Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/060305

Generalized Fibonacci Operational tau Algorithm for Fractional Bagley-Torvik Equation

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Received: 15 Sep. 2018, Revised: 16 Jan. 2019, Accepted: 18 Jan. 2019
Published online: 1 Jul. 2020

Abstract: In the present paper, a numerical technique for solving the fractional Bagley-Torvik equation with homogeneous boundary conditions is investigated. The nonhomogeneous conditions were transformed to homogeneous ones. The technique is based on applying the tau method to reduce the solution of fractional Bagley-Torvik equation for a system of algebraic equations. The latter equations may be solved using the Gaussian elimination procedure. The convergence and error analysis of the generalized Fibonacci basis are discussed in detail. Some explanatory examples are provided to assert the validity, accuracy and applicability of this technique.

Keywords: Fractional differential equations, Fractional Bagley-Torvik equation, Generalized Fibonacci polynomial, Spectral methods.

2010 Mathematics Subject Classification. Primary 11B39, 26A33; Secondary 41A25, 65M70.

1 Introduction

Fractional differential equations (FDEs) have been extensively addressed in many fields , such as applied mathematics, finance, engineering and other areas of applications [1]-[3]. In the last few decades, a mass generalization for classical models to their fractional version has been witnessed . This occurred due to the new properties of these fractional differential and integral operators. Many fractional derivative operators have been widely used in scientific research ,such as Caputo operator, Riemann-Liouville operator and Hadamard operator, etc [4,5]. Many problems in various fields are modeled by FDEs, such as fractional Bagley-Torvik (B-T) equation [6]. Simultaneously, they have been explored both analytically and numerically [4]. In addition, many researchers have solved this equation numerically. For example, in [7] the fractional B-T equation was solved by Abu Arqub and Maayah using iterative reproducing kernel algorithm, in [8] it was solved by Cenesiz et al. using generalized Taylor collocation method and in [9] it was solved by Krishnasamy and Razzaghi using the fractional Taylor method.The exact solution can not be obtained in most of the FDEs. Thus, various numerical methods have been utilized to obtain approximate solutions of the FDEs ,such as finite element methods [10], differential transform method [11] and spectral methods [12], etc. The most important methods are spectral methods. They consist of three versions namely: tau, collocation and Galerkin methods. Spectral methods are used to solve many equations, including partial differential equations, ordinary differential equations and FDEs. These versions have been widely used by many authors, including Abd-Elhameed and Youssri [13] introduced an approximate approach based on the tau method to solve coupled system of FDEs. Abd-Elhameed et al. [14] employed a collocation method to solve second-order nonlinear two-point boundary value problems. Abd-Elhameed [15] adopted Galerkin method to solve third-order linear two-point boundary value problems. Generalized Fibonacci polynomial (GFP) is one of the most important polynomials in special function. It has grabbed considerable interest due to its wide applications in many areas. Therefore, it has been theoretically investigated by several authors. For some of these studies, see Panwar et al. [16], Nalli and Haukkanen [17] and Gupta et al. [18]. On the other hand, some practical studies addressed it. For example, the collocation algorithm based on employing GFPs has analyzed multi-term fractional differential equations in [19]. The

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tau algorithm based on employing GFPs has analyzed coupled system of Caputo fractional differential equations in [13]. Accordingly, the present paper aims to introduce numerical technique for the fractional B-T equation by applying the spectral tau method. The advantage of using this approach is that it can decrease the solution of the equation with its homogeneous boundary conditions into a system of algebraic equations which may be solved using a suitable solver.
The rest of this paper is organized as follows: Section Two is devoted to the preliminaries and relations that will be used throughout the paper. Section Three is dedicated to the basis function and solving the fractional B-T equation using the tau method. Section Four handles convergence and error analysis of the suggested method. In section Five the present method is applied to various examples. Conclusion is presented in section Six.

2 Preliminaries and Notations

In this section, some important definitions are introduced. Moreover, an overview on GFPs and some properties and relations are presented.

