

Hot duplication versus survivor equivalence in Gamma-Weibull distribution

Tibor K. Pogány, Vinko Tomas, Mato Tudor

Faculty of Maritime Studies, University of Rijeka, 51000 Rijeka, Studentska 2, Croatia

Received: 14 Jul 2012, Revised: 25 Oct. 2012, Accepted: 27 Oct. 2012

Published online: 1 Mar. 2013

Abstract: The reliability of composite system (series, parallel) is improved by (i) reduction method, and by (ii) hot duplication, considering the systems survivor function. Related survivor equivalence functions and pointwise survivor equivalence factors are derived in all cases when the components lifetime distribution follow the gamma–Weibull distribution introduced recently by Leipnik and Pearce.

Keywords: Composite series/parallel system, Hot duplication method, Reduction method, Reliability function, Survivor equivalence function, Survivor function, gamma–Weibull distribution, Incomplete Fox–Wright Ψ –function.

1 Introduction

The *reliability equivalence* has been introduced by Råde [6], who developed this concept to improve the reliability of various systems [7]. Following Sarhan [9] and Xia & Zhang [14], the reliability equivalence factor (REF) is a *factor by which the failure rates of some of the system's components should be reduced in order to reach equality of the reliability of another better system*.

Detailed account and various generalizations of Råde's ideas can be find in Sarhan's articles [8–10]. He studied among others the reliability of composite, i.i.d. series/parallel systems by the failure rates decreasing, by hot and cold duplication. He also consider parameter estimation in composite systems and related questions (mainly when the life distribution of components is exponential), see Sarhan's cited articles and the references therein.

Råde discussed three different methods to improve the systems reliability: **1.** Improving the quality of $r \leq n$ components by decreasing their hazard rates; **2.** adding a hot component to the system, and **3.** adding a cold (redundant) component to the system [6], [7]. However, Sarhan [8] introduced more general methods in systems reliability improvement: **a)** modifying **1.** by introducing a factor $\rho \in (0, 1)$; **b)** assuming cold redundant standby components connected with some components by random switches. All mentioned results concern components of exponential life distributions. The only exceptions are the work by Xia and Zhang [14], where the case of the parallel system of gamma–(life) distributed components has been discussed and the recent paper [13] in which the modified Weibull distribution's parameters have been estimated; these estimation results could be help in system identification questions, consult also [10] and the references therein.

It is well–known that the hazard rate (or otherwords failure rate) is constant only for exponential life distribution; the gamma–distribution has a functional hazard rate [14]. So, Sarhan's n –component parallel system results are generalized in [14] by taking the more general gamma–distribution. In the same time Xia and Zhang unified the concept of REF, where now F means *function* instead of earlier *factor*. To get more general and substantially simpler approach we introduce the *survivor equivalence function*.

Definition 1. *The survivor equivalence function (SEF) is a function by which the survivor function of the considered system has to be multiplied in order to reach pointwise equality of the survivor function of another better system.*

* Corresponding author e-mail: poganj@pfri.hr (T.K. Pogány)

In this article we obtain the SEF in general case, when each components life distribution is described by a random variable (r.v.) ξ defined on a fixed probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, with the *cumulative distribution function* (CDF)

$$F(x) = \mathbb{P}\{\xi < x\}, \quad x \in \mathbb{R},$$

and the *probability distribution function* (PDF)

$$f(x) = F'(x),$$

such that exists (in general) for absolutely continuous CDF. The related *reliability function* one defines by

$$R(x) = 1 - F(x), \quad x \in \mathbb{R}.$$

We study the composite systems **(i)** with *independent identically distributed components* (IIDC) in series connected (S), and **(ii)** composite system (P) which components are connected in parallel. Now, denote $R_j(x)$ the reliability function of the j th component in a composite system and define the systems survivor function as its reliability function. We have

$$S_S(x) = \prod_{j=1}^n R_j(x), \quad (1)$$

$$S_P(x) = 1 - \prod_{j=1}^n [1 - R_j(x)], \quad (2)$$

where S_S, S_P denote the survivor functions of the series and the parallel systems respectively.

The number of changed components has been calculated by the criteria of improved reliability when the reduction method is combined with hot duplication method. It is shown in section 2 that for the series composite systems *the number of components treated by reduction method has to be equal to the number of hot duplicated components*. Moreover, parallel systems work without such limitations.

