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New inequalities of Hermite-Hadamard type for h-convex functions via generalized fractional integrals

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Abstract: In this paper, we establish new inequalities of Hermite-Hadamard type for h-convex functions using generalized fractional integral. The results are an extension of a previous research.

Keywords: Hermite-Hadamard inequalities, generalized fractional integral, h-convex functions.

1 Introduction

Because of its indisputable applications in several sectors of science and engineering, the subject of fractional calculus (integrals and derivatives) has grown in quality and importance during the last three decades. The fractional integral is useful for a variety of problems involving mathematical science special functions, as well as their extensions and generalizations in one and multiple variables. This subject is extensively investigated by several authors, (see [1,2,3,4,5, 6,7,8,9,10,11,12,13]). One among the vital applications of fractional integrals is Hermite-Hadamard integral inequality, see [14,15,16]. First, let's recall the fundamental expression of the classical Hermite-Hadamard inequality, as follows:

For a convex mapping $F : I \subseteq \mathbb{R} \to \mathbb{R}$ on the interval $I = [\kappa_1, \kappa_2]$, we have the following double inequality [15]:

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)dx \leq \frac{F(\kappa_1)+F(\kappa_2)}{2}. \tag{1}$$

The Hermite-Hadamard inequality is an inequality that linked to the integral mean of a convex function.

In [17], Dragomir and Agarwal established the following useful results related to the right hand side of (1):

Lemma 1. *Let $F : I = [\kappa_1, \kappa_2] \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on $I^\circ = (\kappa_1, \kappa_2)$, then following equality holds:*

$$\frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)dx = \frac{\kappa_2-\kappa_1}{2}\int_0^1 (1-2\xi)F'(\xi\kappa_1+(1-\xi)\kappa_2)d\xi. \tag{2}$$

Theorem 1. *Let $F : I = [\kappa_1, \kappa_2] \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on $I^\circ = (\kappa_1, \kappa_2)$. If $\left|F'\right|$ is convex, then we have the following inequality:*

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)dx\right| \leq \frac{(\kappa_2-\kappa_1)}{8}\left(\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right). \tag{3}$$

In [13], Sarikaya established the fundamental Hermite-Hadamard type inequalities for fractional integrals, as follows:

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Theorem 2. *Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a positive function with $L_1([\kappa_1, \kappa_2])$, with $\kappa_1 < \kappa_2$. If F is convex function on the given interval, then we have the following double inequalities for fractional integrals:*

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\kappa_2-\kappa_1)^\alpha}\left[J^{\alpha}_{\kappa_1+}F(\kappa_2)+J^{\alpha}_{\kappa_2-}\right] \leq \frac{F(\kappa_1)+F(\kappa_2)}{2}, \tag{4}$$

with $\alpha > 0$.

Definition 1. *[17] A non negative function $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ is said to be convex function, if the following inequality holds:*

$$F(\xi\kappa_1 + (1-\xi)\kappa_2) \leq \xi F(\kappa_1) + (1-\xi)F(\kappa_2),$$

for all ξ in $[0,1]$.

Definition 2. *[18] A non negative function $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ is said to be $s-$convex function in second sense, if the following inequality holds:*

$$F(\xi\kappa_1 + (1-\xi)\kappa_2) \leq \xi^s F(\kappa_1) + (1-\xi)^s F(\kappa_2),$$

for all ξ in $[0,1]$ and s in $(0,1]$.

In [19], Varošanec defined generalization of convex, $s-$convex, Godunova-Levin functions and $P-$convex functions as follows:

Definition 3. *[19] Let $h : J \to \mathbb{R}$ be non negative function and $h \neq 0$. A non negative function $F : I = [\kappa_1, \kappa_2] \to \mathbb{R}$ is called $h-$convex function, if the following inequalities holds:*

$$F(\xi\kappa_1 + (1-\xi)\kappa_2) \leq h(\xi)F(\kappa_1) + h(1-\xi)F(\kappa_2),$$

for all ξ in $[0,1]$. For more discussion about the $h-$convex function, see [19].

