\frac{1}{\gamma_{i_1}+1}\frac{1}{\gamma_{i_2}+1}\cdots\frac{1}{\gamma_{i_p}+1}\ -\ \frac{p!}{a^p}\sum_{i_1=r+1}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_p=i_{p-1}}^{s-1}\frac{1}{\gamma_{i_1}+1}\frac{1}{\gamma_{i_2}+1}\cdots\frac{1}{\gamma_{i_p}+1}$$
$$=\frac{p!}{a^p}\frac{1}{\gamma_{r+1}}\sum_{i_2=r}^{s-1}\sum_{i_3=i_2}^{s-1}\cdots\sum_{i_p=i_{p-1}}^{s-1}\frac{1}{\gamma_{i_1}+1}\frac{1}{\gamma_{i_2}+1}\cdots\frac{1}{\gamma_{i_p}+1}$$
$$+\ \frac{p!}{a^p}\sum_{i_1=r+1}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_p=i_{p-1}}^{s-1}\frac{1}{\gamma_{i_1}+1}\frac{1}{\gamma_{i_2}+1}\cdots\frac{1}{\gamma_{i_p}+1}\ -\ \frac{p!}{a^p}\sum_{i_1=r+1}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_p=i_{p-1}}^{s-1}\frac{1}{\gamma_{i_1}+1}\frac{1}{\gamma_{i_2}+1}\cdots\frac{1}{\gamma_{i_p}+1}$$
$$=\frac{p!}{a^p}\frac{1}{\gamma_{r+1}}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_{p-1}=i_{p-2}}^{s-1}\frac{1}{\gamma_{i_1}+1}\frac{1}{\gamma_{i_2}+1}\cdots\frac{1}{\gamma_{i_{p-1}}+1}.$$

Therefore (16), becomes

$$\frac{f(x)}{F(x)}=a\ h'(x).$$

Hence the sufficiency part.

Remark 2.3. At $m=0$ and $k=1$, (10) is the result for order statistics

$$E[\{h(X_{s:n})-h(X_{r:n})\}^p|X_{r:n}=x]=\frac{p}{a^p}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_p=i_{p-1}}^{s-1}\frac{1}{(n-i_1)}\frac{1}{(n-i_2)}\cdots\frac{1}{(n-i_p)},$$

if and only if $\quad \bar{F}(x)=e^{-ah(x)}, \qquad x\in(\alpha,\beta).$

Remark 2.4. At $m=-1$ and $k=1$, (10) reduces to the case of upper record values

$$E[\{h(X_{U(s)})-h(X_{U(r)})\}^p|X_{U(r)}=x]=\frac{p}{a^p}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_p=i_{p-1}}^{s-1}$$

$$=\frac{1}{a^p}(p+s-r-1)(p+s-r-2)\cdots(s-r),$$

$$=\frac{1}{a^p}\frac{\Gamma(p+s-r)}{\Gamma(s-r)},$$

if and only if $\quad \bar{F}(x)=e^{-ah(x)}, \qquad x\in(\alpha,\beta),$

as obtained by Noor and Athar [15].

For various choice of a and $h(x)$ several distributions are listed (Khan and Abu-Salih [10]).

Theorem 2.3. Let X be a random variable with an absolutely continuous df $F(x)$ and pdf $f(x)$ in the interval (α,β), where α and β may be finite or infinite, then for $1\le r<s\le n$,

$$E[\{h(X(s,n,m,k))-h(X(r,n,m,k))\}^p|X(r,n,m,k)=x]=\xi_{r,s,p}(x)$$

$$=\left(\frac{a\,h(x)+b}{a}\right)^p\sum_{i=0}^{p}(-1)^{i+p}\binom{p}{i}\prod_{j=r+1}^{s}\left(\frac{c\gamma_j}{i+c\gamma_j}\right), \tag{17}$$

if and only if

$$\bar{F}(x)=[ah(x)+b]^c, \qquad a\neq 0, \tag{18}$$

where $h(x)$ be a continuous, differentiable and monotonic function of x.

Proof. First, we shall prove (18) implies (17).

