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Identities for Certain Products of Theta Functions with Applications to Modular Relations

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Abstract: In this paper, we derive general formulas to express the product of two theta functions as linear combinations of other products of theta functions. Several known theta function identities follow immediately from our formulas. In fact, we derive many identities for certain products of special cases of theta functions defined by Ramanujan and also we establish several modular relations for Rogers-Ramanujan functions, septic, nonic and dodecic analogues of the Rogers-Ramanujan functions.

Keywords: Rogers-Ramanujan functions, theta function, Jacobi triple product identity, modular relations.

1 Introduction

Ramanujan's forty identities motivated many mathematicians to find general formulas to express the product of theta functions as linear combinations of other products of theta functions. For example, L. J. Rogers [28] has found some general identities based on Schröter-type theta function identities. This approach has been generalized by many mathematicians including D. Bressoud [10], [11], H. Yesilyurt [33] and more recently L.-C. Zhang [34] slightly generalized this approach but unfortunately was not able to find any new examples to illustrate any of his general theorems. S.-L. Chen and S.-S. Huang [14] derived two theorems for certain products of theta functions using G. Watson's technique [31] which was used to establish some of the Ramanujan's forty identities. Q. Yan [32] has used the technique of L. Carlitz and M. V. Subbarao [13], to establish a general identity for expanding the product of two Jacobi's triple products found in [16]. Recently, Z. Cao [12] gave a general theorem to write a product of n theta functions as linear combinations of other products of theta functions. Cao showed that many known identities for products of theta functions are special cases of his main theorem.

The purpose of the present paper is to derive new general formulas for the product of two theta functions as linear combinations of other products of theta functions. To establish our main results namely the four identities in Theorem 1, we use the idea of Watson [31] which he has used to prove some of the Ramanujan's forty identities. Many identities in the literature and also some new identities follow as special cases of our main theorem. For example, in Section 3, we find many identities for the products of theta functions $\varphi(q)$, $\psi(q)$, $f(-q)$ and $f_*(q)$ defined in Section 2, and we have recovered many identities established in [12], [14] and [32]. Many known identities for products of two theta functions found in Chapter 16 [3] also follow as special cases of our identities. In Section 4, we use our identities to establish several modular relations for the Rogers-Ramanujan functions established by Ramanujan [24]. We also deduce some identities involving septic analogues of the Rogers-Ramanujan functions obtained by Hahn [20] using Bressoud's approach and some new modular relations for these functions. Similar things have been done for the nonic [6] and dodecic analogues [8] of the Rogers-Ramanujan functions.

2 Some Definitions and Preliminary Results

Throughout the paper, we use the customary notation $(a;q)_0 := 1$,

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geq 1,$$

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$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \qquad |q|<1,$$

$$(a_1,a_2,\ldots,a_n;q)_\infty := \prod_{i=1}^{n}(a_i;q)_\infty.$$

Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab|<1. \tag{1}$$

The Jacobi triple product identity [3, Entry 19] in Ramanujan's notation is

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2}$$

Using (1), we define

$$f_\delta(a,b) = \begin{cases} f(a,b) & \text{if } \delta\equiv 0 \pmod 2,\\ f(-a,-b) & \text{if } \delta\equiv 1 \pmod 2.\end{cases}$$

The function $f(a,b)$ satisfies the following basic properties [3]:

$$f(a,b) = f(b,a), \tag{3}$$

$$f(1,a) = 2f(a,a^3), \tag{4}$$

$$f(-1,a) = 0, \tag{5}$$

and, if n is an integer,

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}). \tag{6}$$

Ramanujan has defined the following three special cases of (1) [3, Entry 22]:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{7}$$

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{n(2n+1)} = \sum_{n=-\infty}^{\infty} q^{n(2n-1)}, \tag{8}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2} = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}. \tag{9}$$

Also, after Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty.$$

We define

$$f_*(q) := f(q,q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}. \tag{10}$$

The following identity is an easy consequence of Entry 31 [3] when $n=2$:

$$f(a,b) = f(a^3b,ab^3) + af(b/a,a^5b^3). \tag{11}$$

The following identity can be found in [3, Entry 30(iv)]

$$f(a,b)f(-a,-b) = f(-a^2,-b^2)\varphi(-ab). \tag{12}$$

The famous quintuple product identity, in Ramanujan's notation [3, Entry 28(iv)] is

$$\frac{f(-a^2,-a^{-2}q)}{f(-a,-a^{-1}q)} = \frac{1}{f(-q)}\left\{f(-a^3q,-a^{-3}q^2) + af(-a^{-3}q,-a^3q^2)\right\}. \tag{13}$$

For convenience, we define

$$f_n := f(-q^n) = (q^n;q^n)_\infty,$$

for a positive integer n.
The following lemma is a consequence of (2) and Entry 24 of [3].

Lemma 1. *We have*

$$\varphi(q) = \frac{f_2^5}{f_1^2f_4^2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad f(q) = \frac{f_2^3}{f_1f_4},$$

$$\varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(-q) = \frac{f_1f_4}{f_2}, \qquad \chi(q) = \frac{f_2^2}{f_1f_4},$$

$$\chi(-q) = \frac{f_1}{f_2}, \qquad f(-q,-q^5) = \frac{f_1f_6^2}{f_2f_3},$$

$$f(q,q^5) = \frac{f_2^2f_3f_{12}}{f_1f_4f_6} \quad \text{and} \quad f_*(q) = \frac{f_2f_3^2}{f_1f_6}.$$

We use the following lemma very often in this paper:

Lemma 2. *Let* $m = \left[\frac{s}{s-r}\right]$, $l = m(s-r)-r$, $k = -m(s-r)+s$ *and* $h = mr - \frac{m(m-1)(s-r)}{2}$, $0\le r<s$. *Here* $[x]$ *denote the largest integer less than or equal to* x. *Then,*
(i) $f(q^{-r},q^s) = q^{-h}f(q^l,q^k)$,
(ii) $f(-q^{-r},-q^s) = (-1)^m q^{-h} f(-q^l,-q^k)$.

For a proof of Lemma 2, see [2].

3 Main Results

In this section, we prove the following main theorem by a method similar to the one used by Watson [31] to prove some of Ramanujan's identities. As special cases of our theorem, we recover many theorems found in [12], [14] and [32].

Theorem 1. *For any integers g, h, u, v, l and $k>1$ with $g+h=S_1>0$, $g-h=D_1\geq 0$, $u+v=S_2>0$, $u-v=D_2\geq 0$, and $S_1=l(k-l)S_2$, we have*

$$\begin{aligned}
& f(\varepsilon_1 q^g,\varepsilon_1 q^h)f(\varepsilon_2 q^u,\varepsilon_2 q^v) = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{(S_2a^2+aD_2)/2} \\
& \times f_{\delta_1+\delta_2(k-l)}\left(q^{\{(kS_2+D_2)(k-l)+D_1+2aS_2(k-l)\}/2}, q^{\{(kS_2-D_2)(k-l)-D_1-2aS_2(k-l)\}/2}\right) \\
& \times f_{\delta_1+\delta_2 l}\left(q^{\{S_2lk+2aS_2l-D_1+lD_2\}/2}, q^{\{S_2lk-2aS_2l+D_1-lD_2\}/2}\right) \qquad (14)\\
& = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{(S_2a^2+aD_2)/2} \\
& \times f_{\delta_1+\delta_2(k-l)}\left(q^{\{(kS_2+D_2)(k-l)-D_1+2aS_2(k-l)\}/2}, q^{\{(kS_2-D_2)(k-l)+D_1-2aS_2(k-l)\}/2}\right) \\
& \times f_{\delta_1+\delta_2 l}\left(q^{\{S_2lk+2aS_2l+D_1+lD_2\}/2}, q^{\{S_2lk-2aS_2l-D_1-lD_2\}/2}\right) \qquad (15)\\
& = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{(S_2a^2-aD_2)/2} \\
& \times f_{\delta_1+\delta_2(k-l)}\left(q^{\{(kS_2-D_2)(k-l)-D_1+2aS_2(k-l)\}/2}, q^{\{(kS_2+D_2)(k-l)+D_1-2aS_2(k-l)\}/2}\right) \\
& \times f_{\delta_1+\delta_2 l}\left(q^{\{S_2lk+2aS_2l+D_1-lD_2\}/2}, q^{\{S_2lk-2aS_2l-D_1+lD_2\}/2}\right) \qquad (16)\\
& = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{(S_2a^2-aD_2)/2} \\
& \times f_{\delta_1+\delta_2(k-l)}\left(q^{\{(kS_2-D_2)(k-l)+D_1+2aS_2(k-l)\}/2}, q^{\{(kS_2+D_2)(k-l)-D_1-2aS_2(k-l)\}/2}\right) \\
& \times f_{\delta_1+\delta_2 l}\left(q^{\{S_2lk+2aS_2l-D_1-lD_2\}/2}, q^{\{S_2lk-2aS_2l+D_1+lD_2\}/2}\right). \qquad (17)
\end{aligned}$$

Here $\varepsilon_i\in\{-1,1\}$, $\delta_i=\dfrac{1-\varepsilon_i}{2}$ for $i=1,2$ and $[x]$ denote the greatest integer less than or equal to x.