In [20], Sarikaya and Ertuğral defined a new left-sided and right-sided generalized fractional integrals, as follows:

$$_{\kappa_1+}I_\varphi F(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\kappa_1}^{\varkappa} \frac{\varphi(\varkappa-\xi)}{\varkappa-\xi}F(\xi)d\xi, \ \ \varkappa > \kappa_1$$

$$_{\kappa_2-}I_\varphi F(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\kappa_2} \frac{\varphi(\xi-\varkappa)}{\xi-\varkappa}F(\xi)d\xi, \ \ \varkappa < \kappa_2$$

respectively, where $\varphi : [0,\infty) \to [0,\infty)$ a function which satisfies $\int_0^1 \frac{\varphi(\xi)}{\xi}d\xi < \infty$.

Some of the special cases of these generalized fractional operators are given, as follows:

Remark. If we choose $\varphi(\xi) = \xi$, $\varphi(\xi) = \frac{1}{\Gamma(\alpha)}\xi^\alpha$, $\varphi(\xi) = \frac{1}{k\Gamma_k(\alpha)}\xi^{\frac{\alpha}{k}}$, $k > 0$, $\varphi(\xi) = \xi(\varkappa-\xi)^{\alpha-1}$ and $\varphi(\xi) = \frac{\xi}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}\xi\right), \alpha \in (0,1)$, then we obtain classical Riemann integral, Riemann-Liouville fractional integral, k-Riemann-Liouville fractional [6], conformable fractional integrals [21] and fractional integral operators with exponential kernel [22], respectively.

The present paper aims to establish inequalities of Hermite-Hadamard type for the h-convex functions via generalized fractional integrals. We also discuss some special cases of our main results.

2 Main results

In this study, we suppose

$$\Lambda(\varkappa) = \int_0^{\varkappa} \frac{\varphi\left(\frac{(\kappa_2-\kappa_1)\xi}{2}\right)}{\xi}d\xi.$$

Lemma 2.*Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $\kappa_1, \kappa_2 \in I$, with $\kappa_1 < \kappa_2$. If $F' \in L[\kappa_1, \kappa_2]$, then we have the following identity that holds for generalized fractional integrals:*

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}I_{\varphi}F(\kappa_2) + {}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}I_{\varphi}F(\kappa_1)\right]$$
$$= \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1 \Lambda(\xi)\left[F'\left((1-\xi)\kappa_1 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right) - F'\left((1-\xi)\kappa_2 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right]d\xi, \tag{5}$$

and

$$\frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + {}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]$$
$$\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1 \Lambda(\xi)\left[F'\left(\xi\kappa_2 + (1-\xi)\frac{\kappa_1+\kappa_2}{2}\right) - F'\left(\xi\kappa_1 + (1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right]d\xi. \tag{6}$$

*Proof.*Integrating by parts, we have

$$\int_0^1 \Lambda(\xi)\left[F'\left((1-\xi)\kappa_1 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right) - F'\left((1-\xi)\kappa_2 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right]d\xi$$
$$= \int_0^1 \Lambda(\xi)F'\left((1-\xi)\kappa_1 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)d\xi - \int_0^1 \Lambda(\xi)F'\left((1-\xi)\kappa_2 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)d\xi$$
$$= \frac{2}{\kappa_2-\kappa_1}\left[\Lambda(\xi)F\left((1-\xi)\kappa_1 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\Big|_0^1 - \int_0^1 \frac{\varphi\left(\frac{(\kappa_2-\kappa_1)\xi}{2}\right)}{\xi}F\left((1-\xi)\kappa_1 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)d\xi\right]$$
$$+\frac{2}{\kappa_2-\kappa_1}\left[\Lambda(\xi)F\left((1-\xi)\kappa_2 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\Big|_0^1 - \int_0^1 \frac{\varphi\left(\frac{(\kappa_2-\kappa_1)\xi}{2}\right)}{\xi}F\left((1-\xi)\kappa_2 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)d\xi\right]$$
$$= \frac{2}{\kappa_2-\kappa_1}\left[\Lambda(1)F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \int_0^1 \frac{\varphi\left(\frac{(\kappa_2-\kappa_1)\xi}{2}\right)}{\xi}F\left((1-\xi)\kappa_1 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)d\xi\right]$$
$$+\frac{2}{\kappa_2-\kappa_1}\left[\Lambda(1)F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \int_0^1 \frac{\varphi\left(\frac{(\kappa_2-\kappa_1)\xi}{2}\right)}{\xi}F\left((1-\xi)\kappa_2 + \xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)d\xi\right].$$