We have,

$$E[\{h(X(s,n,m,k))-h(X(r,n,m,k))\}^p|X(r,n,m,k)=x]$$

$$=\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_x^\beta(h(y)-h(x))^p\left[1-\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}^{m+1}\right]^{s-r-1}\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_s-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{19}$$

Let $\frac{\bar{F}(y)}{\bar{F}(x)}=u$, then $(h(y)-h(x))^p=(-1)^p\left(\frac{ah(x)+b}{a}\right)^p(1-u^{\frac{1}{c}})^p$.

Therefore, RHS of (19) becomes

$$=(-1)^p\left(\frac{ah(x)+b}{a}\right)^p\frac{C_{s-1}}{C_{r-1}(s-r-1)!\,(m+1)^{s-r-1}}\int_0^1(1-u^{\frac{1}{c}})^p\left(1-u^{m+1}\right)^{s-r-1}u^{\gamma_s-1}du. \tag{20}$$

Now set $u^{m+1}=t$, to get

$$E[\{h(X(s,n,m,k))-h(X(r,n,m,k))\}^p|X(r,n,m,k)=x]$$

$$=(-1)^p\left(\frac{ah(x)+b}{a}\right)^p\frac{C_{s-1}}{C_{r-1}(s-r-1)!\,(m+1)^{s-r}}\int_0^1(1-t^{\frac{1}{c(m+1)}})^p(1-t)^{s-r-1}t^{\frac{\gamma_s}{m+1}-1}dt. \tag{21}$$

Expanding $(1-t^{\frac{1}{c(m+1)}})^p$ binomially in (21), we get

$$=(-1)^p\left(\frac{ah(x)+b}{a}\right)^p\frac{C_{s-1}}{C_{r-1}(s-r-1)!\,(m+1)^{s-r}}\sum_{i=0}^{p}(-1)^i\binom{p}{i}\int_0^1(1-t)^{s-r-1}t^{\frac{\gamma_s+\frac{i}{c}}{m+1}-1}dt.$$

$$= (-1)^p \left(\frac{ah(x)+b}{a}\right)^p \frac{C_{s-1}}{C_{r-1}(s-r-1)!\,(m+1)^{s-r}} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \beta\left(s-r, \frac{\gamma_s + \frac{i}{c}}{m+1}\right). \tag{22}$$

Simplification of (22) leads to necessary part.

Now to prove sufficiency part, let

$$E[\{h(X(s,n,m,k)) - h(X(r,n,m,k))\}^p | X(r,n,m,k) = x] = \xi_{r,s,p}(x)$$

or

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \int_x^\beta (h(y)-h(x))^p \left[[\bar{F}(x)]^{m+1} - [\bar{F}(y)]^{m+1}\right]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y)dy$$

$$= \xi_{r,s,p}(x)\,[\bar{F}(x)]^{\gamma_{r+1}}. \tag{23}$$

Differentiating (23) w.r.t x and rearranging the terms, we get

$$\frac{f(x)}{\bar{F}(x)} = \frac{1}{\gamma_{r+1}} \frac{\xi'_{r,s,p}(x) + p\,h'(x)\xi_{r,s,p-1}(x)}{[\xi_{r,s,p}(x) - \xi_{r+1,s,p}(x)]} \tag{24}$$

or

$$\frac{f(x)}{\bar{F}(x)} = \frac{1}{\gamma_{r+1}} \frac{p\,h'(x)\left(\frac{a\,h(x)+b}{a}\right)^{p-1}}{(-1)^p\left(\frac{a\,h(x)+b}{a}\right)^p} \frac{\left[(-1)^p\sum_{i=0}^p(-1)^i\binom{p}{i}\prod_{j=r+1}^s\left(\frac{c\gamma_j}{i+c\gamma_j}\right) + (-1)^{p-1}\sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}\prod_{j=r+1}^s\left(\frac{c\gamma_j}{i+c\gamma_j}\right)\right]}{\left[\sum_{i=0}^p(-1)^i\binom{p}{i}\prod_{j=r+1}^s\left(\frac{c\gamma_j}{i+c\gamma_j}\right) - \sum_{i=0}^p(-1)^i\binom{p}{i}\prod_{j=r+2}^s\left(\frac{c\gamma_j}{i+c\gamma_j}\right)\right]}$$