Proof. Using (1), we have

$$\begin{aligned}
f(\varepsilon_1 q^g,\varepsilon_1 q^h)f(\varepsilon_2 q^u,\varepsilon_2 q^v) &= \sum_{m,n=-\infty}^{\infty} \varepsilon_1^m\varepsilon_2^n q^{(S_1m^2+D_1m+S_2n^2+D_2n)/2} && (18)\\
&= \sum_{m,n=-\infty}^{\infty} \varepsilon_1^m\varepsilon_2^n q^{(S_1m^2-D_1m+S_2n^2+D_2n)/2} && (19)\\
&= \sum_{m,n=-\infty}^{\infty} \varepsilon_1^m\varepsilon_2^n q^{(S_1m^2-D_1m+S_2n^2-D_2n)/2} && (20)\\
&= \sum_{m,n=-\infty}^{\infty} \varepsilon_1^m\varepsilon_2^n q^{(S_1m^2+D_1m+S_2n^2-D_2n)/2}. && (21)
\end{aligned}$$

In these representations, we make the change of indices by setting

$$lm+n=kM+a \quad \text{and} \quad (l-k)m+n=kN+b,$$

where a and b have values selected from the complete set of residues modulo k given by

$$\left\{\left[\frac{2-k}{2}\right],\left[\frac{4-k}{2}\right],\ldots,-1,0,1,\ldots,\left[\frac{k-2}{2}\right],\left[\frac{k}{2}\right]\right\}.$$

Then

$$m = M - N + \frac{(a-b)}{k} \quad \text{and} \quad n = (k-l)M + lN + \frac{(k-l)a+lb}{k}.$$

It follows easily by the above equations that $a = b$, and so $m = M - N$ and $n = (k-l)M + lN + a$, where $\left[\frac{2-k}{2}\right] \leq a \leq \left[\frac{k}{2}\right]$. Thus, there is one-to-one correspondence between the set of all pairs of integers (m,n), $-\infty < m,n < \infty$, and triples of integers (M,N,a), $-\infty < M,N < \infty$, $\left[\frac{2-k}{2}\right] \leq a \leq \left[\frac{k}{2}\right]$. From (18), we find that

$$f(\varepsilon_1 q^g, \varepsilon_1 q^h) f(\varepsilon_2 q^u, \varepsilon_2 q^v) = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} \sum_{M,N=-\infty}^{\infty} \varepsilon_1^{M+N} \varepsilon_2^{(k-l)M+lN+a} q^{(S_1(M-N)^2+D_1(M-N)+S_2((k-l)M+lN+a)^2+D_2((k-l)M+lN+a))/2}. \tag{22}$$

Since $S_1 = l(k-l)S_2$ and $\varepsilon_i = (-1)^{\frac{1-\varepsilon_i}{2}}$, for $i = 1,2$, the above identity can be written as

$$f(\varepsilon_1 q^g, \varepsilon_1 q^h) f(\varepsilon_2 q^u, \varepsilon_2 q^v) = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{\delta_2 a} q^{\frac{S_2 a^2 + aD_2}{2}} \sum_{M=-\infty}^{\infty} (-1)^{(\delta_1+\delta_2(k-l))M} q^{\left((k(k-l)S_2)M^2+(D_1+2a(k-l)S_2+(k-l)D_2)M\right)/2}$$
$$\times \sum_{N=-\infty}^{\infty} (-1)^{(\delta_1+\delta_2 l)N} q^{\left((lkS_2)N^2+(2alS_2-D_1+lD_2)N\right)/2},$$

which is same as (14).
In a similar way, on using (19)-(21), we obtain (15)-(17).

Since $S_1 > 0$, $S_2 > 0$, $k > 1$ and $S_1 = l(k-l)S_2$, one can easily show that $k > l > 0$. Observe that (15) can be obtained from (14) on changing l to $k-l$. Throughout the paper, we use the following notations:

$$\varepsilon_i = \pm 1, \quad \delta_i = \frac{1-\varepsilon_i}{2}, \quad \text{for} \quad i = 1,2.$$

Theorem 2. *For any positive integers k and l, such that $0 < l < k$, we have*

$$\begin{aligned}\varphi(\varepsilon_1 q^{l(k-l)})\varphi(\varepsilon_2 q) =& \varphi((-1)^{\delta_1+\delta_2(k-l)} q^{k(k-l)})\varphi((-1)^{\delta_1+\delta_2 l} q^{lk}) \\ &+ (-1)^{\delta_2 \frac{k}{2}} \frac{1}{2} q^{k^2/4}(1+(-1)^k)(1+(-1)^{\delta_1+\delta_2(k-l)})(1+(-1)^{\delta_1+\delta_2 l})\psi(q^{2k(k-l)})\psi(q^{2lk}) \\ &+ 2\sum_{a=1}^{\left[\frac{k-1}{2}\right]} (-1)^{a\delta_2} q^{a^2} f_{\delta_1+\delta_2(k-l)}(q^{(k-l)(k+2a)}, q^{(k-l)(k-2a)}) f_{\delta_1+\delta_2 l}(q^{l(k+2a)}, q^{l(k-2a)}).\end{aligned} \tag{23}$$

Proof. Setting $g = h = l(k-l)$ and $u = v = 1$ in (14) and using (7), we find that

$$\varphi(\varepsilon_1 q^{l(k-l)})\varphi(\varepsilon_2 q) = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{a^2} f_{\delta_1+\delta_2(k-l)}(q^{(k-l)(k+2a)}, q^{(k-l)(k-2a)}) f_{\delta_1+\delta_2 l}(q^{l(k+2a)}, q^{l(k-2a)}). \tag{24}$$

Now we consider two cases according to the parity of k.
Case 1. When k is even, we have

$$\begin{aligned}\mathscr{A}(a) :=& q^{a^2} f_{\delta_1+\delta_2(k-l)}(q^{(k-l)(k+2a)}, q^{(k-l)(k-2a)}) f_{\delta_1+\delta_2 l}(q^{l(k+2a)}, q^{l(k-2a)}) \\ =& \begin{cases} \varphi((-1)^{\delta_1+\delta_2(k-l)} q^{k(k-l)})\varphi((-1)^{\delta_1+\delta_2 l} q^{lk}), & \text{if } a = 0; \\ q^{k^2/4} f_{\delta_1+\delta_2(k-l)}(1, q^{2k(k-l)}) f_{\delta_1+\delta_2 l}(1, q^{2lk}), & \text{if } a = \frac{k}{2}; \\ \mathscr{A}(-a), & \text{if } 0 < a < \frac{k}{2}. \end{cases}\end{aligned}$$

Now, using (4), (5) and (8), the identity (24) can be written in the form

$$\begin{aligned}\varphi(\varepsilon_1 q^{l(k-l)})\varphi(\varepsilon_2 q) =& \varphi((-1)^{\delta_1+\delta_2(k-l)} q^{k(k-l)})\varphi((-1)^{\delta_1+\delta_2 l} q^{lk}) \\ &+ (-1)^{\delta_2 \frac{k}{2}} q^{k^2/4}(1+(-1)^{\delta_1+\delta_2(k-l)})(1+(-1)^{\delta_1+\delta_2 l})\psi(q^{2k(k-l)})\psi(q^{2lk}) \\ &+ 2\sum_{a=1}^{\frac{k-2}{2}} (-1)^{a\delta_2} q^{a^2} f_{\delta_1+\delta_2(k-l)}(q^{(k-l)(k+2a)}, q^{(k-l)(k-2a)}) f_{\delta_1+\delta_2 l}(q^{l(k+2a)}, q^{l(k-2a)}).\end{aligned} \tag{25}$$

The identity (25) is same as (23) when k is even.
Case 2. When k is odd, with the similar arguments, we derive

$$\begin{aligned}\varphi(\varepsilon_1 q^{l(k-l)})\varphi(\varepsilon_2 q) =&\varphi((-1)^{\delta_1+\delta_2(k-l)}q^{k(k-l)})\varphi((-1)^{\delta_1+\delta_2 l}q^{lk})\\&+2\sum_{a=1}^{\frac{k-1}{2}}(-1)^{a\delta_2}q^{a^2}f_{\delta_1+\delta_2(k-l)}(q^{(k-l)(k+2a)},q^{(k-l)(k-2a)})f_{\delta_1+\delta_2 l}(q^{l(k+2a)},q^{l(k-2a)}),\end{aligned}$$

which is same as (23) when k is odd.
Combining these two cases, we complete the proof of the theorem.