Changing the variable, we have our required identity (5).

Similarly, integrating by parts, we have

$$\int_0^1 \Lambda(\xi)\left[F^{'}\left(\xi\kappa_2+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)-F^{'}\left(\xi\kappa_1+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right]d\xi$$

$$=\int_0^1 \Lambda(\xi)F^{'}\left(\xi\kappa_2+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)d\xi-\int_0^1 \Lambda(\xi)F^{'}\left(\xi\kappa_1+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)d\xi$$

$$=\frac{2}{\kappa_2-\kappa_1}\left[\Lambda(\xi)F\left(\xi\kappa_2+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\Big|_0^1-\int_0^1\frac{\varphi\left(\frac{(\kappa_2-\kappa_1)\xi}{2}\right)}{\xi}F\left(\xi\kappa_2+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)d\xi\right]$$

$$+\frac{2}{\kappa_2-\kappa_1}\left[\Lambda(\xi)F\left(\xi\kappa_1+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\Big|_0^1-\int_0^1\frac{\varphi\left(\frac{(\kappa_2-\kappa_1)\xi}{2}\right)}{\xi}F\left(\xi\kappa_1+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)d\xi\right]$$

$$=\frac{2}{\kappa_2-\kappa_1}\left[\Lambda(1)F(\kappa_2)-\int_0^1\frac{\varphi\left(\frac{(\kappa_2-\kappa_1)\xi}{2}\right)}{\xi}F\left(\xi\kappa_2+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)d\xi\right]$$

$$+\frac{2}{\kappa_2-\kappa_1}\left[\Lambda(1)F(\kappa_1)-\int_0^1\frac{\varphi\left(\frac{(\kappa_2-\kappa_1)\xi}{2}\right)}{\xi}F\left(\xi\kappa_1+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)d\xi\right].$$

Changing the variables, we have our desired identity (6).
This completes the proof.

Theorem 3. *Let $F: I=[\kappa_1,\kappa_2]\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $F^{'}\in L([\kappa_1,\kappa_2])$. If $\left|F^{'}\right|$ is h-convex, then the following inequality holds for the generalized fractional integrals:*

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}I_\varphi F(\kappa_2)+{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}I_\varphi F(\kappa_1)\right]\right| \tag{7}$$

$$\leq\frac{(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\left|F^{'}(\kappa_1)\right|+\left|F^{'}(\kappa_2)\right|\right)\int_0^1\Lambda(\xi)(h(1-\xi))d\xi+\frac{(\kappa_2-\kappa_1)}{2\Lambda(1)}\left|F^{'}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\int_0^1\Lambda(\xi)h(\xi)d\xi,$$

and

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_\varphi F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_\varphi F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \tag{8}$$

$$\leq\frac{(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\left|F^{'}(\kappa_1)\right|+\left|F^{'}(\kappa_2)\right|\right)\int_0^1\Lambda(\xi)h(\xi)d\xi+\frac{(\kappa_2-\kappa_1)}{2\Lambda(1)}\left|F^{'}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\int_0^1\Lambda(\xi)h(1-\xi)d\xi.$$

Proof. From (5) of Lemma 2 and since $\left|F^{'}\right|$ is h-convex function, we have the following inequality

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}I_\varphi F(\kappa_2)+{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}I_\varphi F(\kappa_1)\right]\right|$$

$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left|F^{'}\left((1-\xi)\kappa_1+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|d\xi+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left|F^{'}\left((1-\xi)\kappa_2+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|d\xi$$