2.1 The fractional derivative in the Caputo sense

Definition 1. *As shown in Podlubny [4], the Caputo definition of the fractional-order derivative is defined as:*

$$D^{\beta} h(x) = \frac{1}{\Gamma(k-\beta)} \int_0^x (x-t)^{k-\beta-1} h^{(k)}(t)dt, \quad \beta > 0, \quad x > 0, \tag{1}$$

$$k-1 \leqslant \beta < k, \quad k \in \mathbb{N},$$

the following properties are satisfied by the operator D^{β} for $\quad k-1 \leqslant \beta < k, \quad k \in \mathbb{N},$

$$D^{\beta} C = 0, \ (C\, is\, a\, constant) \tag{2}$$

$$D^{\beta} x^k = \begin{cases} 0, & \text{if } k \in \mathbb{N}_0 \quad and \quad k < \lceil \beta \rceil, \\ \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)} x^{k-\beta}, & \text{if } k \in \mathbb{N}_0 \quad and \quad k \geq \lceil \beta \rceil. \end{cases} \tag{3}$$

Where $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. In addition, the notation $\lceil \beta \rceil$ represents the ceiling function.

2.2 Some properties and relations of the GFPs

The sequence of GFPs $\{\phi_j^{a,b}(x)\}$ is generated by the following recurrence relation:

$$\phi_j^{a,b}(x) = a\,x\,\phi_{j-1}^{a,b}(x) + b\,\phi_{j-2}^{a,b}(x), \quad j \geq 2, \tag{4}$$

subject to: $\quad \phi_0^{a,b}(x) = 1$ and $\phi_1^{a,b}(x) = ax$.
The analytic form of $\phi_j^{a,b}(x)$ may be written as:

$$\phi_j^{a,b}(x) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j-r}{r} (ax)^{j-2r} (b)^r, \tag{5}$$

which can be expressed alternatively as:

$$\phi_j^{a,b}(x) = \sum_{k=0}^{j} \eta_{j+k}\, a^k\, b^{\frac{j-k}{2}} \binom{\frac{j+k}{2}}{\frac{j-k}{2}} x^k, \tag{6}$$

where

$$\eta_r = \begin{cases} 1, & \text{if } r \, \text{even}, \\ 0, & \text{if } r \, \text{odd}. \end{cases} \tag{7}$$

For more properties about GFPs, see [16]-[18].

3 Choice of the basis function and numerical treatment of the fractional B-T equation

In this section, the basis function is chosen. In addition, the solution of the fractional B-T equation is presented. Furthermore, the nonhomogeneous boundary conditions are converted to the homogeneous ones.

3.1 Choice of the basis function

Suppose that $u(x) \in L^2(0,\ell)$ satisfies the following homogenous boundary conditions:

$$u(0) = u(\ell) = 0.$$

The function $u(x)$ may be written as:

$$u(x) = \sum_{j=0}^{\infty} e_j \psi_j^{a,b}(x), \tag{8}$$

where

$$\psi_j^{a,b}(x) = x(\ell - x)\phi_j^{a,b}(x). \tag{9}$$

Assuming that $u(x)$ can be approximated as:

$$u(x) \approx u_M(x) = \sum_{k=0}^{M} e_k \psi_k^{a,b}(x). \tag{10}$$

In the following section, we address and prove an important theorem related to the basis functions $\psi_j^{a,b}(x)$. We give a new formula for the fractional derivatives of the polynomials $\psi_j^{a,b}(x)$ in the sense of Caputo.

Theorem 1. *Let $\beta \in]1,2]$. Then, the following fractional derivative formula holds.*

$$D^{\beta}\psi_i^{a,b}(x) = x^{-\beta}\left[\sum_{\substack{k=1\\ i+k\,even}}^{i} \frac{a^k b^{\frac{i-k}{2}}\,(\frac{i+k}{2})!\,(k+1)}{\Gamma(k+2-\beta)\,(\frac{i-k}{2})!}\left(\ell x^{k+1} - \frac{(k+2)}{(k+2-\beta)}x^{k+2}\right) + x^2\,\delta_i\right], \tag{11}$$

where

$$\delta_i = \begin{cases} 0, & \text{if } i \,\text{odd},\\ \frac{-2b^{\frac{i}{2}}}{(2-\beta)!}, & \text{if } i \,\text{even}. \end{cases} \tag{12}$$