Corollary 1.*(Cao [12, (2.7) p. 4485] when $\varepsilon=1$, Yan [32, Theorem 3.1] and Chen-Huang [14, Theorem 1] when $\varepsilon=1$)*

$$\begin{aligned}\varphi(\varepsilon q^m)\varphi(q) =&\varphi(\varepsilon q^{(m+1)})\varphi(\varepsilon q^{m(m+1)})+\delta q^{\frac{(m+1)^2}{4}}\psi(q^{2(m+1)})\psi(q^{2m(m+1)})\\&+2\sum_{a=1}^{[\frac{m}{2}]}q^{a^2}f(\varepsilon q^{m+1-2a},\varepsilon q^{m+1+2a})f(\varepsilon q^{m(m+1)-2ma},\varepsilon q^{m(m+1)+2ma}),\end{aligned}\tag{26}$$

where $\delta=\begin{cases}4, \text{ if } \varepsilon=1 \text{ and } m \text{ odd;}\\ 0, \text{ otherwise.}\end{cases}$

*Proof.*Setting $k=m+1$, $l=1$, $\varepsilon_1=\varepsilon$ and $\varepsilon_2=1$ in Theorem 2, we deduce (26).

Corollary 2.*(Cao [12, (2.8), p. 4485] when $\varepsilon_1=\varepsilon_2=1$)*

$$\begin{aligned}\varphi(\varepsilon_1 q^{l^2})\varphi(\varepsilon_2 q) =&\varphi^2((-1)^{\delta_1+\delta_2 l}q^{2l^2})+(-1)^{\delta_2 l}2(1+(-1)^{\delta_1+\delta_2 l})q^{l^2}\psi^2(q^{4l^2})\\&+2\sum_{a=1}^{l-1}(-1)^{a\delta_2}q^{a^2}f^2_{\delta_1+\delta_2 l}(q^{2l^2+2la},q^{2l^2-2la}).\end{aligned}\tag{27}$$

*Proof.*Putting $k=2l$ in Theorem 2, we obtain (27).

On employing the identities $f(q,q^5)=\chi(q)\psi(-q^3)$ and $f(q,q^2)=\varphi(-q^3)/\chi(-q)$ in Theorem 2, we easily deduce the following corollary.

Corollary 3.

$$\begin{aligned}
\varphi(-q)\varphi(q)=&\varphi(-q^2),\\
\varphi^2(-q)=&\varphi(q^2)-4q\psi^2(q^4),\\
\varphi(-q^2)\varphi(q)=&\varphi(-q^3)\varphi\left(-q^6\right)+2q\psi(q^3)\psi\left(q^6\right)\chi(-q)\chi(-q^2),\\
\varphi(-q^3)\varphi(q)=&2q\psi\left(-q^6\right)\psi(-q^2)+\varphi(-q^{12})\varphi(-q^4),\\
\varphi(-q^3)\varphi(-q)=&4q^4\psi(q^8)\psi(q^{24})-2q\psi(q^2)\psi\left(q^6\right)+\varphi(q^4)\varphi(q^{12}),\\
\varphi(-q^4)\varphi(-q)=&\varphi^2(-q^8)-2q\psi^2(-q^4),\\
\varphi(q^4)\varphi(-q)=&\varphi^2(q^8)-2q^2\psi^2(q^4)+4q^4\psi^2\left(q^{16}\right),\\
\varphi\left(-q^5\right)\varphi(q)=&\varphi\left(-q^6\right)\varphi(-q^{30})+2q^4\psi\left(q^6\right)\psi(q^{30})\chi(-q^2)\chi(-q^{10})+2qf(-q^{20})f(-q^4),\\
\varphi(-q^8)\varphi(q)=&\varphi(-q^{12})\varphi(-q^{24})+2q^4\psi(q^{12})\psi(q^{24})\chi(-q^4)\chi(-q^8)+2qf\left(-q^{16}\right)f(-q^8),\\
\varphi(q^2)\varphi(q)=&\varphi(q^3)\varphi\left(q^6\right)+2q\psi(-q^3)\psi\left(-q^6\right)\chi(q)\chi(q^2),\\
\varphi\left(q^5\right)\varphi(q)=&\varphi\left(q^6\right)\varphi(q^{30})+2q^4\psi\left(-q^6\right)\psi(-q^{30})\chi(q^2)\chi(q^{10})+2\frac{q\varphi(-q^{60})\varphi(-q^{12})}{\chi(-q^{20})\chi(-q^4)}+4q^9\psi\left(q^{60}\right)\psi(q^{12}),\\
\varphi(q^8)\varphi(q)=&\varphi(q^{12})\varphi(q^{24})+2q^4\psi(-q^{12})\psi(-q^{24})\chi(q^4)\chi(q^8)+2\frac{q\varphi(-q^{48})\varphi(-q^{24})}{\chi(-q^{16})\chi(-q^8)}+4q^9\psi(q^{48})\psi(q^{24}),\\
\varphi(q^9)\varphi(q)=&\varphi^2(q^{18})+2q^4\chi^2\left(q^6\right)\psi^2(-q^{18})+2\frac{q\varphi^2(-q^{36})}{\chi^2(-q^{12})}+4q^9\psi^2\left(q^{36}\right).
\end{aligned}$$

Theorem 3. *For any positive integers k and l, such that $0<l<k$, we have*

$$\begin{aligned}\psi(\varepsilon_1 q^{l(k-l)})\psi(\varepsilon_2 q) =& f_\alpha\left(q^{(k-l)(2k+1-l)},q^{(k-l)(2k-1+l)}\right) f_\beta\left(q^{l(3k-l+1)},q^{l(k+l-1)}\right)\\ &+(-1)^{\beta+\delta_2\frac{k}{2}}\frac{1}{2}\left(1+(-1)^k\right)q^{(k^2+k-2kl+2l^2-2l)/2} f_\alpha\left(q^{(k-l)(4k+1-l)},q^{(k-l)(l-1)}\right)\\ &\times f_\beta\left(q^{l(k-l+1)},q^{l(3k+l-1)}\right)+\sum_{a=1}^{\left[\frac{k-1}{2}\right]}(-1)^{a\delta_2}q^{2a^2-a}\\ &\times\left\{f_\alpha\left(q^{(k-l)(2k+1-l-4a)},q^{(k-l)(2k-1+l+4a)}\right)f_\beta\left(q^{l(3k-l+1-4a)},q^{l(k+l-1+4a)}\right)\right.\\ &\left.+q^{2a}f_\alpha\left(q^{(k-l)(2k+1-l+4a)},q^{(k-l)(2k-1+l-4a)}\right)f_\beta\left(q^{l(3k-l+1+4a)},q^{l(k+l-1-4a)}\right)\right\}, \end{aligned}\tag{28}$$

where $\alpha=\delta_1+\delta_2(k-l)$ and $\beta=\delta_1+\delta_2 l$.

Proof. Setting $g=3l(k-l)$, $h=l(k-l)$, $u=3$ and $v=1$ in (15), we find that

$$\psi(\varepsilon_1 q^{l(k-l)})\psi(\varepsilon_2 q)=\sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]}(-1)^{a\delta_2}q^{2a^2+a}f_\alpha\left(q^{(k-l)(2k+1-l+4a)},q^{(k-l)(2k-1+l-4a)}\right)f_\beta\left(q^{l(3k-l+1+4a)},q^{l(k+l-1-4a)}\right).\tag{29}$$

Now we consider two cases according to the parity of k.
Case 1. Suppose k is even. Define

$$\mathscr{A}(a):=q^{2a^2+a}f_\alpha\left(q^{(k-l)(2k+1-l+4a)},q^{(k-l)(2k-1+l-4a)}\right)f_\beta\left(q^{l(3k-l+1+4a)},q^{l(k+l-1-4a)}\right).$$

Then

$$\begin{aligned}\mathscr{A}(0)=&f_\alpha\left(q^{(k-l)(2k+1-l)},q^{(k-l)(2k-1+l)}\right)\times f_\beta\left(q^{l(3k-l+1)},q^{l(k+l-1)}\right),\\ \mathscr{A}\left(\frac{k}{2}\right)=&(-1)^\beta q^{(k^2+k-2lk+2l^2-2l)/2}f_\alpha\left(q^{(k-l)(4k+1-l)},q^{(k-l)(l-1)}\right)f_\beta\left(q^{l(k-l+1)},q^{l(3k+l-1)}\right).\end{aligned}$$