2 Reduction and Hot Duplication

Let us improve the reliability $R(x)$ of a component by some intervention, calling this procedure reduction method; introducing a reduction factor $\rho \in (0, 1)$, the improved components reliability will be

$$R(\rho x) \geq R(x).$$

Applying the reduction method to any r , $1 \leq r \leq n$ components in (S), (P), we get improved composite systems $(S_r), (P_r)$; being R monotone nonincreasing, the concept needs only technological support, the introduced mathematical model is well defined.

The related survivor functions are

$$S_{S_r}^\rho(x) = [R(\rho x)]^r [R(x)]^{n-r}, \quad (3)$$

$$S_{P_r}^\rho(x) = 1 - [1 - R(\rho x)]^r [1 - R(x)]^{n-r}. \quad (4)$$

According to Definition 1 multiplying the original survivor functions $S_S(x), S_P(x)$ by SEFs $r_{S_r}^\rho(x), r_{P_r}^\rho(x)$ respectively, we reduce r arguments of R_j 's to $\rho x, \rho \in (0, 1)$ in (3) and (4). Hence

$$r_{S_r}^\rho(x) S_S(x) = S_{S_r}^\rho(x),$$

$$r_{P_r}^\rho(x) S_P(x) = S_{P_r}^\rho(x),$$

that is

$$r_{S_r}^\rho(x) = \left[\frac{R(\rho x)}{R(x)} \right]^r,$$

$$r_{P_r}^\rho(x) = \frac{1 - [1 - R(\rho x)]^r [1 - R(x)]^{n-r}}{1 - [1 - R(x)]^n}.$$

Prescribing $\rho \in (0, 1)$ and $r \in \{1, \dots, n\}$ we can work with exact SEF functions associated with the reduction method.

Now, assume that $q, 1 \leq q \leq n$ components of the considered systems (S), (P) are hot duplicated, that is q components has been in parallel switched with another new IIDC. This procedure results in transformed systems $(S_q^H), (P_q^H)$ such that are composed now by $n + q$ independent identical components. The reliability of a pair of hot duplicated components' has reliability function equal to

$$1 - [1 - R(x)]^2 = R(x)[2 - R(x)],$$

therefore the related survivor functions become

$$\begin{aligned} S_{S_q^H}(x) &= [R(x)]^n [2 - R(x)]^q, \\ S_{P_q^H}(x) &= 1 - [1 - R(x)]^{n+q}. \end{aligned} \tag{5}$$

According to Definition 1, one obtains the pointwise SEF in the form

$$\begin{aligned} r_{S_r}(x)S_S(x) &= S_{S_r}^{\rho}(x) = [R(\rho x)]^r [R(x)]^{n-r} \\ &= S_{S_q^H}^{\rho}(x) = [R(x)]^n [2 - R(x)]^q, \end{aligned}$$

that is, we get the following equation in ρ :

$$R(\rho x) = R(x) [2 - R(x)]^{q/r}. \tag{6}$$

Now, the following crucial question arises:

- Has to be the ratio q/r of q hot duplicated components and r reduced reliability components in a series composite system (S) bounded?

It can be easily shown that $q = r$. Indeed, being probability of some random event, the right-hand expression in (6) has to be less than or equal to 1. So, considering the function $f(x) = x(2 - x)^{q/r}$ on the range $x \in [0, 1]$, we conclude that

$$\max_{0 \leq x \leq 1} f(x) = f\left(\frac{2}{1 + q/r}\right) = \frac{2}{1 + q/r} \left(\frac{2}{1 + r/q}\right)^{q/r}.$$

Obviously, the last expression's minimum equals 1 at $q = r, q, r \in \{0, 1, \dots, n\}$, else (6) is senseless. On the other hand, the reliability function R is of bounded variation ($R(-\infty) = 1 \geq R(x) \geq 0 = R(\infty), x \in \mathbb{R}$), monotone nonincreasing ($R(x) \geq R(y)$ whenever $x < y$) and left-continuous ($\lim_{h \rightarrow 0} R(x - h) = R(x)$). Any such function R possesses a so-called *generalized inverse*

$$\mathfrak{R}^*(z) := \inf\{x : R(x) \geq z\}, \quad 0 \leq z < 1.$$

Moreover, if R is strongly monotone decreasing, that is $R(x) > R(y)$ for all $0 < x < y$, then \mathfrak{R}^* coincides with the usual 'ordinary' inverse R^{-1} .