$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left[h(1-\xi)\left|F^{'}(\kappa_1)\right|+h(\xi)\left|F^{'}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\right]d\xi$$

$$+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left[h(1-\xi)\left|F^{'}(\kappa_2)\right|+h(\xi)\left|F^{'}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\right]d\xi$$

$$=\frac{(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\left|F^{'}(\kappa_1)\right|+\left|F^{'}(\kappa_2)\right|\right)\int_0^1\Lambda(\xi)(h(1-\xi))d\xi+\frac{(\kappa_2-\kappa_1)}{2\Lambda(1)}\left|F^{'}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\int_0^1\Lambda(\xi)h(\xi)d\xi,$$

which is our required inequality (7).

Now from (6) of Lemma 2 and since $\left|F'\right|$ is h-convex function, we have the following inequality

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$

$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1 \Lambda(\xi)\left|F'\left(\xi\kappa_2+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right|d\xi+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1 \Lambda(\xi)\left|F'\left(\xi\kappa_1+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right|d\xi$$

$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1 \Lambda(\xi)\left[h(\xi)\left|F'(\kappa_2)\right|+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\right]d\xi$$

$$+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1 \Lambda(\xi)\left[h(\xi)\left|F'(\kappa_1)\right|+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\right]d\xi$$

$$=\frac{(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right)\int_0^1 \Lambda(\xi)h(\xi)d\xi+\frac{(\kappa_2-\kappa_1)}{2\Lambda(1)}\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\int_0^1 \Lambda(\xi)h(1-\xi)d\xi,$$

which is our required inequality (8).

Remark. Under the assumption of Theorem (3) $\varphi(\xi)=\frac{\xi^{\alpha}}{\Gamma(\alpha)}$, we have Theorem 2.1 of [23].

Remark. Under the assumption of Theorem (3) $\varphi(\xi)=\frac{\xi^{\alpha}}{\Gamma(\alpha)}$ and $h(\xi)=\xi$, we have Corollary 1 of [23].

Remark. Under the assumption of Theorem (3) $\varphi(\xi)=\frac{\xi^{\alpha}}{\Gamma(\alpha)}$ and $h(\xi)=\xi^{s}$, we have Corollary 2 of [23].

Corollary 1. *Let $F:I=[\kappa_1,\kappa_2]\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $F'\in L([\kappa_1,\kappa_2])$. If $\left|F'\right|$ is h-convex, then the following inequality holds for the k-Riemann-Liouville fractional integrals:*

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,k}F(\kappa_2)+I^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,k}F(\kappa_1)\right]\right| \tag{9}$$

$$\leq \frac{(\kappa_2-\kappa_1)}{4}\left(\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right)\int_0^1 \xi^{\frac{\alpha}{k}}(h(1-\xi))d\xi+\frac{(\kappa_2-\kappa_1)}{2}\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\int_0^1 \xi^{\frac{\alpha}{k}}h(\xi)d\xi,$$

and

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[I_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+I_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \tag{10}$$

$$\leq \frac{(\kappa_2-\kappa_1)}{4}\left(\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right)\int_0^1 \xi^{\frac{\alpha}{k}}h(\xi)d\xi+\frac{(\kappa_2-\kappa_1)}{2}\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\int_0^1 \xi^{\frac{\alpha}{k}}h(1-\xi)d\xi.$$

Proof. In Theorem (3), if we use $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have our desired inequalities (9) and (10).

Remark. If we use $h(\xi)=\xi$ and $h(\xi)=\xi^{s}$ in Corollary 1, then we obtain new inequalities involving k-Riemann-Liouville fractional for convex functions and s-convex functions in second sense, receptively.