Proof. The alternative formula for $\phi_i^{a,b}(x)$ in Eq. (6) enables one to write $\psi_i^{a,b}(x)$ as:

$$\psi_i^{a,b}(x) = \sum_{k=0}^{i} \eta_{i+k}\, a^k b^{\frac{i-k}{2}} \binom{\frac{i+k}{2}}{\frac{i-k}{2}} (\ell x^{k+1} - x^{k+2}), \tag{13}$$

which can be rewritten in the form

$$\psi_i^{a,b}(x) = \eta_i\, b^{\frac{i}{2}}\,[\ell x - x^2] + \sum_{\substack{k=1\\ i+k\,\text{even}}}^{i} a^k b^{\frac{i-k}{2}} \binom{\frac{i+k}{2}}{\frac{i-k}{2}} (\ell x^{k+1} - x^{k+2}), \tag{14}$$

applying the operator D^{β} given in Eq. (3) to both sides of Eq. (14), one gets

$$\begin{aligned} D^{\beta}\psi_i^{a,b}(x) = {}& \eta_i\, b^{\frac{i}{2}}\left[\frac{-2}{(2-\beta)!}x^{2-\beta}\right] \\ & + \sum_{\substack{k=1\\ i+k\,\text{even}}}^{i} \frac{a^k b^{\frac{i-k}{2}}\,(\frac{i+k}{2})!}{(\frac{i-k}{2})!\,k!}\left[\ell\,\frac{(k+1)!}{(k+1-\beta)!}x^{k+1-\beta} - \frac{(k+2)!}{(k+2-\beta)!}x^{k+2-\beta}\right], \end{aligned} \tag{15}$$

then

$$D^{\beta}\psi_i^{a,b}(x) = x^{-\beta}\left[\sum_{\substack{k=1\\ i+k\,\text{even}}}^{i} \frac{a^k b^{\frac{i-k}{2}}\,(\frac{i+k}{2})!\,(k+1)}{\Gamma(k+2-\beta)\,(\frac{i-k}{2})!}\left(\ell x^{k+1} - \frac{(k+2)}{(k+2-\beta)}x^{k+2}\right) + x^2\,\delta_i\right], \tag{16}$$

which completes the proof of theorem.

3.2 Numerical treatment of the fractional B-T equation

Consider the following linear fractional B-T equation:

$$a_1 D^{(2)} v(x) + a_2 D^{(\frac{3}{2})} v(x) + a_3 v(x) = h(x), \quad x \in (0,\ell), \tag{17}$$

where ($a_1 \neq 0$ *and* $a_2, a_3 \in \mathbb{R}$) arises, for example, in the modelling of the motion of a rigid plate immersed in a Newtonian fluid [6]. In addition, $h(x)$ symbolizes the external force and $v(x)$ is the displacement of the plate. The above-mentioned equation is subject to the following nonhomogeneous boundary conditions

$$v(0) = \alpha, \quad v(\ell) = \mu. \tag{18}$$

To convert the nonhomogeneous boundary conditions (18) to a homogeneous boundary ones, the following transformation is used

$$u(x) = v(x) - (1 - \frac{x}{\ell}) v(0) - \frac{x}{\ell} v(\ell). \tag{19}$$

Combining Eqs. (17)-(19), one gets

$$a_1 D^{(2)} u(x) + a_2 D^{(\frac{3}{2})} u(x) + a_3 u(x) = k(x), \quad x \in (0,\ell) \tag{20}$$

where

$$k(x) = h(x) - a_3 \left[(1 - \frac{x}{\ell}) \alpha + \frac{x}{\ell} \mu \right], \tag{21}$$