Applying now the generalized inverse operator \mathfrak{R}^* to (6) we deduce the *pointwise hot-duplication reduction factor* for the system (S):

$$\rho_S^H = x^{-1} \mathfrak{R}^*\left(R(x)[2 - R(x)]\right), \quad x > 0. \tag{7}$$

Also, the following question turns out immediately:

- Has to be the ratio q/r of q hot duplicated components and r reduced reliability components in a parallel composite system (P) bounded?

To answer, we derive the equation

$$S_{P_r}^{\rho}(x) = S_{P_q^H}^H(x),$$

such that corresponds to (P). By virtue of (4) and (5) this equation becomes

$$R(\rho x) = 1 - [1 - R(x)]^{1+q/r}.$$

Because

$$1 - [1 - R(x)]^{1+q/r} \leq 1, \quad q, r \in \{0, 1, \dots, n\},$$

no limitation turns out for q/r in the case of (P). Hence, the SEF for parallel system (P) becomes

$$\rho_P^H = x^{-1} \mathfrak{R}^*\left(1 - [1 - R(x)]^{1+q/r}\right), \quad x > 0. \tag{8}$$

Thus, we obtain the following result.

Theorem 1. Assume that r IIDC components of (S) are improved by reduction method, and $q = r$ components (in general not the same ones) are improved by hot duplication in (S) . Then the pointwise hot duplication SEF associated with composite system (S_r) is given by

$$r_{S_r}^{\rho}(x) = \left[\frac{R(\rho_S^H x)}{R(x)} \right]^q$$

where ρ_S^H is described by (7).

Moreover, the pointwise hot-duplication SEF associated with parallel system (P_r) is

$$r_{P_r}^{\rho}(x) = \frac{1 - [1 - R(\rho_P^H x)]^r [1 - R(x)]^{n-r}}{1 - [1 - R(x)]^n}$$

for all $q, r \in \{0, 1, \dots, n\}$. Here ρ_P^H is given by (8).

The strong monotone (increasing) CDF $F(x)$ means strong monotone (decreasing) reliability function $R(x)$. This ensures the existence of the 'ordinary inverse' $R^{-1}(x)$, that is, the generalized inverse \mathfrak{R}^* becomes R^{-1} in (7) and (8). So the following result.

Corollary 1. Assume that the probability distribution function F is strong monotone, i.e. $F(x) < F(y)$ whenever $x < y$. Then

$$r_{S_r}^{\rho}(x) = [2 - R(x)]^q, \quad q = r; q, r \in \{1, \dots, n\}; \quad (9)$$

moreover we have

$$r_{P_r}^{\rho}(x) = \frac{1 - [1 - R(x)]^{n+q}}{1 - [1 - R(x)]^n}. \quad (10)$$

At this point the question of approximate solution of equation (6) arises, when there exists the PDF $f(x) = F'(x)$. Under $q = r$ equation (6) becomes

$$R(\rho x) = R(x) [2 - R(x)]. \quad (11)$$

Applying Maclaurin approximation, that is

$$R(x) = R(0_+) + R'(0_+)x + o(x)$$

as x approaches zero from the right, to both sides to (11). Thus

$$\rho = \frac{R(0_+)(1 - R(0_+))}{R'(0_+)} \frac{1}{x} + (2 - 2R(0_+)) + o(1).$$

Hence, we get the following result.

Corollary 2. For small values of the argument $x > 0$, we have

$$\rho_{S_r}^H = \frac{R(0_+)(1 - R(0_+))}{R'(0_+)} \frac{1}{x} + (2 - 2R(0_+)) + o(1). \quad (12)$$

Endly, to control the behaviour of the reduction factor $\rho_{S_r}^H$ near to the origin we make the common sense assumption $R(0_+) = 1$. Hence, we arrive at

$$\rho_{S_r}^H = o(1),$$

that is, near to the origin $\rho_{S_r}^H \approx 0$. Also we point out that if $R(x)$ is continuously differentiable at some $x = a > 0$, we can easily extend (12) easily by the Taylor series expansion to the neighborhood of a , leaving this to the interested reader.