Theorem 4. *Let $F:I=[\kappa_1,\kappa_2]\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $F'\in L([\kappa_1,\kappa_2])$. If $\left|F'\right|^{q}$ is h-convex, then the following inequality holds for the generalized fractional integrals:*

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}I_{\varphi}F(\kappa_2)+{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}I_{\varphi}F(\kappa_1)\right]\right| \tag{11}$$

$$\leq \frac{(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^{p}d\xi\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(h(1-\xi)\left|F'(\kappa_1)\right|^{q}+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^{q}\right)d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(h(1-\xi)\left|F'(\kappa_2)\right|^{q}+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^{q}\right)d\xi\right)^{\frac{1}{q}}\right],$$

and

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \tag{12}$$
$$\leq \frac{(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(h(\xi)\left|F'(\kappa_1)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$
$$\left.\left(\int_0^1\left(h(\xi)\left|F'(\kappa_2)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $p,q>1$.

*Proof.*From identity (5) of Lemma 2 and using the well known Hölder inequality, we have

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}I_{\varphi}F(\kappa_2)+{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}I_{\varphi}F(\kappa_1)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left|F'\left((1-\xi)\kappa_1+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|d\xi+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left|F'\left((1-\xi)\kappa_2+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|d\xi$$
$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|F'\left((1-\xi)\kappa_1+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|^q d\xi\right)^{\frac{1}{q}}$$
$$+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|F'\left((1-\xi)\kappa_2+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|^q d\xi\right)^{\frac{1}{q}}$$
$$\leq \frac{M(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(h(1-\xi)\left|F'(\kappa_1)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}$$
$$+\frac{M(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(h(1-\xi)\left|F'(\kappa_2)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}$$
$$=\frac{M(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(h(1-\xi)\left|F'(\kappa_1)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left(h(1-\xi)\left|F'(\kappa_2)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right],$$

which is our required inequality (11).

From identity (6) of Lemma 2 and using well known Hölder inequality, we have the following inequality

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$

$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1 \Lambda(\xi)\left|F'\left(\xi\kappa_2+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right|d\xi+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1 \Lambda(\xi)\left|F'\left(\xi\kappa_1+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right|d\xi$$

$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|F'\left(\xi\kappa_2+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right|^q d\xi\right)^{\frac{1}{q}}$$

$$+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|F'\left(\xi\kappa_1+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right|^q d\xi\right)^{\frac{1}{q}}$$

$$\leq \frac{M(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(h(\xi)\left|F'(\kappa_1)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}$$

$$+\frac{M(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(h(\xi)\left|F'(\kappa_2)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}$$

$$=\frac{M(\kappa_2-\kappa_1)}{4\Lambda(1)}\left(\int_0^1(\Lambda(\xi))^p d\xi\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(h(\xi)\left|F'(\kappa_1)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.\left(\int_0^1\left(h(\xi)\left|F'(\kappa_2)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right],$$

which gives the inequality (12).
This completes the proof.

*Remark.*Under the assumption of Theorem 4 $\varphi(\xi)=\frac{\xi^\alpha}{\Gamma(\alpha)}$, we have Theorem 2.2 of [23].

Corollary 2.*Let $F: I=[\kappa_1,\kappa_2]\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $F'\in L([\kappa_1,\kappa_2])$. If $\left|F'\right|^q$ is h-convex, then the following inequality holds for the k-Riemann-Liouville fractional integrals:*

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,k}F(\kappa_2)+I^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,k}F(\kappa_1)\right]\right| \tag{13}$$

$$\leq\frac{(\kappa_2-\kappa_1)}{4}\left(\frac{k}{p\alpha-k}\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(h(1-\xi)\left|F'(\kappa_1)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(h(1-\xi)\left|F'(\kappa_2)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right],$$

and

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[I_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+I_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \tag{14}$$

$$\leq\frac{(\kappa_2-\kappa_1)}{4}\left(\frac{k}{p\alpha-k}\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(h(\xi)\left|F'(\kappa_1)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.\left(\int_0^1\left(h(\xi)\left|F'(\kappa_2)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p,q>1$.

Proof. In Theorem (4), if we use $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have our desired inequalities (13) and (14).