in accordance to the homogeneous boundary conditions

$$u(0) = 0, \quad u(\ell) = 0. \tag{22}$$

Since

$$u(x) \approx u_M(x) = \sum_{i=0}^{M} e_i \psi_i^{a,b}(x),$$

Eq. (20) takes the form

$$a_1 D^{(2)} \sum_{i=0}^{M} e_i \psi_i^{a,b}(x) + a_2 D^{(\frac{3}{2})} \sum_{i=0}^{M} e_i \psi_i^{a,b}(x) + a_3 \sum_{i=0}^{M} e_i \psi_i^{a,b}(x) = k(x), \tag{23}$$

with the aid of Eqs. (11) and (13), the residual of Eq. (23) becomes:

$$\begin{aligned} R(x) = a_1 \sum_{i=0}^{M} e_i & \left[\sum_{\substack{k=1 \\ i+k \text{even}}}^{i} \frac{a^k b^{\frac{i-k}{2}} (\frac{i+k}{2})! (k+1)}{\Gamma(k) (\frac{i-k}{2})!} \left(\ell x^{k-1} - \frac{(k+2)}{k} x^k \right) + \varepsilon_{1i} \right] \\ + a_2 \sum_{i=0}^{M} e_i & \left[\sum_{\substack{k=1 \\ i+k \text{even}}}^{i} \frac{a^k b^{\frac{i-k}{2}} (\frac{i+k}{2})! (k+1)}{\Gamma(k+\frac{1}{2}) (\frac{i-k}{2})!} \left(\ell x^{k-\frac{1}{2}} - \frac{(k+2)}{(k+\frac{1}{2})} x^{k+\frac{1}{2}} \right) + x^{\frac{1}{2}} \varepsilon_{2i} \right] \\ + a_3 \sum_{i=0}^{M} \sum_{\substack{k=0 \\ i+k \text{even}}}^{i} & e_i \frac{a^k b^{\frac{i-k}{2}} (\frac{i+k}{2})!}{(k)! (\frac{i-k}{2})!} \left(\ell x^{k+1} - x^{k+2} \right) \\ - k(x), & \end{aligned} \tag{24}$$

where

$$\varepsilon_{1i} = \begin{cases} 0, & \text{if } i \text{odd}, \\ -2 b^{\frac{i}{2}}, & \text{if } i \text{even}, \end{cases} \tag{25}$$

and

$$\varepsilon_{2i} = \begin{cases} 0, & \text{if } i \text{odd}, \\ \frac{-4 b^{\frac{i}{2}}}{\sqrt{\pi}}, & \text{if } i \text{even}. \end{cases} \tag{26}$$

The application of the tau method yields

$$\int_0^{\ell} R(x)\,\psi_i^{a,b}(x)\,dx = 0, \qquad 0 \le i \le M. \tag{27}$$

Eq. (27) forms a system of algebraic equations in the unknown coefficients e_i, which can be solved via the Gaussian elimination procedure.

4 Convergence and error analysis

Using similar argument as given in [13], the convergence and error analysis for the technique given in section 3 is investigated as follows:

Lemma 1. *According to Abd-Elhameed and Youssri [13], let $f(x)$ be an infinitely differentiable function at $s \in \mathbb{R}$, it has the following expansion*

$$f(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{f^{(i+j)}(s)\,\zeta_{i+j,i}}{(i+j)!}\,\phi_i^{a,b}(x), \tag{28}$$

where

$$\zeta_{i,j} = (j+1)\,i!\begin{cases} \dfrac{(-b)^{\frac{i-j}{2}}\,a^{-i}}{(\frac{i-j}{2})!\,(\frac{i+j}{2}+1)!}\,{}_2F_1\left(\frac{-i-j-2}{2}, \frac{j-i}{2}; \frac{1}{2}; \frac{-a^2 s^2}{4b}\right), & \text{if } (i+j)\text{ even},\\[2ex] \dfrac{(-b)^{\frac{i-j-1}{2}}\,a^{-i+1}}{(\frac{i-j-1}{2})!\,(\frac{i+j+1}{2})!}\,{}_2F_1\left(\frac{-i-j-1}{2}, \frac{j-i+1}{2}; \frac{3}{2}; \frac{-a^2 s^2}{4b}\right), & \text{if } (i+j)\text{ odd}. \end{cases} \tag{29}$$

where ${}_2F_1$ is the hypergeometric function.