3 Gamma–Weibull distribution and related survivor functions

Leipnik and Pearce [3] introduced recently the so-called *gamma–Weibull distribution* (gW); in fact, they renormalize the multiplied densities of the gamma– and the Weibull distributions. Nadarajah and Kotz [4] pointed out that it is enough to take four parameters to define the $gW(\theta)$ distribution having (PDF)

$$f_{gW}(x) = \begin{cases} Kx^{\alpha-1}e^{-\mu x-ax^\kappa} & x > 0 \\ 0 & x \leq 0 \end{cases}, \tag{13}$$

where $\theta := (\alpha, \mu, a, \kappa) > 0$ and K stands for the normalizing constant. So, in this case the r.v. ξ is said to have $gW(\theta)$

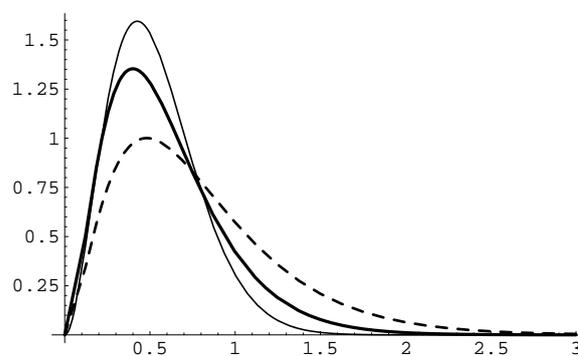


Fig. 1 Different patterns of the gamma–Weibull probability density function $f_{gW}(x)$, with $\alpha = 3, \mu = 3, a = 2; \kappa = 0.363$ dashed line, $\kappa = 1$ solid line and $\kappa = 2$ thin solid line.

distribution, such that we write $\xi \sim gW(\theta)$. Because the linear and the fractional power parts in the exponent and the polynomial factor of the PDF it is very flexible for different modeling purposes. So, besides the straightforward cases when $gW(\theta)$ covers the gamma– and the Weibull distributions, the case $\theta = (1, K, 0, \kappa)$ stands for the exponential $\mathcal{E}(K)$ distribution with parameter $K > 0$ and $\theta = (2, 0, K/2, 2)$ gives the celebrated Rayleigh distribution’s PDF among others.

The normalizing constant K and the reliability function related to the r.v. $\xi \sim gW(\theta)$, such that reads

$$R_{gW}(x) := \begin{cases} 1 & x \leq 0 \\ \int_x^\infty f_{gW}(t) dt & x > 0 \end{cases},$$

are expressed in terms of the so-called *Upper incomplete confluent Fox–Wright generalized hypergeometric function* ${}_1\Psi_0$, which has been introduced recently by Srivastava and Pogány [11]. Here we give in brief the main properties of this higher transcendental function.

The Fox–Wright generalization ${}_p\Psi_q$ of the generalized hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters is the series

$${}_p\Psi_q \left[\begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \middle| x \right] := \sum_{m=0}^\infty \frac{\prod_{j=1}^p \Gamma(a_j + \alpha_j m)}{\prod_{j=1}^q \Gamma(b_j + \beta_j m)} \frac{x^m}{m!}, \tag{14}$$

where (u_r, Υ_r) stands for the parameter r -tuple $(u_1, \Upsilon_1), \dots, (u_r, \Upsilon_r)$; $a_j \in \mathbb{C}, b_k \in \mathbb{C} \setminus \mathbb{Z}_0^-, \alpha_j, \beta_k > 0, j = \overline{1, p}, k = \overline{1, q}$ and the empty product means unity. The convergence of the above series is ensured when

$$\Delta := 1 + \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j > 0 \tag{15}$$

for suitably bounded values of $|x|$. In the case $\Delta = 0$, the convergence holds in the open disc

$$|z| < \varepsilon = \prod_{j=1}^q \beta_j^{\beta_j} \cdot \prod_{j=1}^p \alpha_j^{-\alpha_j},$$

see e.g. [12].

The *Upper incomplete gamma-function* [1, **8.350 2.**] we define as the integral

$$\Gamma(s, z) := \int_z^{\infty} t^{s-1} e^{-t} dt; \quad \lim_{z \rightarrow 0} \Gamma(s, z) = \Gamma(s).$$

Replacing in series expansion (14) all Gamma-function terms by upper incomplete Gamma-function terms with identical second variable z , we get the *Upper Incomplete Fox-Wright Psi-Function* [11]:

$${}_p\Psi_q \left[\begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \middle| \Gamma(x, z) \right] := \sum_{m=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(a_j + \alpha_j m, z)}{\prod_{j=1}^q \Gamma(b_j + \beta_j m, z)} \frac{x^m}{m!}.$$

for all $z \geq 0$ having on mind the parameter constraint (15).