In the last step, we have used (6) with $n=-1$. Thus, identity (29) is equivalent to

$$\begin{aligned}\psi(\varepsilon_1 q^{l(k-l)})\psi(\varepsilon_2 q) =& f_\alpha\left(q^{(k-l)(2k+1-l)},q^{(k-l)(2k-1+l)}\right) f_\beta\left(q^{l(3k-l+1)},q^{l(k+l-1)}\right)\\ &+(-1)^{\beta+\delta_2\frac{k}{2}}q^{(k^2+k-2kl+2l^2-2l)/2} f_\alpha\left(q^{(k-l)(4k+1-l)},q^{(k-l)(l-1)}\right)\\ &\times f_\beta\left(q^{l(k-l+1)},q^{l(3k+l-1)}\right)+\sum_{a=1}^{\frac{k-2}{2}}(-1)^{a\delta_2}q^{2a^2-a}\\ &\times\left\{f_\alpha\left(q^{(k-l)(2k+1-l-4a)},q^{(k-l)(2k-1+l+4a)}\right)f_\beta\left(q^{l(3k-l+1-4a)},q^{l(k+l-1+4a)}\right)\right.\\ &\left.+q^{2a}f_\alpha\left(q^{(k-l)(2k+1-l+4a)},q^{(k-l)(2k-1+l-4a)}\right)f_\beta\left(q^{l(3k-l+1+4a)},q^{l(k+l-1-4a)}\right)\right\}. \end{aligned}\tag{30}$$

Identity (30) is same as (28) when k is even.
Case 2. Suppose k is odd. As in *Case* 1, we establish that

$$\begin{aligned}\psi(\varepsilon_1 q^{l(k-l)})\psi(\varepsilon_2 q) =& f_\alpha\left(q^{(k-l)(2k+1-l)},q^{(k-l)(2k-1+l)}\right) f_\beta\left(q^{l(3k-l+1)},q^{l(k+l-1)}\right)\\ &+\sum_{a=1}^{\frac{k-1}{2}}(-1)^{a\delta_2}q^{2a^2-a}\\ &\times\left\{f_\alpha\left(q^{(k-l)(2k+1-l-4a)},q^{(k-l)(2k-1+l+4a)}\right)f_\beta\left(q^{l(3k-l+1-4a)},q^{l(k+l-1+4a)}\right)\right.\\ &\left.+q^{2a}f_\alpha\left(q^{(k-l)(2k+1-l+4a)},q^{(k-l)(2k-1+l-4a)}\right)f_\beta\left(q^{l(3k-l+1+4a)},q^{l(k+l-1-4a)}\right)\right\}. \end{aligned}\tag{31}$$

Combining (30) with (31), we obtain (28). This complete the proof of the theorem.

Corollary 4._(Yan [32, Theorem 3.4]), (Chen and Huang [14, Theorem 5] when $\varepsilon = 1$)_

$$\psi(q)\psi(\varepsilon q^m) = \psi(\varepsilon q^{m+1})\varphi(\varepsilon q^{2^{\delta}m(m+1)}) + \sum_{a=1}^{\left[\frac{m}{2}\right]} q^{2a^2-a} f(q^{2a}, \varepsilon q^{m+1-2a}) f(\varepsilon q^{2m(m+1)-4ma}, \varepsilon q^{2m(m+1)+4ma}), \tag{32}$$

where $\delta = \begin{cases} -1, & \textit{if } \varepsilon = 1 \textit{ and } m \textit{ odd;} \\ 1, & \textit{otherwise.} \end{cases}$

*Proof.*Setting $k = m+1$, $l = 1$, $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 1$ in Theorem 3 with considering two cases according to the parity of m (i.e., even or odd) and then using (11), we arrive at (32) after some simplifications.

On employing the identities $f(q,q^5) = \chi(q)\psi(-q^3)$ and $f(q,q^2) = \varphi(-q^3)/\chi(-q)$ in Theorem 3, we easily deduce the following corollary.

Corollary 5.

$$\begin{aligned}
\psi(-q)\psi(q) &= \varphi(-q^4)\psi(-q^2), \\
\varphi(q) &= \varphi(q^4) + 2q\psi(q^8), \\
\psi(-q^2)\psi(q)\chi(q) &= \varphi(-q^{12})\psi(-q^3)\chi(q) + q\chi(-q^4)\psi(q^{12})\varphi(q^3), \\
\psi(q^2)\psi(q)\chi(-q) &= \varphi(q^{12})\psi(q^3)\chi(-q) + q\chi(q^4)\psi(-q^{12})\varphi(-q^3), \\
\psi(-q^2)\psi(q) &= f(-q^8)f(q), \\
\psi(q^2)\psi(q) &= \frac{\varphi(-q^3)\varphi(-q^{24})}{\chi(-q)\chi(-q^8)} + 2q^3\psi(q^9)\psi(q^{24}), \\
\psi(-q^3)\psi(q) &= \varphi(-q^{24})\psi(-q^4) + q\varphi(-q^8)\psi(-q^{12}), \\
\psi(q^3)\psi(q) &= \varphi\left(q^6\right)\psi(q^4) + q\varphi(q^2)\psi(q^{12}).
\end{aligned}$$

Corollary 6.

$$\varphi(\varepsilon_1 q^{2l(k-l)})\psi(\varepsilon_2 q) = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{2a^2+a} f_\alpha\left(q^{(k-l)(2k+1+4a)}, q^{(k-l)(2k-1-4a)}\right) f_\beta\left(q^{l(2k+1+4a)}, q^{l(2k-1-4a)}\right), \tag{33}$$

$$(1+\varepsilon_2)\varphi(\varepsilon_1 q^{l(k-l)})\psi(q^2) = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{a^2+a} f_\alpha\left(q^{(k-l)(k+1+2a)}, q^{(k-l)(k-1-2a)}\right) f_\beta\left(q^{l(k+1+2a)}, q^{l(k-1-2a)}\right), \tag{34}$$

$$\begin{aligned}
(1+\varepsilon_1)(1+\varepsilon_2)\psi(q^{l(k-l)})\psi(q) = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} & (-1)^{a\delta_2} q^{(a^2+a)/2} f_\alpha\left(q^{\frac{1}{2}(k-l)(k+l+1+2a)}, q^{\frac{1}{2}(k-l)(k-l-1-2a)}\right) \\
& \times f_\beta\left(q^{\frac{1}{2}l(l+1+2a)}, q^{\frac{1}{2}l(2k-l-1-2a)}\right),
\end{aligned} \tag{35}$$

where $\alpha = \delta_1 + \delta_2(k-l)$ _and_ $\beta = \delta_1 + \delta_2 l$.

*Proof.*Setting $g = h = 2l(k-l)$, $u = 3$ and $v = 1$ in (14), we obtain (33).
To prove (34), we put $g = h = l(k-l)$, $u = 2$ and $v = 0$ in (14). Identity (35) follows from (14) by setting $g = l(k-l)$, $h = 0$ $u = 1$ and $v = 0$.

Using Corollary 6, one can easily deduce the following theta function identities:

Corollary 7.

$$\varphi(-q)\,\psi(q)=f(-q^2)\,f(-q),$$
$$\varphi(q)\,\psi(q)\,\chi(-q)\,\chi(-q^2)=\varphi(-q^3)\,\varphi\left(-q^6\right)+2q\psi(q^3)\,\psi\left(q^6\right)\chi(-q)\,\chi(-q^2),$$
$$\psi^2(q)=\psi(q^2)\,\varphi(q),$$
$$2\,\psi(q^2)\,\psi(q)\,\chi(-q)=\varphi(q^3)\,\psi(q^3)\,\chi(-q)+\varphi(-q^3)\,\psi(-q^3)\,\chi(q),$$
$$\psi(q^3)\,\psi(q)=\varphi\left(q^6\right)\psi(q^4)+q\varphi(q^2)\,\psi(q^{12}).$$

The proof of the following corollary is similar to that of Corollary 6:

Corollary 8.

$$\psi(\varepsilon_1 q^{l(k-l)})\varphi(\varepsilon_2 q^2)=\sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]}(-1)^{a\delta_2}q^{2a^2}f_\alpha\left(q^{(k-l)(2k+l+4a)},q^{(k-l)(2k-l-4a)}\right)f_\beta\left(q^{l(k+l+4a)},q^{l(3k-l-4a)}\right),\quad(36)$$

$$(1+\varepsilon_1)\psi(q^{2l(k-l)})\varphi(\varepsilon_2 q)=\sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]}(-1)^{a\delta_2}q^{a^2}f_\alpha\left(q^{(k-l)(k+l+2a)},q^{(k-l)(k-l-2a)}\right)f_\beta\left(q^{l(l+2a)},q^{l(2k-l-2a)}\right),\quad(37)$$

$$(1+\varepsilon_1)\psi(q^{4l(k-l)})\psi(\varepsilon_2 q)=\sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]}(-1)^{a\delta_2}q^{2a^2+a}f_\alpha\left(q^{(k-l)(2k+2l+1+4a)},q^{(k-l)(2k-2l-1-4a)}\right)$$
$$\times f_\beta\left(q^{l(2l+1+4a)},q^{l(4k-2l-1-4a)}\right),\quad(38)$$

$$(1+\varepsilon_2)\psi(\varepsilon_1 q^{l(k-l)})\psi(q^4)=\sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]}(-1)^{a\delta_2}q^{2a^2+2a}f_\alpha\left(q^{(k-l)(2k+l+2+4a)},q^{(k-l)(2k-l-2-4a)}\right)$$
$$\times f_\beta\left(q^{l(k+l+2+4a)},q^{l(3k-l-2-4a)}\right).\quad(39)$$

where $\alpha=\delta_1+\delta_2(k-l)$ and $\beta=\delta_1+\delta_2 l$.