Lemma 2. *If $g(x)$ is a solution of Eq. (20) at any real number s, it can be rewritten as:*

$$g(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{f^{(i+j)}(s)\,\zeta_{i+j,i}}{(i+j)!}\,\psi_i^{a,b}(x), \tag{30}$$

where $g(x) = x(\ell - x)f(x)$.

Proof. The proof is easily found by multiplying both sides of Lemma 1 by $x(\ell - x)$ to get the desired equation.

Lemma 3. *The following inequality holds:*

$$|\psi_k^{a,b}(x)| \le \frac{\ell^2}{4}\rho^k, \quad x \in [0,\ell], \forall \ell > 0 \quad \text{and}\ \rho = \sqrt{a^2\,\ell^2 + 2\,|b|}. \tag{31}$$

Proof. It is evident that

$$|\psi_k^{a,b}(x)| = |x(\ell - x)\,\phi_k^{a,b}(x)| \le \frac{\ell^2}{4}\,|\phi_k^{a,b}(x)|,$$

Using lemma 5 in Abd-Elhameed and Youssri [13], one gets

$$|\phi_k^{a,b}(x)| \le \rho^k,$$

then

$$|\psi_k^{a,b}(x)| \le \frac{\ell^2}{4}\rho^k.$$

Theorem 2. *If $g(x)$ is defined on $[0,\ell]$ and $|g^{(i)}(s)| \le \lambda^i \ , i > 0$, where $s \in (0,\ell)$, λ is a positive constant and $g(x) = \sum_{k=0}^{\infty} c_k\,\psi_k^{a,b}(x)$, the following conclusions are obtained:*

1. $|c_k| < \frac{6\,\theta\,|b|\,(\frac{\lambda}{|a|})^{k+1}}{k!\,\lambda\,|a|\,s^2}\cosh\left(\frac{2\lambda\sqrt{|b|}}{|a|}\right)$,
where $\theta = Li_6(\frac{a^2 s^2}{3\,|b|})$ and $Li_n(x)$ is the well-known polylogarithm function [20].

2. The series converges absolutely.

Proof. As shown in Abd-Elhameed and Youssri [13], the first part is proved.
To prove the series $\sum_{k=0}^{\infty} c_k\, \psi_k^{a,b}(x)$ is absolutely convergent, the absolute convergence must be shown. With the aid of the first part and Lemma 3, one gets

$$\sum_{k=0}^{\infty} |c_k\, \psi_k^{a,b}(x)| < \sum_{k=0}^{\infty} \frac{3\,\ell^2\,\theta\,|b|\,(\frac{\rho\,\lambda}{|a|})^{k+1}\cosh\left(\frac{2\,\lambda\,\sqrt{|b|}}{|a|}\right)}{2\,k!\,\lambda\,|a|\,s^2}.$$

In accordance with the ratio test, the series $\sum_{k=0}^{\infty} \frac{3\,\ell^2\,\theta\,|b|\,(\frac{\rho\,\lambda}{|a|})^{k+1}\cosh\left(\frac{2\,\lambda\,\sqrt{|b|}}{|a|}\right)}{2\,k!\,\lambda\,|a|\,s^2}$ is convergent, then by applying the comparison test, the series $\sum_{k=0}^{\infty} |c_k\, \psi_k^{a,b}(x)|$ is convergent. Hence, the series is absolutely convergent.

Theorem 3. *If $g(x)$ satisfies the hypothesis of Theorem 2, and $e_n(x) = \sum_{k=n+1}^{\infty} c_k\, \psi_k^{a,b}(x)$, the following error estimation is satisfied:*

$$|e_n(x)| \leq \frac{3\,e^{\tau}\,\ell^2\,\theta\,|b|\,\cosh\left(\frac{2\,\lambda\,\sqrt{|b|}}{|a|}\right)\,\tau^{n+1}}{2\,|a|^2\,s^2\,(n+1)!}, \tag{32}$$

where $\tau = \frac{\lambda\,\rho}{|a|}$.