We point out that the Fox-Wright function can be easily evaluated by in-built routines for hypergeometric functions, e.g. *Mathematica's* Gamma $[a, z]$.

Now, the normalizing constant $K = K(\theta)$ of the PDF (13) is [5, Eq. (9)]

$$K^{-1} = \begin{cases} \mu^{-\alpha} {}_1\Psi_0 \left[\begin{matrix} (\alpha, \kappa) \\ - \end{matrix} \middle| -\frac{a}{\mu^\kappa} \right] & 0 < \kappa < 1 \\ \frac{\Gamma(\alpha)}{(\mu+a)^\alpha} & \kappa = 1 \\ \frac{1}{\kappa a^{1/\kappa}} {}_1\Psi_0 \left[\begin{matrix} (\frac{\alpha}{\kappa}, \frac{1}{\kappa}) \\ - \end{matrix} \middle| -\frac{\mu}{a^{1/\kappa}} \right] & \kappa > 1 \end{cases},$$

where ${}_1\Psi_0[\cdot]$ is the *confluent Fox-Wright Psi-function*.

The reliability function $R_{gW}(x)$ of a component behaved according to the $gW(\theta)$ life-distribution becomes

$$R_{gW}(x) = \begin{cases} \frac{{}_1\Psi_0 \left[\begin{matrix} (\alpha, \kappa) \\ - \end{matrix} \middle| \Gamma(-a\mu^{-\kappa}, \mu x) \right]}{{}_1\Psi_0 \left[\begin{matrix} (\alpha, \kappa) \\ - \end{matrix} \middle| -\frac{a}{\mu^\kappa} \right]} \\ \frac{\Gamma(\alpha, (\mu+a)x)}{\Gamma(\alpha)} \\ \frac{{}_1\Psi_0 \left[\begin{matrix} (\frac{\alpha}{\kappa}, \frac{1}{\kappa}) \\ - \end{matrix} \middle| \Gamma(-\mu a^{-1/\kappa}, ax^\kappa) \right]}{{}_1\Psi_0 \left[\begin{matrix} (\frac{\alpha}{\kappa}, \frac{1}{\kappa}) \\ - \end{matrix} \middle| -\frac{\mu}{a^{1/\kappa}} \right]} \end{cases} \quad (16)$$

where the cases appear for $0 < \kappa < 1$, $\kappa = 1$, $\kappa > 1$ respectively for any $x > 0$, while for $x \leq 0$ it is $R_{gW}(x) \equiv 1$, see Fig 2. Here ${}_1\Psi_0$ denotes the *confluent upper incomplete Fox-Wright Psi-function*.

Indeed, assume $\kappa > 1$. Then we have

$$R_{gW}(x) = K \int_x^\infty t^{\alpha-1} e^{-\mu t - at^\kappa} dt \tag{17}$$

$$= K \sum_{n=0}^\infty \frac{(-\mu)^n}{n!} \int_x^\infty t^{\alpha+n-1} \exp\{-at^\kappa\} dt \tag{18}$$

$$= \frac{K}{\kappa a^{\frac{\alpha+1}{\kappa}}} \sum_{n=0}^\infty \frac{\left(-\frac{\mu}{a^{\frac{1}{\kappa}}}\right)^n}{n!} \int_{ax^\kappa}^\infty y^{\frac{\alpha+n}{\kappa}-1} e^{-y} dy \tag{19}$$

$$= \frac{K}{\kappa a^{\frac{\alpha+1}{\kappa}}} \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\alpha+n}{\kappa}, ax^\kappa\right)}{n!} \left(-\frac{\mu}{a^{\frac{1}{\kappa}}}\right)^n, \tag{20}$$

such that proves (16). In the case $\kappa \in (0, 1)$ one expands the term e^{-ax^κ} into the Maclaurin series and then integrate termwise. Finally, $\kappa = 1$ results in the Gamma distribution.