Using Corollary 8, we can easily derive the following theta function identities:

$$\psi(q^4)\,\varphi(q)\,\chi(-q^4)=\chi(q)\,\psi(-q^3)\,\varphi(-q^{12})+q\psi(q^{12})\,\varphi(q^3)\,\chi(-q^4),$$
$$\psi(q^8)\,\varphi(q)=\psi^2(q^4)+q\varphi(q^8)\,\psi\left(q^{16}\right),$$
$$\psi\left(q^{16}\right)\varphi(q)=q\varphi(q^{24})\,\psi(q^{24})+q^4\psi(q^{48})\,\varphi(q^{12})$$
$$+\frac{\varphi(-q^{48})\,\psi(-q^{12})\,\chi(q^4)}{\chi(-q^{16})}+\frac{q\varphi(-q^{24})\,\psi(-q^{24})\,\chi(q^8)}{\chi(-q^8)},$$
$$\psi(q^4)\,\psi(q)=f(q,q^7)\,f\left(q^3,q^5\right).$$

Corollary 9.*(Cao [12, (2.12), (2.14), p. 4486] when $\varepsilon=1$ and Yan [32, Theorem 3.8])*

$$\varphi(\varepsilon q^{2m})\psi(q)=\sum_{a=\left[\frac{1-m}{2}\right]}^{\left[\frac{m+1}{2}\right]}q^{2a^2-a}f\left(\varepsilon q^{2m^2+m+4ma},\varepsilon q^{2m^2+3m-4ma}\right)f(\varepsilon q^{2m+1+4a},\varepsilon q^{2m+3-4a}),\quad(40)$$

$$\psi(\varepsilon q^{m})\psi(q)=\sum_{a=\left[\frac{1-m}{2}\right]}^{\left[\frac{m+1}{2}\right]}q^{2a^2-a}f\left(\varepsilon q^{2m^2+4ma},\varepsilon q^{2m^2+4m-4ma}\right)f(\varepsilon q^{3m+1+4a},\varepsilon q^{m+3-4a}).\quad(41)$$

*Proof.*To prove (40), put $g=h=2m$, $u=3$, $v=1$, $k=m+1$, $l=1$, $\varepsilon_1=\varepsilon$ and $\varepsilon_2=1$ in (16) (or in (17)). Setting $g=3m$, $h=m$, $u=3$, $v=1$, $k=m+1$, $l=1$, $\varepsilon_1=\varepsilon$ and $\varepsilon_2=1$ in (16), we get (41) immediately.

Corollary 10.

$$\psi(\varepsilon q^m)\varphi(q^2) = \sum_{a=\left[\frac{1-m}{2}\right]}^{\left[\frac{m+1}{2}\right]} q^{2a^2} f\left(\varepsilon q^{2m^2+3m+4ma}, \varepsilon q^{2m^2+m-4ma}\right) f(\varepsilon q^{m+2+4a}, \varepsilon q^{3m+2-4a}), \tag{42}$$

$$2\psi(q^{2m})\varphi(q) = \sum_{a=\left[\frac{1-m}{2}\right]}^{\left[\frac{m+1}{2}\right]} q^{2a^2} f\left(q^{m^2+2m+2ma}, q^{m^2-2ma}\right) f(q^{2a+1}, q^{2m+1-2a}), \tag{43}$$

$$2\varphi(\varepsilon q^m)\psi(q^2) = \sum_{a=\left[\frac{1-m}{2}\right]}^{\left[\frac{m+1}{2}\right]} q^{a^2-a} f\left(\varepsilon q^{m^2+2ma}, \varepsilon q^{m^2+2m-2ma}\right) f(\varepsilon q^{m+2a}, \varepsilon q^{m+2-2a}), \tag{44}$$

$$\psi(\varepsilon_1 q^{l^2})\psi(\varepsilon_2 q) = \sum_{a=1-l}^{l} (-1)^{a\delta_2} q^{a^2+2a} f_{\delta_1+\delta_2(k-l)}\left(q^{l(3l+1+4a)}, q^{l(5l-1-4a)}\right) f_{\delta_1+\delta_2 l}(q^{l(5l+1+4a)}, q^{l(3l-1-4a)}), \tag{45}$$

$$\varphi(\varepsilon q^{2l^2})\psi(q) = \sum_{a=1-l}^{l} q^{2a^2-a} f^2\left(\varepsilon q^{l(4l-1+4a)}, \varepsilon q^{l(4l+1-4a)}\right), \tag{46}$$

$$2\psi(q^{4l^2})\psi(q) = \sum_{a=1-l}^{l} q^{2a^2-a} f\left(q^{l(2l-1+4a)}, q^{l(6l+1-4a)}\right) f(q^{l(6l-1+4a)}, q^{l(2l+1-4a)}), \tag{47}$$

$$2\varphi(\varepsilon q^l)\psi(q) = \sum_{a=-l}^{l} q^{(a^2-a)/2} f\left(\varepsilon q^{2l(l+a)}, \varepsilon q^{2l(l+1-a)}\right) f(\varepsilon q^{l+a}, \varepsilon q^{l+1-a}). \tag{48}$$

All the identities in the corollary 10 follow easily from Theorem 1. Identities (42), (43) and (44) can be found in [32] as Theorems 3.11, 3.12, and 3.16 respectively. Identities (43) and (46)-(48) appear as the identities (2.13) and (2.9)-(2.11), respectively, in [12] when $\varepsilon = 1$. Identity (45) seems to be new.

Here and in the sequel, we use the notation

$$f(\varepsilon q) := \begin{cases} f(-q) & \text{if } \varepsilon = -1, \\ f_*(q) & \text{if } \varepsilon = 1, \end{cases} \tag{49}$$

where $f(-q)$ and $f_*(q)$ are defined by (9) and (10), respectively.
Setting $g = 2l(k-l)$, $h = l(k-l)$, $u = 2$ and $v = 1$ in (14) and (15), we obtain the following new corollary:

Corollary 11.

$$\begin{aligned} f(\varepsilon_1 q^{l(k-l)}) f(\varepsilon_2 q) &= \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{(3a^2+a)/2} f_\alpha\left(q^{(k-l)(3k+l+1+6a)/2}, q^{(k-l)(3k-l-1-6a)/2}\right) \\ &\quad \times f_\beta\left(q^{l(2k+l+1+6a)/2}, q^{l(4k-l-1-6a)/2}\right), \end{aligned} \tag{50}$$

$$\begin{aligned} &= \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{(3a^2+a)/2} f_\alpha\left(q^{(k-l)(3k-l+1+6a)/2}, q^{(k-l)(3k+l-1-6a)/2}\right) \\ &\quad \times f_\beta\left(q^{l(4k-l+1+6a)/2}, q^{l(2k+l-1-6a)/2}\right), \end{aligned} \tag{51}$$

where $\alpha = \delta_1 + \delta_2(k-l)$ *and* $\beta = \delta_1 + \delta_2 l$.

Setting $k = 2l$ in (50) or (51), we obtain

$$f(\varepsilon_1 q^{l^2}) f(\varepsilon_2 q) = \sum_{a=1-l}^{l} (-1)^{a\delta_2} q^{(3a^2+a)/2} f_{\delta_1+\delta_2 l}\left(q^{l(7l+1+6a)/2}, q^{l(5l-1-6a)/2}\right) f_{\delta_1+\delta_2 l}\left(q^{l(5l+1+6a)/2}, q^{l(7l-1-6a)/2}\right). \tag{52}$$

Using Corollary 11, we can easily derive the following theta function identities:

$$\begin{aligned} f(q) &= f\left(-q^2\right)\chi(q), \\ f^2(-q)\chi\left(-q^2\right) &= \varphi\left(-q^6\right)\varphi\left(q^3\right) - 2q\chi(q)\chi\left(-q^2\right)\psi\left(-q^3\right)\psi\left(q^6\right), \\ f\left(-q^2\right)f(-q)\chi\left(-q^3\right) &= \varphi^2\left(-q^9\right) - q\varphi\left(-q^9\right)\psi\left(q^9\right)\chi\left(-q^3\right) - 2q^2\psi^2\left(q^9\right)\chi^2\left(-q^3\right), \\ \varphi\left(-q^6\right)f(-q) &= f\left(-q^3\right)\chi\left(-q^2\right)\left\{\varphi\left(q^9\right) - q\psi\left(-q^9\right)\chi\left(q^3\right)\right\}, \\ f\left(-q^3\right)\varphi\left(-q^3\right) &= q\psi\left(-q^9\right)f(q)\chi(-q) + f\left(-q^4\right)\varphi\left(-q^{18}\right)\chi(-q). \end{aligned}$$