Proof. From Theorem 2, one gets

$$|e_n(x)| \leq \frac{3\,\ell^2\,\theta\,|b|\,\cosh\left(\frac{2\,\lambda\,\sqrt{|b|}}{|a|}\right)}{2\,\lambda\,|a|\,s^2} \sum_{k=n+1}^{\infty} \frac{(\frac{\lambda}{|a|})^{k+1}}{k!}\,\rho^k, \tag{33}$$

in other words,

$$|e_n(x)| \leq \frac{3\,\ell^2\,\theta\,|b|\,\cosh\left(\frac{2\,\lambda\,\sqrt{|b|}}{|a|}\right)}{2\,|a|^2\,s^2} \sum_{k=n+1}^{\infty} \frac{\tau^k}{k!}, \tag{34}$$

since

$$\sum_{k=n+1}^{\infty} \frac{\tau^k}{k!} = \frac{\gamma(n+1,\tau)}{\Gamma(n+1)}\,e^{\tau}, \tag{35}$$

where $\gamma(n+1,\tau)$ is the lower incomplete gamma function [21], then

$$|e_n(x)| \leq \frac{3\,e^{\tau}\,\ell^2\,\theta\,|b|\,\cosh\left(\frac{2\,\lambda\,\sqrt{|b|}}{|a|}\right)}{2\,|a|^2\,s^2\,\Gamma(n+1)} \int_0^{\tau} t^n\,e^{-t}\,dt. \tag{36}$$

It is evident that $e^{-t} \leq 1, \quad \forall t \geq 0$, the inequality (36) may be formulated as

$$|e_n(x)| \leq \frac{3\,e^{\tau}\,\ell^2\,\theta\,|b|\,\cosh\left(\frac{2\,\lambda\,\sqrt{|b|}}{|a|}\right)\,\tau^{n+1}}{2\,|a|^2\,s^2\,(n+1)!}. \tag{37}$$

5 Numerical results

In order to show the convenience and validity of the above-mentioned technique, the present procedure is illustrated throughout some numerical examples.

Example 1. As given in Mdallal et al.[22], consider the fractional B-T equation

$$D^{(2)}v(x) + D^{(\frac{3}{2})}v(x) + v(x) = 2 + 4\sqrt{\frac{x}{\pi}} + x^2, \qquad x \in (0,5), \tag{38}$$

subject to

$$v(0)=0, \quad v(5)=25, \tag{39}$$

where the exact solution is $v(x)=x^2$. Using the transformation (19), Eqs. (38) and (39) become

$$D^{(2)}u(x)+D^{(\frac{3}{2})}u(x)+u(x)=2-5x+4\sqrt{\frac{x}{\pi}}+x^2, \qquad x\in(0,5), \tag{40}$$

subject to

$$u(0)=0, \quad u(5)=0, \tag{41}$$

and the exact solution $u(x)=x^2-5x$. Applying the present method to Eq. (40), the residual for the case that corresponds to $M=1$ and $a=b=1$ takes the form

$$R(x)=x(5+5e_0-6e_1)-2(1+e_0-5e_1)-x^2(1+e_0-5e_1)-x^3e_1-\frac{4\sqrt{x}\,(1+e_0+(-5+2x)e_1)}{\sqrt{\pi}} \tag{42}$$

The application of the present method yields the following two equations

$$576\sqrt{\frac{5}{\pi}}+18\left(-35+32\sqrt{\frac{5}{\pi}}\right)e_0+5\left(-315+64\sqrt{\frac{5}{\pi}}\right)e_1=630 \tag{43}$$

and

$$704\sqrt{\frac{5}{\pi}}+11\left(-63+64\sqrt{\frac{5}{\pi}}\right)e_0+12\left(-121+80\sqrt{\frac{5}{\pi}}\right)e_1=693. \tag{44}$$

Eqs. (43) and (44) may be solved to give $e_0=-1, e_1=0,$
and consequently $u_M(x)=x^2-5x$. Hence, the exact solution would be $v_M(x)=x^2$.