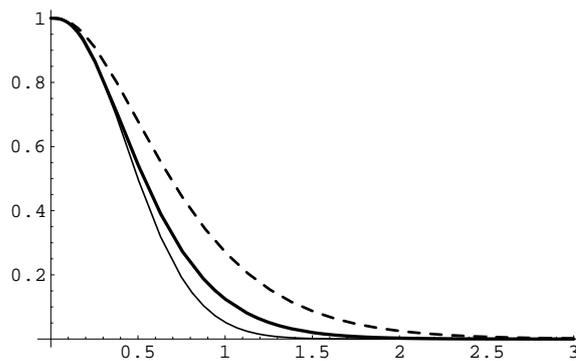


Fig. 2 Different patters of the gamma–Weibull reliability functions $R_{gW}(x)$ with $\alpha = 3, \mu = 3, a = 2$; $\kappa = 0.363$ dashed line, $\kappa = 1$ solid line and $\kappa = 2$ thin solid line.

Theorem 2. Let us consider (S),(P) consisting from n IIDC such that have $gW(\theta)$ life–distributions. Then the related survivor functions have the form

$$S_{gW,S}(x) = [R_{gW}(x)]^n$$

$$S_{gW,P}(x) = 1 - [1 - R_{gW}(x)]^n,$$

where $R_{gW}(x)$ is displayed in (16).

Proof. By virtue of (1) we build the survivor functions of systems (S),(P) applying n i.i.d. replicæ of a r.v. $\xi \sim gW(\theta)$ such that describes the life–distribution of all involved components.

Now, we obtain the pointwise SEF r^H and survivor equivalence factors ρ^H for both - series and parallel composite systems. The proof of all relations can be realized combining Corollary 1 and Theorem 2.

Theorem 3. Suppose that n IID components having $gW(\theta), \theta = (\alpha, \mu, a, \kappa) > 0$ life distribution, are connected in series forming a composite system (S_r), and connected in parallel to form a composite system (P_r) in which r components have been improved by reduction method. Improving the pointwise reliability of $1 \leq q \leq n$ components by hot–duplication, the

related pointwise SEF $r_A^H(x)$ and the related pointwise survivor equivalence factors ρ_A^H , $A \in \{S, P\}$ become

$$r_{S_r}^H(x) = [2 - R_{gW}(x)]^q, \quad (21)$$

$$\rho_S^H = x^{-1} R_{gW}^{-1} \left(R_{gW}(x) [2 - R_{gW}(x)] \right); \quad (22)$$

$$r_{P_r}^H(x) = \frac{1 - [1 - R_{gW}(x)]^{n+q}}{1 - [1 - R_{gW}(x)]^n}, \quad (23)$$

$$\rho_P^H = x^{-1} R_{gW}^{-1} \left(1 - [1 - R_{gW}(x)]^{1+q/r} \right). \quad (24)$$

when $q = r$ for (S), however no restrictions appear upon q/r for (P); $R_{gW}(x)$ is described in (16) and R_{gW}^{-1} is the inverse of R_{gW} .

4 Numerical results and conclusion

To illustrate how the theory, which were obtained in the previous sections, can be applied, three different parameter cases are presented in this section.

The $gW(\theta)$ lifetime–distribution's PDF takes three analytically different forms depending on the κ , compare Fig 1. So do the associated reliability functions as illustrates Fig 2. *via* (16). Therefore we decide to study the PDF (13) when $(\alpha, \mu, a) = (3, 3, 2)$ and $\kappa \in \{0.363, 1, 2\}$ as shown in Fig 1.

Table 1 Numerical simulation results for $\theta = (3, 3, 2, 0.363)$.

x	R_{gW}	$S_{gW,S}$	$S_{gW,P}$	ρ_S^H	ρ_P^H	$r_{S_3}^H$	$r_{P_2}^H$
0.1	0.98947	0.91878	1.00000	0.1850	0.0900	1.03193	1.00000
0.4	0.77863	0.13509	0.99999	0.4758	0.3440	1.82199	1.00053
0.7	0.48882	0.00326	0.99534	0.6293	0.5200	3.45102	1.03133
1.0	0.27098	0.00003	0.92022	0.7234	0.6367	5.16891	1.15884
1.3	0.13941	$1.43 \cdot 10^{-7}$	0.69913	0.7852	0.7160	6.44099	1.32423
1.6	0.06829	$4.73 \cdot 10^{-10}$	0.43216	0.8274	0.7713	7.20813	1.45063