The following new identities involving product of two theta functions, follow directly form (14) and (15), where $\alpha := \delta_1 + \delta_2(k-l)$ and $\beta := \delta_1 + \delta_2 l$:

$$\varphi(\varepsilon_1 q^{3l(k-l)})f(\varepsilon_2 q^2) = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{3a^2+a} f_\alpha\left(q^{(k-l)(3k+1+6a)}, q^{(k-l)(3k-1-6a)}\right) f_\beta\left(q^{l(3k+1+6a)}, q^{l(3k-1-6a)}\right), \tag{53}$$

$$f(\varepsilon_1 q^{2l(k-l)})\varphi(\varepsilon_2 q^3) = \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{3a^2} f_\alpha\left(q^{(k-l)(3k+l+6a)}, q^{(k-l)(3k-l-6a)}\right) f_\beta\left(q^{l(2k+l+6a)}, q^{l(4k-l-6a)}\right), \tag{54}$$

$$\begin{aligned} f(\varepsilon_1 q^{4l(k-l)})\psi(\varepsilon_2 q^3) &= \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{6a^2+3a} f_\alpha\left(q^{(k-l)(6k+2l+3+12a)}, q^{(k-l)(6k-2l-3-12a)}\right) \\ &\quad \times f_\beta\left(q^{l(4k+2l+3+12a)}, q^{l(8k-2l-3-12a)}\right) &(55) \\ &= \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{6a^2+3a} f_\alpha\left(q^{(k-l)(6k-2l+3+12a)}, q^{(k-l)(6k+2l-3-12a)}\right) \\ &\quad \times f_\beta\left(q^{l(8k-2l+3+12a)}, q^{l(4k+2l-3-12a)}\right), &(56) \end{aligned}$$

$$\begin{aligned} (1+\varepsilon_2)f(\varepsilon_1 q^{l(k-l)})\psi(q^3) &= \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{3a(a+1)/2} f_\alpha\left(q^{\frac{1}{2}(k-l)(3k+l+3+6a)}, q^{\frac{1}{2}(k-l)(3k-l-3-6a)}\right) \\ &\quad \times f_\beta\left(q^{\frac{1}{2}l(2k+l+3+6a)}, q^{\frac{1}{2}l(4k-l-3-6a)}\right) &(57) \\ &= \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{3a(a+1)/2} f_\alpha\left(q^{\frac{1}{2}(k-l)(3k-l+3+6a)}, q^{\frac{1}{2}(k-l)(3k+l-3-6a)}\right) \\ &\quad \times f_\beta\left(q^{\frac{1}{2}l(4k-l+3+6a)}, q^{\frac{1}{2}l(2k+l-3-6a)}\right), &(58) \end{aligned}$$

$$\begin{aligned} \psi(\varepsilon_1 q^{3l(k-l)})f(\varepsilon_2 q^4) &= \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{6a^2+2a} f_\alpha\left(q^{(k-l)(6k+3l+2+12a)}, q^{(k-l)(6k-3l-2-12a)}\right) \\ &\quad \times f_\beta\left(q^{l(3k+3l+2+12a)}, q^{l(9k-3l-2-12a)}\right) &(59) \\ &= \sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]} (-1)^{a\delta_2} q^{6a^2+2a} f_\alpha\left(q^{(k-l)(6k-3l+2+12a)}, \varepsilon q^{(k-l)(6k+3l-2-12a)}\right) \\ &\quad \times f_\beta\left(q^{l(9k-3l+2+12a)}, q^{l(3k+3l-2-12a)}\right), &(60) \end{aligned}$$

$$(1+\varepsilon_1)\psi(q^{3l(k-l)})f(\varepsilon_2 q)=\sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]}(-1)^{a\delta_2}q^{(3a^2+a)/2}$$
$$\times f_\alpha\left(q^{\frac{1}{2}(k-l)(3k+3l+1+6a)},q^{\frac{1}{2}(k-l)(3k-3l-1-6a)}\right)$$
$$\times f_\beta\left(q^{\frac{1}{2}l(3l+1+6a)},q^{\frac{1}{2}l(6k-3l-1-6a)}\right) \quad (61)$$
$$=\sum_{a=\left[\frac{2-k}{2}\right]}^{\left[\frac{k}{2}\right]}(-1)^{a\delta_2}q^{(3a^2+a)/2}$$
$$\times f_\alpha\left(q^{\frac{1}{2}(k-l)(3k-3l+1+6a)},q^{\frac{1}{2}(k-l)(3k+3l-1-6a)}\right)$$
$$\times f_\beta\left(q^{\frac{1}{2}l(6k-3l+1+6a)},q^{\frac{1}{2}l(3l-1-6a)}\right). \quad (62)$$

The following theta functions identities follow easily from (57):

$$f(-q)=\chi(-q)\,f(-q^2),$$
$$2f(-q)\,\psi(q)=\psi(-q)\,f(q)+f(-q^4)\,\varphi(-q^2),$$
$$\varphi(-q^3)\,\psi(q)\,\chi(-q^4)\chi(q)=\varphi(-q^{12})\,\varphi\left(-q^6\right)$$
$$+2q\psi\left(-q^6\right)\psi(q^4)\,\chi^2(-q^4).$$

4 Applications to Modular Relations

In this section, we find many applications for our identities established in Section 3. We also establish many modular relations for several functions of the Rogers-Ramanujan type.

4.1 Modular relations for the Rogers-Ramanujan functions

The well-known Rogers-Ramanujan functions [22,23,26], are defined by

$$G(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty}=\frac{f(-q^2,-q^3)}{f_1}, \quad (63)$$

and

$$H(q):=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}=\frac{f(-q,-q^4)}{f_1}. \quad (64)$$

In his Lost Notebook [24], Ramanujan recorded forty beautiful modular relations involving the Rogers-Ramanujan functions without proof. For a beautiful history of the 40 identities, we refer the reader to see the excellent monograph [9] of B. C. Berndt, et al.

Here we establish the following list of modular relations for the Rogers-Ramnujan functions:

$$G(q)G(q^4)+qH(q)H(q^4)=\chi^2(q)=\frac{\varphi(q)}{f(-q^2)}, \quad (65)$$
$$G(q^{16})H(q)-q^3G(q)H(q^{16})=\chi(q^2), \quad (66)$$
$$G(q)G(q^9)+q^2H(q)H(q^9)=\frac{f^2(-q^3)}{f(-q)f(-q^9)}, \quad (67)$$
$$G(q^2)G(q^3)+qH(q^2)H(q^3)=\frac{\chi(-q^3)}{\chi(-q)}, \quad (68)$$
$$G(q^6)H(q)-qG(q)H(q^6)=\frac{\chi(-q)}{\chi(-q^3)}, \quad (69)$$
$$G(q^8)H(q^3)-qG(q^3)H(q^8)=\frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}, \quad (70)$$
$$G(q)G(q^{24})+q^5H(q)H(q^{24})=\frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)}, \quad (71)$$
$$G(q^9)H(q^4)-qG(q^4)H(q^9)=\frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}, \quad (72)$$
$$G(q)H(-q)+G(-q)H(q)=\frac{2}{\chi^2(-q^2)}$$
$$=\frac{2\psi(q^2)}{f(-q^2)}, \quad (73)$$
$$G(q)G(q^8)H^2(q^2)+qH(q)H(q^8)G^2(q^2)$$
$$=\frac{f(-q^5)}{f(-q^2)f(-q^4)\varphi(-q^5)}$$
$$\times\left\{2\psi(q)\psi(q^4)-\ \psi(q^5)\varphi(q^{10})\right\}, \quad (74)$$
$$G(q)G(q^6)H(q^3)H(q^4)+H(q)H(q^6)G(q^3)G(q^4)$$
$$=\frac{f(-q^{15})}{f(-q^2)f(-q^6)\varphi(-q^{15})}$$
$$\times\left\{2\psi(q)\psi(q^6)-q\ \ \psi(q^{15})\varphi(q^5)\right\}. \quad (75)$$

Identities (65)-(73) are due to Ramanujan [24], which can be found in [9] as Entries 3.2, 3.5, 3.6, 3.7, 3.8, 3.11, 3.12, 3.13 and 3.20, respectively. Identity (65) has many proofs, first proof was by Rogers [28], also proved by Watson [31], Gugg [18], Son [30], Yan [32], Chu and Yan [17]. Andrews [4, p. 27] has shown that (65) follows from a very general identity in three variables found in Ramanujan's lost notebook. The identities (66)-(70) first proved by Rogers [28]. W. Chu [15] gave a new proof of (66). Identity (66) was also established by N. D. Baruah, J. Bora, and N. Saikia [8] and Baruah and Bora [7]. Generalizing slightly the approach of Rogers [28], Zhang [34] proved four general theorems from which the

identities (65), (66), (67), (70), and (71) follow as special cases. Unfortunately, he was not able to find any new examples to illustrate any of his general theorems. Identities (71) and (72) were first proved by Bressoud in his doctoral dissertation [10]. Watson [31] gave a proof of (73) which established later by Chu [15], Gugg [18], Son [30] and M. D. Hirschhorn [21]. Here we present new proofs of these identities as an application of our results established in the previous section.
Before proceeding to prove the above results, we prove the following lemma:

Lemma 3.

$$f(-q^2,-q^3) = f(q^9,q^{11}) - q^2 f(q,q^{19}), \tag{76}$$
$$f(-q,-q^4) = f(q^7,q^{13}) - q f(q^3,q^{17}), \tag{77}$$
$$f(-q^2,q^3) = f(-q^9,-q^{11}) - q^2 f(-q,-q^{19}), \tag{78}$$
$$f(q,-q^4) = f(-q^7,-q^{13}) + q f(-q^3,-q^{17}), \tag{79}$$
$$f(-q^2)G(q) = f(-q^{13},-q^{17}) + q f(-q^7,-q^{23}), \tag{80}$$
$$f(-q^2)H(q) = f(-q^{11},-q^{19}) + q^3 f(-q,-q^{29}), \tag{81}$$
$$f(-q)G(q^2) = f(q^7,q^8) - q f(q^2,q^{13}), \tag{82}$$
$$f(-q)H(q^2) = f(q^4,q^{11}) - q f(q,q^{14}), \tag{83}$$
$$f(q^3,q^7) = f(-q^2)G(q^4)H(q) \tag{84}$$
$$f(q,q^9) = f(-q^2)H(q^4)G(q).$$

Proof. Identities (76)-(79) follow easily from (11). Identities (80)-(83) follow from (13) with replacing q by q^{10}, q^{10}, q^5,q^5 and a by q, q^7, $-q$, $-q^2$, respectively with using (2). Proofs of last two identities can be found in [30, Theorem 3.2]. Second identity in (84) was also proved by the authors in [1, (3.13)].

Proof of (65)-(75). To prove (65), we set $k=5$, $l=3$, $\varepsilon_1=-1$ and $\varepsilon_2=1$ in (54), and then we use (80) and (81). Identities (66) and (70) follow from Theorem 3 on setting $k=5$, $l=4$ and $\varepsilon_1=\varepsilon_2=-1$ and $k=5$, $l=3$ and $\varepsilon_1=\varepsilon_2=-1$, respectively, and then employing (76) and (77) in the resulting identities. Putting $k=5$, $l=2$ $\varepsilon_1=-1$, $\varepsilon_2=1$ and $k=5$, $l=1$ $\varepsilon_1=-1$, $\varepsilon_2=1$ in (50) and then employing (80) and (81) in the resulting identities, we obtain (67) and (68). Setting $k=5$, $l=1$, $g=10$, $h=2$, $u=2$, $v=1$ and $\varepsilon_1=\varepsilon_2=-1$ in (14) and then employing (82) and (83) in the resulting identity, we obtain (69). To prove (71), set $k=5$, $l=1$, $g=16$, $h=8$, $u=5$, $v=1$, $\varepsilon_1=-1$ and $\varepsilon_2=1$ in (14), and then use (80) and (81) in the resulting identity. Setting $k=5$, $l=3$, $g=15$, $h=3$, $u=2$, $v=1$, $\varepsilon_1=1$ and $\varepsilon_2=-1$ in (14) and then using (82) and (83) in the resulting identity, we deduce (72). Setting $k=5$, $l=4$, $\varepsilon_1=-1$ and $\varepsilon_2=1$ in (39), and then using (78) and (79) in the resulting identity, we obtain (73). Last two identities (74) and (75) follow from (35), by setting $k=5$, $l=4$, 2, and $\varepsilon_1=\varepsilon_2=1$, respectively, and then applying (12) and (84) in the resulting identities. This completes the proof.

4.2 Modular relations for septic analogues of the Rogers-Ramanujan functions

In [20], H. Hahn considered the following septic analogues of the Rogers-Ramanujan functions:

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7,q^3,q^4;q^7)_\infty}{(q^2;q^2)_\infty} = \frac{f(-q^3,-q^4)}{f(-q^2)}, \tag{85}$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7,q^2,q^5;q^7)_\infty}{(q^2;q^2)_\infty} = \frac{f(-q^2,-q^5)}{f(-q^2)}, \tag{86}$$

and

$$C(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{(q^7,q,q^6;q^7)_\infty}{(q^2;q^2)_\infty} = \frac{f(-q,-q^6)}{f(-q^2)}. \tag{87}$$

Identities (85), (86) and (87) are due to Rogers [27], Later, Slater [29] offered different proofs of these identities. Hahn [20] established several modular relations involving $A(q)$, $B(q)$ and $C(q)$ including (88) and (89). In this section, we not only give new proofs of (88) and (89) but also we establish two new modular relations ((90) and (91)) for septic analogues of the Rogers-Ramanujan functions. Following Hahn, for simplicity, we use the following notations. For positive integer n, let

$$A_n := A(q^n), \quad B_n := B(q^n), \quad C_n := C(q^n),$$

and

$$A_n^* := A(-q^n), \quad B_n^* := B(-q^n), \quad C_n^* := C(-q^n).$$

Theorem 4. *We have*

$$A_{24}C_1 - q^3 B_{24}A_1 + q^{10}C_{24}B_1 = \frac{\chi(-q^6)\chi(-q^{24})}{\chi(q)}, \tag{88}$$

$$A_{16}C_3 - qB_{16}A_3 + q^6 C_{16}B_3 = \frac{\chi(-q)\chi(-q^4)\chi(-q^8)\chi(-q^{16})}{\chi(-q^6)\chi(-q^{24})}, \tag{89}$$

$$A_5B_8 - qC_5A_8 - q^3B_5C_8 = \frac{\chi(-q)\chi(-q^4)\chi(-q^8)}{\chi(-q^{20})}, \tag{90}$$

$$A_3C_1^* + B_3A_1^* + qC_3B_1^* = \frac{2}{\chi(-q)\chi(q^3)}. \tag{91}$$

To prove our Theorem 4, we need the following lemma, which can be proved easily using (11):

Lemma 4. *We have*

$$f(q,-q^6)=f(-q^9,-q^{19})+qf(-q^5,-q^{23}),$$
$$f(-q^2,q^5)=f(-q^{11},-q^{17})-q^2f(-q^3,-q^{25}),$$
$$f(q^3,-q^4)=f(-q^{13},-q^{15})+q^3f(-q,-q^{27}).$$

Proof of (88)-(91). Setting $k=7$, $\varepsilon_1=-1$, $\varepsilon_2=1$ and $l=6,3,2$, respectively, in Theorem 3 and then using Lemma 4 in the resulting identities, we obtain (88)-(90). Setting $k=7$, $l=4$, $\varepsilon_1=-1$ and $\varepsilon_2=1$ in (39), and then employing Lemma 4 in the resulting identity, we deduce (91).

4.3 Modular relations for nonic analogues of the Rogers-Ramanujan functions

Baruah and Bora [7] considered the following nonic analogues of the Rogers-Ramanujan functions:

$$D(q):=\sum_{n=0}^{\infty}\frac{(q;q)_{3n}q^{3n^2}}{(q^3;q^3)_n(q^3;q^3)_{2n}}=\frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty}=\frac{f(-q^4,-q^5)}{f_3},\tag{92}$$

$$E(q):=\sum_{n=0}^{\infty}\frac{(q;q)_{3n}(1-q^{3n+2})q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}}=\frac{(q^2,q^7,q^9;q^9)_\infty}{(q^3;q^3)_\infty}=\frac{f(-q^2,-q^7)}{f_3},\tag{93}$$

and

$$F(q):=\sum_{n=0}^{\infty}\frac{(q;q)_{3n+1}q^{3n(n+1)}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}}=\frac{(q,q^8,q^9;q^9)_\infty}{(q^3;q^3)_\infty}=\frac{f(-q,-q^8)}{f_3}.\tag{94}$$

Identities (92), (93) and (94) are due to W. N. Bailey [5]. In [7], Baruah and Bora established several modular relations involving $D(q),E(q)$ and $F(q)$. They also established some modular relations involving quotients of these functions. Some of these relations are connected with the Rogers-Ramanujan and Göllnitz-Gordon functions. Here, we present different proofs for some of these modular relations using our identities, as well as we establish several new modular relations for $D(q)$, $E(q)$, $F(q)$.

We define, for any positive integer n, $D_n:=D(q^n)$, $E_n:=E(q^n)$, $F_n:=F(q^n)$, $D_n^*:=D(-q^n)$, $E_n^*:=E(-q^n)$ and $F_n^*:=F(-q^n)$.