or

$$\frac{f(x)}{\bar{F}(x)} = \frac{1}{\gamma_{r+1}} \frac{a\,h'(x)}{(a\,h(x)+b)} \frac{p\sum_{i=0}^p(-1)^i\binom{p}{i}\prod_{j=r+1}^s\left(\frac{c\,\gamma_j}{i+c\,\gamma_j}\right) - \sum_{i=0}^p(-1)^i\binom{p}{i}(p-i)\prod_{j=r+1}^s\left(\frac{c\,\gamma_j}{i+c\,\gamma_j}\right)}{\sum_{i=0}^p(-1)^i\binom{p}{i}\prod_{j=r+1}^s\left(\frac{c\,\gamma_j}{i+c\,\gamma_j}\right)\left(1 - \frac{i+c\,\gamma_{r+1}}{c\,\gamma_{r+1}}\right)}$$

$$\frac{f(x)}{\bar{F}(x)} = -\frac{a\,c\,h'(x)}{(a\,h(x)+b)} \frac{\sum_{i=0}^p(-1)^i\binom{p}{i}\,i\,\prod_{j=r+1}^s\left(\frac{c\,\gamma_j}{i+c\,\gamma_j}\right)}{\sum_{i=0}^p(-1)^i\binom{p}{i}\,i\,\prod_{j=r+1}^s\left(\frac{c\,\gamma_j}{i+c\,\gamma_j}\right)}$$

$$= -\frac{a\,c\,h'(x)}{(a\,h(x)+b)}.$$

Hence (18).

Remark 2.5. At $m=0$ and $k=1$, (17) reduces for order statistics

$$E[\{h(X_{s:n}) - h(X_{r:n})\}^p | X_{r:n} = x] = (-1)^p\left(\frac{ah(x)+b}{a}\right)^p \sum_{i=0}^p (-1)^i \binom{p}{i} \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)$$

if and only if $\quad \bar{F}(x) = [ah(x)+b]^c, \qquad x \in (\alpha,\beta).$

Further at $p=1$, we have

$$E[h(X_{s:n})|X_{r:n} = x] = a^*h(x) + b^*$$

if and only if $\quad \bar{F}(x) = [ah(x)+b]^c, \qquad x \in (\alpha,\beta).$

where, $a^* = \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)$ and $b^* = -\frac{b}{a}(1-a^*)$,

as obtained by Khan and Abouammoh [9].

Remark 2.6. If $m=-1$ and $k=1$, then (17) becomes the case for record values

$$E[\{h(X_{U(s)}) - h(X_{U(r)})\}^p | X_{U(r)} = x] = \left(\frac{ah(x)+b}{a}\right)^p \sum_{i=0}^p (-1)^{i+p}\binom{p}{i}\left(\frac{c}{i+c}\right)^{s-r} \tag{25}$$

if and only if $\quad \bar{F}(x) = [ah(x)+b]^c, \qquad x \in (\alpha,\beta).$

Further, at $p=1$, (25) reduces to

$$E[\{h(X_{U(s)})-h(X_{U(r)})\}|X_{U(r)}=x]=-\left(\frac{a\,h(x)+b}{a}\right)\sum_{i=0}^{1}(-1)^i\binom{1}{i}\prod_{j=r+1}^{s}\left(\frac{c}{i+c}\right) \tag{26}$$

if and only if $\quad \bar{F}(x)=[ah(x)+b]^c, \qquad x\in(\alpha,\beta).$

Expression (26) may also be written as

$$E[\{h(X_{U(s)})\}|X_{U(r)}=x]=a^*h(x)+b^* \tag{27}$$

if and only if $\quad \bar{F}(x)=[ah(x)+b]^c, \qquad x\in(\alpha,\beta).$

where, $a^*=\prod_{j=r+1}^{s}\left(\frac{c}{c+1}\right)$ and $b^*=-\frac{b}{a}(1-a^*)$.

Similar results were obtained by Noor and Athar [15] and Khan and Alzaid [11].

With proper choice of a,b,c and $h(x)$ several distributions are characterized. One may refer Khan and Abu-Salih [10] and Khan and Abouammoh [9].

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