Table 2 Numerical simulation results for $\theta = (3, 3, 2, 1)$.

x	R_{gW}	$S_{gW,S}$	$S_{gW,P}$	ρ_S^H	ρ_P^H	$r_{S_3}^H$	$r_{P_2}^H$
0.1	0.98561	0.89053	1.00000	0.2220	0.1150	1.04379	1.00000
0.4	0.67668	0.04396	0.99988	0.5623	0.4395	2.31738	1.00343
0.7	0.32085	0.00011	0.95474	0.7199	0.6316	4.73446	1.11599
1.0	0.12465	$5.83 \cdot 10^{-8}$	0.65530	0.8060	0.7428	6.59547	1.34815
1.3	0.04304	$1.18 \cdot 10^{-11}$	0.29667	0.8567	0.8098	7.49460	1.50254
1.6	0.01375	$1.28 \cdot 10^{-15}$	0.10488	0.8885	0.8520	7.83608	1.56759

Table 3 Numerical simulation results for $\theta = (3, 3, 2, 2)$.

x	R_{gW}	$S_{gW,S}$	$S_{gW,P}$	ρ_S^H	ρ_P^H	$r_{S_3}^H$	$r_{P_2}^H$
0.1	0.98805	0.90829	1.00000	0.2150	0.1020	1.03628	1.00000
0.4	0.65761	0.03497	0.99981	0.6041	0.4585	2.41900	1.00454
0.7	0.23936	0.00001	0.88794	0.7895	0.7183	5.45773	1.19127
1.0	0.05219	$5.50 \cdot 10^{-11}$	0.34872	0.8799	0.8390	7.38992	1.48329
1.3	0.00724	$7.52 \cdot 10^{-18}$	0.05644	0.9250	0.8998	7.91348	1.58280
1.6	0.00066	$3.56 \cdot 10^{-26}$	0.00526	0.9490	0.9320	7.99209	1.59842

Assume that (S),(P) consist from $n = 8$ IID components having $gW(\theta)$ lifetime–distributions, while improving $r = 3, q = 2$ components by reduction method we get $(S_3), (P_2)$ respectively. These systems are now treated by hot duplication. According to our findings (see Theorem 1) exactly 3 components have to be improved by hot duplication in (S); and we have no such limitation for (P).

The numerical simulation results include three cases: $\kappa \in (0, 1), \kappa = 1, \kappa > 1$, presented on Table I, II and III respectively. The tables contain six sampled values of the reliability function $R_{gW}(x)$ of a component; the survivor functions $S_{gW,S}(x), S_{gW,P}(x)$ of series and parallel systems respectively; the survivor equivalence factors ρ_S^H, ρ_P^H and the SEFs $r_{S_r}^H(x), r_{P_r}^H(x)$ all under the same number of reduction–improved components $r = 3, q = 2$. The sample nodes $x = 0.1, 0.4, 0.7, 1.0, 1.3, 1.6$ are used in all cases (by comparison purposes). All these functional characteristics we calculate by (16), Theorems 2 and 3.

As we reported most of recent articles are devoted to various topology composite systems, having preferably exponential $\mathcal{E}(\lambda)$ lifetime distribution, see [8], [9], [10] and the references therein. The Gamma–life–distributed components case is discussed [14] and [13] gives attention to Weibull–distributed components. Although, we generalize these findings introducing the gamma–Weibull $gW(\theta)$ lifetime–distribution for components. Also, there is a considerable interest to extend the here exposed results to another lifetime–distributions coming from the so–called Inverse Weibull distribution family and allied distributions studied recently by Khan and King, see [2] and the references therein.

The SEF study precises the survivor functions for both models’ (S),(P) by Theorems 2 and 3 and the reliability function expression (16) for the gamma–Weibull distribution $gW(\theta)$.

Moreover, we conclude that the hot duplication can be successfully replaced by reliability reduction method using *only the same number of improved components* for (S), while the reduction method is *completely independent from hot duplication* for the parallel systems P), such that clearly expose Corollary 1 by virtue of (9) and (10) respectively. These findings we illustrate on series/parallel composite systems having life–distribution $gW(\theta)$.

Acknowledgement

The present investigation was supported in part by the *Ministry of Science, Education and Sports of Croatia* under Research Project Number 112-1121722-3314.

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