Theorem 5. *We have*

$$D_1D_8+q^3E_1E_8+q^6F_1F_8=\frac{f_2^2f_4^2}{f_1f_3f_8f_{24}}-q,\tag{95}$$
$$D_2D_7+q^3E_2E_7+q^6F_2F_7=\frac{f_2^2f_7^2}{f_1f_6f_{14}f_{21}}-q,\tag{96}$$
$$D_6D_3+q^3E_6E_3+q^6F_6F_3=\frac{f_2^2f_9}{f_1f_{18}^2}-q,\tag{97}$$
$$D_5D_4+q^3E_5E_4+q^6F_5F_4=\frac{f_2^2f_{10}^2}{f_1f_{12}f_{15}f_{20}}-q,\tag{98}$$
$$D_1D_2+qE_1E_2+q^2F_1F_2=\frac{f_2f_3^3}{f_1f_6^3}\tag{99}$$
$$D_{24}F_3+q^6E_{24}D_3-q^{15}F_{24}E_3=\frac{f_1f_4f_{18}}{f_2f_9f_{36}}+q,\tag{100}$$
$$D_{16}F_5+q^2E_{16}D_5-q^9F_{16}E_5=\frac{1}{q}-\frac{f_1f_4f_{20}f_{80}}{qf_2f_{15}f_{40}f_{48}},\tag{101}$$
$$D_7E_8+q^2F_7D_8-q^5E_7F_8=\frac{1}{q}-\frac{f_1f_4f_{14}f_{56}}{qf_2f_{21}f_{24}f_{28}},\tag{102}$$
$$D_{32}F_1+q^{10}E_{32}D_1-q^{21}F_{32}E_1=\frac{f_1f_4f_8f_{32}}{f_2f_3f_{16}f_{96}}+q^3,\tag{103}$$
$$D_{14}F_1+q^4E_{14}D_1-q^9F_{14}E_1=\frac{f_1^2f_{14}^2}{f_2f_3f_7f_{42}}+q,\tag{104}$$
$$D_6F_3+E_6D_3-q^3F_6E_3=\frac{1}{q}-\frac{f_1^2f_{18}^2}{qf_2f_9^2f_{18}},\tag{105}$$
$$D_2E_1-qE_2F_1+qF_2D_1=1,\tag{106}$$
$$D_5F_1+qE_5D_1-q^3F_5E_1=1,\tag{107}$$
$$D_5F_1^*+qE_5D_1^*+q^3F_5E_1^*=2\frac{f_2^2f_3f_5f_{12}f_{20}}{f_1f_6^3f_{10}f_{15}}-1,\tag{108}$$
$$D_2^*E_1+qE_2^*F_1+qF_2^*D_1=2\frac{f_2^3f_6f_8f_{24}}{f_1f_3f_4f_{12}^3}-1.\tag{109}$$

In the above list, identities (95)-(98), (100), (103), (104), (106) and (107) have been established by Baruah and Bora [7] and the others are new.

To prove Theorem 5, we need the following lemma which follows easily from (11):

Lemma 5. *We have*

$$f(-q^2,q^7)=f(-q^{13},-q^{23})-q^2f(-q^5,-q^{31}),$$
$$f(-q^4,q^5)=f(-q^{17},-q^{19})-q^4f(-q,-q^{35}),$$
$$f(q,-q^8)=f(-q^{11},-q^{25})+qf(-q^7,-q^{29}),$$
$$f(q,-q^2)=f(-q^5,-q^7)+qf(-q,-q^{11}).$$

Proof of (95)-(109). Setting $k=9$ and $l=1,\ 2,\ 3,\ 4$ in (34), we obtain (95)-(98), respectively. Putting $k=3$, $l=1$, $\varepsilon_1=-1$ and $\varepsilon_2=1$ in (53), we obtain (99). Setting $k=9$,

$\varepsilon_1=-1$, $\varepsilon_2=1$, and $l=3,4,7,8$, respectively in Theorem 3, and employing Lemma 5 in the resulting identities, and then replacing q by $-q$, we deduce (100)-(103). Similarly, we obtain (104) and (105) using (36), with $k=9$ and $l=2$, 6, respectively. One can establish (106) and (107) using (39) with $k=9$, $l=1,5$, respectively, and $\varepsilon_1=1$, $\varepsilon_2=-1$ and then employing Lemma 5 in the resulting identities. Setting $k=9$, $l=4$, 8, respectively, and $\varepsilon_1=-1$, $\varepsilon_2=1$ in (36) and then employing Lemma 5 in the resulting identities, we obtain (108) and (109).

4.4 Modular relations for dodecic analogues of the Rogers-Ramanujan functions

Here, we consider the following Rogers-Ramanujan type functions of order twelve which called dodecic analogues of the Rogers-Ramanujan functions:

$$X(q):=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n(1-q^{n+1})q^{n(n+2)}}{(q;q)_{2n+2}}=\frac{(q,q^{11},q^{12};q^{12})_\infty}{(q;q)_\infty}=\frac{f(-q,-q^{11})}{f(-q)}, \tag{110}$$

and

$$Y(q):=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n-1}(1+q^n)q^{n^2}}{(q;q)_{2n}}=\frac{(q^5,q^7,q^{12};q^{12})_\infty}{(q;q)_\infty}=\frac{f(-q^5,-q^7)}{f(-q)}, \tag{111}$$

where the later equalities are due to Slater [29]. Robins [25] in his Ph.D. thesis has established four modular relations for dodecic analogues functions using modular forms. Later, Baruah and Bora [6] established many modular relations involving some combinations of $X(q)$ and $Y(q)$, as well as relations that are connected with the Rogers-Ramanujan functions, Göllnitz-Gordon functions, septic analogues and nonic analogues functions. More recently, in his Ph.D. thesis [19], Gugg has given alternative proofs of Robins's four identities.

We establish some of the modular relations for dodecic analogues found by Baruah and Bora [6] and Robins [25]. We also establish some new modular relations involving $X(q)$ and $Y(q)$, using our identities. For simplicity, we define, for any positive integer n, $X_n:=X(q^n)$ and $Y_n:=Y(q^n)$.

$$Y_1+qX_1=\frac{f_2^3}{f_1^2f_4}, \tag{112}$$

$$Y_1-qX_1=\frac{f_4f_6^5}{f_2^2f_3^2f_{12}^2}, \tag{113}$$

$$Y_3^2+q^6X_3^2=\frac{f_4^2f_9^2}{f_2f_3^2f_{18}}-q^2\frac{f_9^2f_{36}^2}{f_3^2f_{18}^2}, \tag{114}$$

$$X_1Y_3-q^2X_3Y_1=\frac{f_2f_{12}^3}{f_1f_3f_4f_6}-q\frac{f_4f_6f_{36}^2}{f_1f_3f_{12}f_{18}}, \tag{115}$$

$$Y_1Y_2+q^3X_1X_2=\frac{f_2f_4^2}{f_1^2f_8}-q\frac{f_3f_{24}}{f_1f_2}, \tag{116}$$

$$Y_1Y_5+q^6X_1X_5=\frac{f_4^2f_5}{f_1f_2f_{10}}-q^2\frac{f_3f_{12}f_{15}f_{60}}{f_1f_5f_6f_{30}}, \tag{117}$$

$$Y_1Y_3-q^4X_1X_3=\frac{f_2f_3^2}{f_1^2f_6}-q\frac{f_2f_{12}^3}{f_1f_3f_4f_6}-q^2\frac{f_4f_6f_{36}^2}{f_1f_3f_{12}f_{18}}, \tag{118}$$

$$Y_1^2+q^2X_1^2=\frac{f_3^2f_4f_6}{f_1^2f_2f_{12}}, \tag{119}$$

$$Y_1^2-q^2X_1^2=\frac{f_2f_6^5}{f_1^2f_3^2f_{12}^2}, \tag{120}$$

$$Y_1^4-q^4X_1^4=\frac{f_4f_6^6}{f_1^4f_{12}^3}, \tag{121}$$

$$Y_4^2-q^8X_4^2=\frac{f_8f_6^5}{f_3^2f_4^2f_{12}}-2q^3\frac{f_8f_{48}}{f_4^2f_{24}} \tag{122}$$

Identities (112)-(117) were established by Baruah and Bora [6] and (115), (119) and (120) have been established by Robins [25] and Gugg [19] and the others seem to be new.

Proof of (112)-(122). Putting $m=2$ and $\varepsilon=-1$ in (42), we obtain (112). To prove (113), we first prove (120). Putting $k=2$, $l=1$, $\varepsilon_1=-1$ and $\varepsilon_2=1$ in (54), we obtain (120). The proof of (113) follows easily from (112) and (120). Setting $k=6$, $\varepsilon_1=-1$, $\varepsilon_2=1$ and $l=1$, 2, 3 in (34), we obtain (117), (116) and (114), respectively. Putting $k=4$, $l=1$, $\varepsilon_1=-1$, $\varepsilon_2=1$, $g=9$, $h=0$, $u=2$ and $v=1$ in (14), we obtain (115). Setting $k=4$, $l=1$, $\varepsilon_1=-1$ and $\varepsilon_2=1$ in (50), we obtain (118). Putting $k=2$, $l=1$, $\varepsilon_1=-1$, $\varepsilon_2=1$ in (53), we obtain (119). Identity (121) follows immediately from (119) and (120). Setting $k=4$ and $l=2$ in (54), we get (122).

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