*Example 2.*Consider the fractional B-T equation

$$D^{(2)}v(x)+D^{(\frac{3}{2})}v(x)+v(x)=e^{\gamma x}[1+\gamma^2+\gamma^{\frac{3}{2}}\,erf(\sqrt{\gamma x})], \qquad x\in(0,1), \tag{45}$$

subject to

$$v(0)=1, \quad v(1)=e^{\gamma}, \tag{46}$$

whose exact solution is given by $v(x)=e^{\gamma x}$, where erf(x) is defined as:

$$\mathrm{erf(x)}=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy.$$

The maximum absolute error (MAE) for various values of M at $\gamma=1$ is given in Table 1. Moreover, Table 2 presents the MAE for several values of M at $\gamma=\pi$. In addition, Figure 1 illustrates the absolute error for the case $M=6$ at $\gamma=1$, while Figure 2 shows the absolute error for the case $M=9$ at $\gamma=\pi$.

Table 1: MAE of Example 2 at $\gamma=1$

M	1	2	3	4	5	6
MAE	1.5×10^{-3}	8×10^{-5}	3×10^{-6}	1.5×10^{-7}	6×10^{-9}	2×10^{-9}

Table 2: MAE of Example 2 at $\gamma=\pi$

M	2	4	6	8
MAE	8×10^{-2}	15×10^{-4}	15×10^{-6}	8×10^{-8}

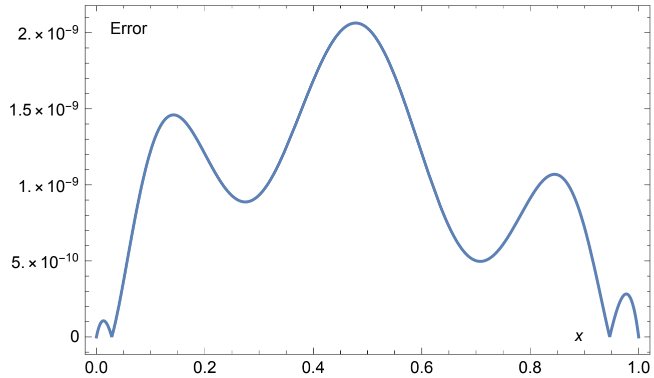

Fig. 1: The absolute error at M=6 and $\gamma = 1$ for Example 2.

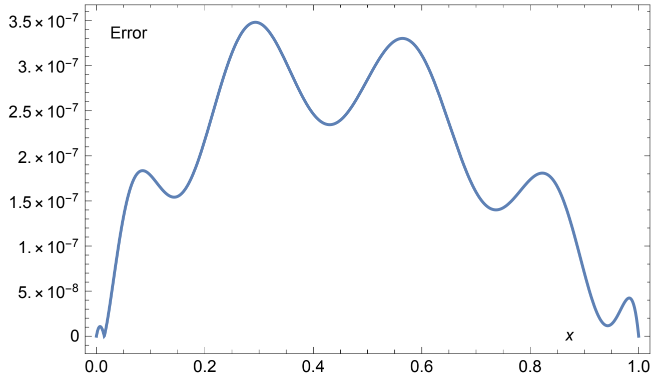

Fig. 2: The absolute error at M=9 and $\gamma = \pi$ for Example 2.

*Example 3.*Consider the fractional B-T equation

$$D^{(2)}v(x) + D^{(\frac{3}{2})}v(x) + v(x) = g(x), \qquad x \in (0,1), \tag{47}$$

subject to

$$v(0) = \frac{1}{\Gamma(\beta)}, \quad v(1) = E_{\alpha,\beta}(1), \tag{48}$$

where $g(x)$ is chosen such that the exact solution of Eq. (47) is $v(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)} = E_{\alpha,\beta}(x)$ and $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function [23] of the two parameters $\alpha, \beta > 0$.
Eq. (47) is solved in two cases corresponding to $\alpha = 2, \beta = 1$ and $\alpha = \frac{1}{2}, \beta = 1$.

Case 1: At $\alpha = 2, \beta = 1$, this exact solution reduces to $v(x) = E_{2,1}(x) = \cosh(x)$. Comparison of the numerical solutions with the exact solution is given in Table 3 for the case corresponding to $M = 2$.