Theorem 5. *Let $F: I=[\kappa_1,\kappa_2]\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $F'\in L([\kappa_1,\kappa_2])$. If $\left|F'\right|^q$, $q\geq 0$, is h-convex, then the following inequality holds for the generalized fractional integrals:*

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^+}I_\varphi F(\kappa_2)+{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^-}I_\varphi F(\kappa_1)\right]\right| \tag{15}$$
$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1\Lambda(\xi)d\xi\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\Lambda(\xi)\left(h(1-\xi)\left|F'(\kappa_1)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\Lambda(\xi)\left(h(1-\xi)\left|F'(\kappa_2)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right],$$

and

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_\varphi F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_\varphi F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \tag{16}$$
$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1\Lambda(\xi)d\xi\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\Lambda(\xi)\left(h(\xi)\left|F'(\kappa_2)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\Lambda(\xi)\left(h(\xi)\left|F'(\kappa_1)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right].$$

Proof. From identity (5) of Lemma 2 and using the well known power mean inequality, we have the following inequality

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^+}I_\varphi F(\kappa_2)+{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^-}I_\varphi F(\kappa_1)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left|F'\left((1-\xi)\kappa_1+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|d\xi+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left|F'\left((1-\xi)\kappa_2+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|d\xi$$
$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1\Lambda(\xi)d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\Lambda(\xi)\left|F'\left((1-\xi)\kappa_1+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|^q d\xi\right)^{\frac{1}{q}}$$
$$+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1\Lambda(\xi)d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\Lambda(\xi)\left|F'\left((1-\xi)\kappa_2+\xi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|^q d\xi\right)^{\frac{1}{q}}.$$

Since $\left|F'\right|^q$ is convex, then we have

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^+}I_\varphi F(\kappa_2)+{}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^-}I_\varphi F(\kappa_1)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1\Lambda(\xi)d\xi\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\Lambda(\xi)\left(h(1-\xi)\left|F'(\kappa_1)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\Lambda(\xi)\left(h(1-\xi)\left|F'(\kappa_2)\right|^q+h(\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right],$$

which is our required inequality (15).

Now, from identity (6) of Lemma 2 and using the well known power mean inequality, we have the following inequality

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_\varphi F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_\varphi F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$

$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left|F'\left(\xi\kappa_2+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right|d\xi+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\int_0^1\Lambda(\xi)\left|F'\left(\xi\kappa_1+(1-\xi)\frac{\kappa_1+\kappa_2}{2}\right)\right|d\xi$$

$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1\Lambda(\xi)\right)^{1-\frac{1}{q}}\left(\int_0^1\Lambda(\xi)\left|F'\left(\xi\kappa_2+(1-\xi)\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|^q d\xi\right)^{\frac{1}{q}}$$

$$+\frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1\Lambda(\xi)d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\Lambda(\xi)\left|F'\left(\xi\kappa_1+(1-\xi)\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)\right|^q d\xi\right)^{\frac{1}{q}}.$$

Since $\left|F'\right|^q$ is convex, then we get

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_\varphi F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_\varphi F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$

$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda(1)}\left(\int_0^1\Lambda(\xi)d\xi\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\Lambda(\xi)\left(h(\xi)\left|F'(\kappa_2)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\Lambda(\xi)\left(h(\xi)\left|F'(\kappa_1)\right|^q+h(1-\xi)\left|F'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)d\xi\right)^{\frac{1}{q}}\right],$$

which is our required inequality (16).

Remark. Under the assumption of Theorem 5 $\varphi(\xi)=\frac{\xi^\alpha}{\Gamma(\alpha)}$, we have Theorem 2.3 of [23].

Remark. If we use $h(\xi)=\xi$ and $h(\xi)=\xi^s$ in Theorem 4, Theorem 5 and Corollary 2, then we obtain new inequalities for convex functions and s-convex functions in second sense, respectively.

3 Conclusion

In this research, we presented some Hermite-Hadamard type inequalities for h-convex functions utilizing the generalized fractional integrals. To validate that their generalized behavior, we showed the relation of our results with the previously published ones.

Conflict of Interest

The authors declare that they have no conflict of interest.

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