Case 2: At $\alpha = \frac{1}{2}, \beta = 1$, this exact solution reduces to $v(x) = E_{\frac{1}{2},1}(x) = e^{x^2}$erfc(-x). Where erfc(-x) is the complementary error function and defined as:

$$\text{erfc(x)} = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = 1 - \text{erf(x)}. \tag{49}$$

In Table 3, we compare the numerical solutions with the exact solution for the case corresponding to $M = 11$. The results displayed in this table show that our method is efficient and accurate.

Table 3: Comparison of the numerical solutions with the exact solution

	$\alpha = 2, \beta = 1, M = 2$		$\alpha = \frac{1}{2}, \beta = 1, M = 11$	
x	Numerical solution	Exact solution	Numerical solution	Exact solution
0.1	-0.00389000537	-0.00389000544	-0.27725465352	-0.27725465387
0.2	-0.00693830825	-0.00693830941	-0.52919399026	-0.52919398766
0.3	-0.00913648701	-0.00913648888	-0.74894479091	-0.74894479138
0.4	-0.01047605886	-0.01047606062	-0.92735807662	-0.92735807561
0.5	-0.01094847984	-0.01094848089	-1.05212955482	-1.05212955119
0.6	-0.01054514477	-0.01054514508	-1.10653393462	-1.10653393672
0.7	-0.00925738735	-0.00925738739	-1.06758395602	-1.06758395397
0.8	-0.00707648007	-0.00707648049	-0.90332289640	-0.90332289522
0.9	-0.00399363424	-0.00399363517	-0.56879775034	-0.56879775065

Example 4. As given in [9,24], consider the fractional B-T equation

$$D^{(2)}v(x) + \frac{8}{17}D^{(\frac{3}{2})}v(x) + \frac{13}{51}v(x) = \frac{x^{-\frac{1}{2}}}{89250\sqrt{\pi}}\left(48\,g(x) + 7\sqrt{\pi x}\,f(x)\right), \qquad x \in]0,1], \tag{50}$$

subject to

$$v(0) = 0, \quad v(1) = 0, \tag{51}$$

where

$$g(x) = 16000x^4 - 32480x^3 + 21280x^2 - 4746x$$

and

$$f(x) = 3250x^5 - 9425x^4 + 264880x^3 - 448107x^2 + 233262x - 34578.$$

The exact solution of Eq. (50) is $v(x) = x^5 - \frac{29}{10}x^4 + \frac{76}{25}x^3 - \frac{339}{250}x^2 + \frac{27}{125}x.$
Applying the present method at $M = 3$ and $a = b = 1$, the following equations are obtained:

$$6.05631\,e_0 + 2.38752\,e_1 + 7.41393\,e_2 + 5.63881\,e_3 = 0.302078 \tag{52}$$

$$1.87181\,e_0 + 1.21582\,e_1 + 2.74908\,e_2 + 3.08073\,e_3 = 0.0360079 \tag{53}$$

$$0.812231\,e_0 + 0.402717\,e_1 + 1.08223\,e_2 + 1.00037\,e_3 = 0.0344076 \tag{54}$$

$$1.4587\,e_0 + 1.01963\,e_1 + 2.23706\,e_2 + 2.64217\,e_3 = 0.0286882 \tag{55}$$

Eqs. (52)-(55) can be solved using Gaussian elimination method to give

$$e_0 = \frac{-421}{250}, e_1 = \frac{43}{50}, e_2 = \frac{19}{10}, e_3 = -1,$$

and $v_M(x) = x^5 - \frac{29}{10}x^4 + \frac{76}{25}x^3 - \frac{339}{250}x^2 + \frac{27}{125}x$, which is the exact solution.

6 Conclusion

In this paper, a new basis of GFPs is successfully employed for solving homogeneous boundary value problem of fractional B-T differential equation. The convergence and error analysis are deeply discussed. Moreover, numerical tests conducted in section Five demonstrated accuracy of this technique.

Acknowledgment

The authors gratefully acknowledge the anonymous reviewers fruitful and insightful comments on the previous version of this paper.

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